On Some Ser's Infinite Product

By Edigles Guedes

August 29, 2018

"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

Abstract. I derive some Ser's infinite product for exponential function and exponential of the digamma function; as well as an integral representation for the digamma function.

2010 Mathematics Subject Classification. Primary 26A09; Secondary 33B10, 33B15.

Key words and phrases. Exponential function, digamma function, infinite product, integral representation.

1. Introduction

In this paper, I prove the following integral representation for digamma function:

$$\psi(z)=\int_0^1 \frac{x^2-x^z\log x-x}{(1-x)x\log x}\mathrm{d}x.$$

Hereafter, I demonstrate some infinite products of the Ser-type:

$$e^{-\frac{1}{z}}=\prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}(z+k)^{(-1)^k\binom{n}{k}}\right)^{1/n},$$

$$\left(1-\frac{1}{z}\right)e^{\frac{1}{z}}=\prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}(z+k)^{(-1)^k\binom{n}{k}}\right)^{(n-1)/n},$$

$$\left(1-\frac{1}{z}\right)e^{-\frac{1}{z}}=\prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}(z+k)^{(-1)^k\binom{n}{k}}\right)^{(n+1)/n}$$

and

$$e^{-\frac{1}{z}-\psi(z)}=\prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}(z+k)^{(-1)^k\binom{n}{k}}\right)^{1/n(n+1)}.$$

Specifically, I derive

$$e^{\gamma-1}=\prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}(k+1)^{(-1)^k\binom{n}{k}}\right)^{1/n(n+1)}$$
$$=\left(\frac{1}{2}\right)^{1/2}\cdot\left(\frac{1\cdot 3}{2^2}\right)^{1/6}\cdot\left(\frac{1\cdot 3^3}{2^3\cdot 4}\right)^{1/12}\cdot\left(\frac{1\cdot 3^6\cdot 5}{2^4\cdot 4^4}\right)^{1/20}\cdot\ldots$$

2. Preliminaries

Lemma 1. *For $z>0$, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(z+k)=\int_0^1\frac{(1-x)^n x^{z-1}-\delta_n}{\log x}\mathrm{d}x, \qquad (1)$$

where $\log x$ *denotes the natural logarithm function and* δ_n *denotes the Kronecker delta, what for* $n=0$, *then* $\delta_n=1$ *and otherwise, then* $\delta_n=0$.

Proof. In [1, page 266], I meet with the integral representation

$$\log(v+1)=\int_0^1 \frac{x^v-1}{\log x}\mathrm{d}x, \tag{2}$$

provided for $v>-1$.

Changing $v=z+k-1$ in (1), multiply by $(-1)^k\binom{n}{k}$ and sum from 0 at n with respect to k

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(z+k)&=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_0^1\frac{x^{z+k-1}-1}{\log x}\mathrm{d}x\\&=\int_0^1\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x^{z+k-1}-1)\frac{\mathrm{d}x}{\log x}\\&=\int_0^1\frac{(1-x)^n x^{z-1}-\delta_n}{\log x}\mathrm{d}x,\end{aligned}$$

which is the desired result. □

Theorem 2. *For* $\mathrm{Re}(z)>0$*, then*

$$\psi(z)=\int_0^1\frac{x^2-x^z\log x-x}{(1-x)x\log x}\mathrm{d}x. \tag{3}$$

Proof. On the other hand, in [2, page 18], I encounter the infinite series

$$\psi(z)=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(z+k), \tag{4}$$

valid for $z>0$.

Multiply the Lemma 1 by $\frac{1}{n+1}$ and sum from 0 at infinity with respect to n and obtain

$$\begin{aligned}\psi(z)&=\sum_{n=0}^{\infty}\frac{1}{n+1}\int_0^1\frac{(1-x)^n x^{z-1}-\delta_n}{\log x}\mathrm{d}x\\&=\int_0^1\left[\sum_{n=0}^{\infty}\frac{(1-x)^n x^{z-1}-\delta_n}{n+1}\right]\frac{\mathrm{d}x}{\log x}\\&=\int_0^1\frac{x^2-x^z\log x-x}{(1-x)x\log x}\mathrm{d}x,\end{aligned}$$

which is the desired result. □

3. Infinite Series and Infinite Product

Theorem 3. *For a real number* $z>0$*, then*

$$-\frac{1}{z}=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(z+k). \tag{5}$$

Proof. Multiply the Lemma 1 by $\frac{1}{n}$ and sum from 1 at infinity with respect to n and obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(z+k)&=\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1\frac{(1-x)^n x^{z-1}-\delta_n}{\log x}\mathrm{d}x\\&=\int_0^1\left[\sum_{n=1}^{\infty}\frac{(1-x)^n x^{z-1}}{n}\right]\frac{\mathrm{d}x}{\log x}\\&=\int_0^1\left[\sum_{n=1}^{\infty}\frac{(1-x)^n}{n}\right]\frac{x^{z-1}}{\log x}\mathrm{d}x\\&=-\int_0^1 x^{z-1}\mathrm{d}x=-\frac{1}{z},\end{aligned}$$

which is the desired result. □

Corollary 4. *For a real number $z>0$, then*

$$e^{-\frac{1}{z}}=\prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}(z+k)^{(-1)^k\binom{n}{k}}\right)^{1/n}. \tag{6}$$

Proof. The exponentiation of the Theorem 3. □

Example 5. Set $z=1$ in Corollary 4

$$\frac{1}{e}=\left(\frac{1}{2}\right)^{1/1}\cdot\left(\frac{1\cdot 3}{2^2}\right)^{1/2}\cdot\left(\frac{1\cdot 3^3}{2^3\cdot 4}\right)^{1/3}\cdot\left(\frac{1\cdot 3^6\cdot 5}{2^4\cdot 4^4}\right)^{1/4}\cdots \tag{7}$$

Example 6. Set $z=2$ in Corollary 4

$$\frac{1}{\sqrt{e}}=\left(\frac{2}{3}\right)^{1/1}\cdot\left(\frac{2\cdot 4}{3^2}\right)^{1/2}\cdot\left(\frac{2\cdot 4^3}{3^3\cdot 5}\right)^{1/3}\cdot\left(\frac{2\cdot 4^6\cdot 6}{3^4\cdot 5^4}\right)^{1/4}\cdots \tag{8}$$

Example 7. Set $z=3$ in Corollary 4

$$\frac{1}{\sqrt[3]{e}}=\left(\frac{3}{4}\right)^{1/1}\cdot\left(\frac{3\cdot 5}{4^2}\right)^{1/2}\cdot\left(\frac{3\cdot 5^3}{4^3\cdot 6}\right)^{1/3}\cdot\left(\frac{3\cdot 5^6\cdot 7}{4^4\cdot 6^4}\right)^{1/4}\cdots \tag{9}$$

Theorem 8. *For a real number $z>1$, then*

$$\frac{1}{z}+\log\left(1-\frac{1}{z}\right)=\sum_{n=1}^{\infty}\frac{n-1}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(z+k). \tag{10}$$

Proof. Multiply the Lemma 1 by $1-\frac{1}{n}$ and sum from 1 at infinity with respect to n and obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(z+k)&=\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)\int_0^1\frac{(1-x)^n x^{z-1}}{\log x}\mathrm{d}x\\&=\int_0^1\left[\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)(1-x)^n\right]\frac{x^{z-1}}{\log x}\mathrm{d}x\\&=\int_0^1\frac{(1-x+x\log x)x^{z-2}\mathrm{d}x}{\log x}=\frac{1}{z}+\log\left(1-\frac{1}{z}\right),\end{aligned}$$

which is the desired result. □

Corollary 9. *For a real number $z>1$, then*

$$\left(1-\frac{1}{z}\right)e^{\frac{1}{z}}=\prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}(z+k)^{(-1)^k\binom{n}{k}}\right)^{(n-1)/n}. \tag{11}$$

Proof. The exponentiation of the Theorem 8. □

Example 10. Set $z=2$ in Corollary 9

$$\sqrt{e}=2\cdot\left(\frac{2\cdot 4}{3^2}\right)^{1/2}\cdot\left(\frac{2\cdot 4^3}{3^3\cdot 5}\right)^{2/3}\cdot\left(\frac{2\cdot 4^6\cdot 6}{3^4\cdot 5^4}\right)^{3/4}\cdot\left(\frac{2\cdot 4^{10}\cdot 6^5}{3^5\cdot 5^{10}\cdot 7}\right)^{4/5}\cdots \tag{12}$$

Example 11. Set $z=3$ in Corollary 9

$$\sqrt[3]{e}=\frac{3}{2}\cdot\left(\frac{3\cdot 5}{4^2}\right)^{1/2}\cdot\left(\frac{3\cdot 5^3}{4^3\cdot 6}\right)^{2/3}\cdot\left(\frac{3\cdot 5^6\cdot 7}{4^4\cdot 6^4}\right)^{3/4}\cdot\left(\frac{3\cdot 5^{10}\cdot 7^5}{4^5\cdot 6^{10}\cdot 8}\right)^{4/5}\cdots \tag{13}$$

Example 12. Set $z=4$ in Corollary 9

$$\sqrt[4]{e}=\frac{4}{3}\cdot\left(\frac{4\cdot 6}{5^2}\right)^{1/2}\cdot\left(\frac{4\cdot 6^3}{5^3\cdot 7}\right)^{2/3}\cdot\left(\frac{4\cdot 6^6\cdot 8}{5^4\cdot 7^4}\right)^{3/4}\cdot\left(\frac{4\cdot 6^{10}\cdot 8^5}{5^5\cdot 7^{10}\cdot 9}\right)^{4/5}\cdots \tag{14}$$

Theorem 13. *For a real number* $z>1$, *then*

$$-\frac{1}{z}+\log\left(1-\frac{1}{z}\right)=\sum_{n=1}^{\infty}\frac{n+1}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(z+k). \tag{15}$$

Proof. Multiply the Lemma 1 by $1+\frac{1}{n}$ and sum from 1 at infinity with respect to n and obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(z+k)&=\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\int_0^1\frac{(1-x)^n x^{z-1}}{\log x}\mathrm{d}x\\&=\int_0^1\left[\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)(1-x)^n\right]\frac{x^{z-1}\,\mathrm{d}x}{\log x}\\&=\int_0^1\frac{(1-x-x\log x)x^{z-2}\,\mathrm{d}x}{\log x}=-\frac{1}{z}+\log\left(1-\frac{1}{z}\right),\end{aligned}$$

which is the desired result. □

Corollary 14. *For a real number* $z>1$, *then*

$$\left(1-\frac{1}{z}\right)e^{-\frac{1}{z}}=\prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}(z+k)^{(-1)^k\binom{n}{k}}\right)^{(n+1)/n}. \tag{16}$$

Proof. The exponentiation of the Theorem 13. □

Example 15. Set $z=2$ in Corollary 14

$$\frac{1}{\sqrt{e}}=2\cdot\left(\frac{2}{3}\right)^{2/1}\cdot\left(\frac{2\cdot 4}{3^2}\right)^{3/2}\cdot\left(\frac{2\cdot 4^3}{3^3\cdot 5}\right)^{4/3}\cdot\left(\frac{2\cdot 4^6\cdot 6}{3^4\cdot 5^4}\right)^{5/4}\cdots \tag{17}$$

Example 16. Set $z=3$ in Corollary 14

$$\frac{1}{\sqrt[3]{e}}=\frac{3}{2}\cdot\left(\frac{3}{4}\right)^{2/1}\cdot\left(\frac{3\cdot 5}{4^2}\right)^{3/2}\cdot\left(\frac{3\cdot 5^3}{4^3\cdot 6}\right)^{4/3}\cdot\left(\frac{3\cdot 5^6\cdot 7}{4^4\cdot 6^4}\right)^{5/4}\cdots \tag{18}$$

Example 17. Set $z=4$ in Corollary 14

$$\frac{1}{\sqrt[4]{e}}=\frac{4}{3}\cdot\left(\frac{4}{5}\right)^{2/1}\cdot\left(\frac{4\cdot 6}{5^2}\right)^{3/2}\cdot\left(\frac{4\cdot 6^3}{5^3\cdot 7}\right)^{4/3}\cdot\left(\frac{4\cdot 6^6\cdot 8}{5^4\cdot 7^4}\right)^{5/4}\cdots \tag{19}$$

Theorem 18. *For a real number* $z\geqslant 1$, *then*

$$-\frac{1}{z}-\psi(z)=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(z+k). \tag{20}$$

Proof. Multiply the Lemma 1 by $\frac{1}{n(n+1)}$ and sum from 1 at infinity with respect to n and obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log(z+k)&=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\int_0^1\frac{(1-x)^n x^{z-1}}{\log x}\mathrm{d}x\\&=\int_0^1\left[\sum_{n=1}^{\infty}\frac{(1-x)^n}{n(n+1)}\right]\frac{x^{z-1}}{\log x}\mathrm{d}x\\&=\int_0^1\frac{(1-x+x\log x)x^{z-1}}{(1-x)\log x}\mathrm{d}x=-\frac{1}{z}-\psi(z),\end{aligned}$$

which is the desired result. ☐

Corollary 19. *For a real number* $z \geqslant 1$*, then*

$$e^{-\frac{1}{z}-\psi(z)} = \prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}(z+k)^{(-1)^k\binom{n}{k}}\right)^{1/n(n+1)}. \qquad (21)$$

Corollary 20. *I have*

$$\begin{aligned} e^{\gamma-1} &= \prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}(k+1)^{(-1)^k\binom{n}{k}}\right)^{1/n(n+1)} \\ &= \left(\frac{1}{2}\right)^{1/2}\cdot\left(\frac{1\cdot 3}{2^2}\right)^{1/6}\cdot\left(\frac{1\cdot 3^3}{2^3\cdot 4}\right)^{1/12}\cdot\left(\frac{1\cdot 3^6\cdot 5}{2^4\cdot 4^4}\right)^{1/20}\cdots \end{aligned} \qquad (22)$$

Proof. Set $z=1$ in Corollary 19. ☐

References

[1] Edwards, Joseph, *A Treatise on Integral Calculus with Applications, Examples and Problems*, Volume II, MacMillan and Co., Limited, London, 1922.

[2] Guillera, Jesús and Sondow, Jonathan, *Double Integrals and Infinite Products for Some Classical Constants via Analytical Continuations of Lerch's Transcendent*, arXiv:math/0506319v3 [math.NT], 5 Aug 2006.