Where Einstein Got It Wrong

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Abstract

In Einstein's General Relativity, gravitation is considered to be only an effect of spacetime geometry. Einstein considered the sources of gravitation to be all varieties of mass and energy excepting that of gravitational fields. This exclusion leaves General Relativity correct only to first order. By considering gravitational fields to be real entities that possess field energy densities and including them as source terms for the Einstein tensor, they contribute to spacetime curvature in a way that prevents the formation of event horizons and also has the effect of accelerating the expansion of the cosmos without any need for a cosmological constant (aka "dark energy").

Key words: general relativity

1 Introduction

When attempting to repair an error of long standing, it is perhaps best to begin by demonstrating clearly the nature of the problem. This will begin at a bedrock undergraduate level of physics. The many asides and caveats that will arise as we proceed will be be deferred to footnotes and appendices.

To begin, if a particle of mass m is displaced quasistatically by $d\vec{r}$ by a force $\vec{F}$, the work done is $\vec{F} \cdot d\vec{r} = c^2 dm$. This merely recognizes the Einstein relationship between energy and mass. If the force is due to a gravitational field for which the potential is U; and rendered dimensionless by division by c^2, such that $\phi = U/c^2$, then the gravitational force on m is given by $\vec{F} = -mc^2\vec{\nabla}\phi$. The change of potential associated with a quasistatic displacement of the particle is the work done by an opposing external force, thus

$$c^2 dm = mc^2 \vec{\nabla}\phi \cdot \vec{dr} = mc^2 d\phi \tag{1}$$

which integrates to $m = m_a e^{(\phi - \phi(a))}$ where the mass would be m_a with the particle at rest at position $\vec{r}_a$. If the potential would be zero at this point we can write

$$m = m_0 e^{\phi} \tag{2}$$

We can consider the same quasistatic process from the standpoint of the the energy-momentum equation, which is slightly modified from the usual result from special relativity.

$$g^{ij} p_i p_j = g_{ij} p^i p^j = m_o^2 c^2 \tag{3}$$

where g^{ij} are metric coefficients (the metric considered here can be regarded as diagonal where $ds^2 = g_{ij} dx^i dx^j$) and p_i the components of momentum of the particle. In this particular case, $p_0 = E/c$ and all other components of momentum are zero[2], thus

$$g^{00} E^2/c^2 = g^{00} m^2 c^2 = m_0^2 c^2 \tag{4}$$

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[2] This state of motion is certainly accessible to a particle that moves under the influence of gravity, for example near the apex of flight of a particle moving in the field of a central mass. There is no need to imagine any other external force. It is only necessary that Eq. 2 apply

Substituting for m from Eq.2 leads immediately to $g^{00}e^{2\phi} = 1$. Noting that $g_{00} = 1/g^{00}$ for a diagonal metric we have

$$\mathbf{g_{00} = e^{2\phi}} \tag{5}$$

While Eq. 5 is a general result, Eq. 2 must be modified if a particle is in motion in a gravitational field. The end result is that the mass given by Eq. 2 needs to be increased by the Lorentz factor $1/\sqrt{1 - v^2/c^2}$, where v is the proper speed of the particle at its current location. See Eq. 58 of Appendix VII where this is discussed in more detail.

2 Where things go wrong

This is where the fun begins. Eq. 5 depends only upon the definition of a potential and the correctness of special relativity. The problem is that this expression for g_{00} is NOT, to my knowledge part of the solution for the metric of any configuration of gravity sources within the context of General Relativity. For example, in the Schwarzschild coordinates and metric for the space exterior to a central mass, M, the potential is $\phi = -GM/c^2r$ and the Hilbert solution of the Einstein equations yields

$$g_{00} = 1 + 2\phi \tag{6}$$

This clearly differs from Eq. 5 by the absence of terms of second and higher order in ϕ. The lack of these terms permits g_{00} to vanish at the Schwarzschild radius, $r = 2GM/c^2$. This occurrence of an event horizon is considered by many to be a mere coordinate singularity that can be removed by a transformation to different coordinates, but that is simply wrong (see Appendix VI). There is a genuine divergence of a physical quantity, the norm of the acceleration vector of a freely falling particle, at an event horizon. It is astonishing to me that otherwise reputable physicists are not troubled by a physical singularity.

It is well known that frequency shifts occur for photons in gravitational fields. If the frequency would be ν_0 at a location where $\phi = 0$, then the frequency at other locations is given by

$$\nu = \nu_0 \; g_{00}^{-1/2} \tag{7}$$

and according to Eq. 5, this would be

$$\nu = \nu_0 \; e^{-\phi} \tag{8}$$

Eq.8 can be be shown to be an EXACT requirement of special relativity and the principle of equivalence (See Appendix I). The photon red shift result of Eq. 8 was stated by Einstein in a 1907 paper (available as translated by H.M. Schwartz, Am. J. Phys, 45, 899, 1977). Although Einstein had first arrived at a first order approximation, he noted that "**in all strictness**" this first order result must be replaced by the exponential form of Eq. 8. For a time after 1907, Einstein maintained that the metric coefficients must be strictly exponential functions in order to conform to the requirements of special relativity, but his final development of general relativity satisfied the requirement only to terms of first order. That was a mistake that needs to be repaired.

3 Gravitational waves

As another exhibit in the case against a theory that that is correct only to first order, consider a topic of considerable recent interest; viz, gravitational waves. We imagine these to be disturbances of spacetime that propagate through a Minkowskian interstellar vacuum. In this case, the Einstein equations are equivalent to those obtained by setting the Ricci tensor components equal to zero.

Consider a metric that might represent gravitational waves propagating along a z-axis with small oscillations of opposite amplitudes in the x and y directions, i.e.;

$$ds^2 = c^2dt^2 - e^{4\phi}dx^2 - e^{-4\phi}dy^2 - dz^2 \tag{9}$$

Where $\phi = \phi(t \pm z/c)$ represents a wavelike distortion of what would otherwise be Minkowskian spacetime. Denoting partial derivatives with respect to z with primes and time derivatives with dots, the Ricci tensor components for the metric are:

$$\begin{gathered} R_{11} = -2e^{4\phi}(\ddot{\phi} - \phi'') \qquad R_{22} = 2e^{-4\phi}(\ddot{\phi} - \phi'') \\ R_{00} = 8(\dot{\phi})^2, \qquad R_{33} = 8(\phi')^2, \qquad R_{30} = -8(\dot{\phi})(\phi') \end{gathered} \tag{10}$$

If set equal to zero, R_{11} and R_{22} obviously yield the wave equation for waves of arbitrary amplitude propagating along the z-axis. Unfortunately, these waves are NOT solutions of the Einstein field equations because none of R_{00}, R_{30} or R_{33} are zero. In fact R_{00}, R_{30} and R_{33} look suspiciously like terms that represent the energy that would propagate with ordinary waves in a medium[3]. If we insist that $R_{ik} = 0$ because we are in matter free space, we are effectively saying that gravitational waves of the kind considered here are not solutions of the Einstein field equations. They would be solutions only for infinitesimal amplitudes and first order. It would make a lot more sense to say that there are energy densities associated with these waves that should be included as source terms in the right members of the Einstein field equations. Given the form of the metric of Eq. 9, and recognizing that the function, ϕ, is a time dependent gravitation potential, it should not require a genius to guess that the squared derivatives of ϕ might have something to do with the energy that propagates with the gravitational waves. All four equations can be satisfied in this way without restricting the waves to be of infinitesimal amplitude. The simple cure for the ills of general relativity consists of regarding gravity fields as real entities that can contribute as sources for the Einstein tensor.

3.1 The Schwarzschild point mass problem

In arriving at Eq. 5, we did not consider the spatial dependence of the metric tensor, but it is necessary to do so in order to see what would be needed for Eq. 5 to be incorporated into the metric tensor. As shown in Appendix II, the following isotropic[4] metric form can be used to describe the static spacetime exterior to a point mass M.

$$ds^2 = e^{2\phi}c^2dt^2 - e^{-2\phi}(dx^2 + dy^2 + dz^2) \tag{11}$$

As discussed in Appendix II, this form was chosen because it permits photons and other waves to pass through this metric spacetime without dispersion. The wave speeds are independent of wavelength or direction of propagation relative to the direction of the gravitational field. In common technical jargon, this metric satisfies a "Harmonic Coordinate" condition, $\partial_k(\sqrt{-g}\ g^{kj}) = 0$ (Weinberg 1972, p. 163) . The Einstein field equations for this metric can be written as

$$R_{ik} = -(8\pi G/c^4)(T_{ik} - g_{ik}T/2) \tag{12}$$

[3]The Einstein equations, in the form $G_i^j = -(8\pi G/c^4)T_i^j$ can be rewritten as $R_{ik} = -(8\pi G/c^4)(T_{ik} - g_{ik}T/2)$, where T_{ik} is the stress-energy tensor that represents matter and/or all other energy densities

[4]The isotropic form with all spatial dimensions affected by gravity in the same way would seem to be necessary for consistency with the Hughes-Drever experiments (Hughes, Robinson & Beltran-Lopez 1960 and Drever 1961) that revealed the isotropy of inertia.

where $T = T_k^k$. Evaluating the Ricci tensor for the metric form of Eq. 11 yields the following results:

$$\begin{aligned} R_{00} = -e^{-4\phi}(\phi'' + 2\phi'/r) \qquad R_{22} = -r^2(\phi'' + 2\phi'/r) \\ R_{33} = -r^2 sin^2\theta(\phi'' + 2\phi'/r) \qquad R_{11} = -(\phi'' + 2\phi'/r) + 2\phi'^2 \end{aligned} \tag{13}$$

Here primes represent derivatives with respect to r. According to the Einstein theory, $T_{ik} = 0$ in the matter free space beyond M. If the first three of these expressions were set equal to zero, $\phi = -GM/c^2r$ would be a solution of the equations. However, if this solution would be substituted into the fourth expression, the result would be $R_{11} = 2(GM/c^2r^2)^2 \neq 0$.

To the electric field, $\vec{E}$ of a point charge we attribute an energy density of $\epsilon_0 E^2/2$. In the Newtonian limit, we attribute a gravitational field, $g = -GM/r^2 = -c^2\phi'$ to the space in the vicinity of mass M and we can attribute to this field an analogous energy density of $g^2/(8\pi G) = (GM/r^2)^2/(8\pi G) = c^4\phi'^2/(8\pi G)$. Thus the last term of the expression for R_{11} is very clearly related to the energy density of the gravitational field. If the gravitational field energy density is regarded as a source in T_{ik}, then it should be represented in the right member of the equation for R_{11}. If $T_0^0 = T_2^2 = T_3^3 = -T_1^1 = g^2e^{2\phi}/(8\pi G)$ then the right member of the R_{11} equation would become $-(8\pi G/c^4)(T_{11} - g_{11}T/2) = 2\phi'^2$ while leaving the right members of the other Ricci components equal to zero.

This all that is required to make the metric of Eq 11 become a solution of the Einstein gravitational field equations. The failure to include field energy density has led to the Hilbert modification of the original Schwarzschild (Schwarzschild 1916) solution with its gravitational time dilation singularity and event horizon at $r = GM/2c^2$, for an isotropic metric.[5]. With $\phi = -GM/c^2r$, $g_{00} \to 0$ as $r \to 0$, but this is just an indicator of the inadequacy of a classical point particle model. It does not represent a pathological singular condition of the metric.

There is a subtle point about the inclusion of field energy density terms in T_{ik}. When a form, such as Eq. 11, is adopted for the metric, the construction of the Einstein tensor will generate terms sufficient to determine the field energy density from the solutions of the equations for some particular distribution of energy-momentum given by the sources in the right member. But this is usually only determined AFTER the solution is obtained. If you do not include obvious and explicit expressions for the field energy density, such as $g^2/8\pi G$ in T_{ik}, you may find spurious and incorrect solutions of the field equations and fool yourself into thinking that there is no field energy density, but that is not the case.

For example, consider the case of an isolated static sphere consisting of a perfect fluid for which the components of T_i^j are $(\rho c^2, -p, -p, -p)$ where ρ is the proper energy density of the fluid and p the internal pressure. For a fluid particle, the symbols, ρc^2 and p represent internal energy densities that actually include the gravitational field energy density in the space occupied by the particle; indeed, the pressure in the sphere owes its non-zero existence to gravity. Only AFTER the field equations have been solved for the metric tensor components, can one calculate the gravitational field energy density and the pressure. Einstein's pseudo-tensor for the gravitational field energy density is expressed in terms of the derivatives of the metric tensor which are then found to be determined by ρc^2 and the sphere radius. If it has been assumed at the outset that ρc^2 does not include any energy density of the gravitational field, this is a contradictory result.

For the perfect fluid sphere, one can imagine partitioning ρc^2 into a rest mass density, σc^2, plus a gravitational field energy density of $-g^2/(8\pi G)$ in T_0^0, but it is not clear just how the pressure

[5]But Schwarzschild's original solution had neither, see translation of original article by Antoci and Linger https://arxiv.org/abs/physics/9905030

terms should be modified for the other principal stress components. In the matter free space outside the fluid sphere, it is clear that $\sigma = 0$ and $p = 0$, but there is still an exterior gravitational field. One surely needs to include the field energy density as a source in the right member of the field equations for the exterior even if $g^2/(8\pi G)$ must be expressed in terms of some of the same derivatives that enter the Einstein tensor. Feynman and others have shown that the Einstein tensor uniquely contains the appropriate energy operator terms, but that does not in any way imply that the right member cannot contain terms depending on parts of the metric tensor or its derivatives. The entire community of general relativist theorists seems to have collectively misunderstood the concept of an energy-momentum source.

4 Tweaking General Relativity

In the current swamp that is relativistic cosmology, it seems that theorists are quite willing to resurrect something that Einstein is said to have considered to be his biggest blunder; namely his cosmological constant. He used it to produce a static cosmology and missed the chance to predict the expansion of the universe. With G_i^j representing the Einstein tensor, T_i^j, the "matter" stress-energy tensor and Λ the cosmological constant the Einstein field equations are:

$$G_i^j = -(8\pi G/c^4)T_i^j - \delta_i^j \Lambda \tag{14}$$

In this form, $\Lambda c^4/(8\pi G)$ is considered to be a constant "dark energy" density of the cosmological vacuum. This makes it quite clear that it is apparently acceptable to have an energy density source in addition to the "matter tensor", T_i^j, in the right member of the Einstein field equations. Λ represents an energy density in the otherwise empty space of the cosmological vacuum for which $T_i^j = 0$ where no mass is present. In the discussion to follow, it will become apparent that there are some better choices than a constant Λ for the right member.

The cosmological constant has added another epicycle to astronomy, but if it represents the ground state oscillations of all of the fields within the cosmos, we might expect its value to be roughly 120 orders of magnitude larger (Carroll 2004, Sec. 4.5) than the $\sim 10^{-8} erg/cm^{-3}$ that is needed to explain the cosmological redshift observations of type 1a supernovae. In addition to this rather glaring discrepancy, the energy density of matter would decrease in an expanding universe, which would allow only one coincidental moment in time in which matter and vacuum energy densities might be of comparable magnitude as they are at present. This coincidence problem and the difficulties associated with the cosmological constant and the dubious concepts of dark energy and non-baryonic dark matter can be removed by replacing the cosmological constant with a variable stress-energy tensor in the right member of Eq. 14. Contrary to Einstein's fiat, if a gravitational field exists in space, there will also be a field energy density. Setting the Einstein or Ricci tensors to zero in matter-free space is simply a mistake because that would also say that there can be no gravitational field energy there.

Huseyin Yilmaz (Yimaz 1958 ... 1992) proposed that the right member of the Einstein field equations be modified by removing the cosmological constant and replacing it with a variable gravitational field stress-energy tensor. The modified equation can be written as:

$$G_i^j = -(8\pi G/c^4)(T_i^j + t_i^j) \tag{15}$$

where t_i^j is the Yilmaz stress-energy tensor of the gravitational field (Yilmaz 1971, 1992.) In 1992 Yilmaz provided general equations for t_i^j that permits its evaluation in terms of the metric tensor components. (These equations are shown below in Appendix III.) They can be used to verify that the right member of Eq. 13 for R_{11} should be $2\phi'^2$.)

Three years ago I applied Eqs. 15 and an exponential metric of the form of Eq. 26 to the calculation of SNe 1a redshifts and showed that the results agree superbly with observations (Robertson 2015). The only adjustable parameter required was the present mean density of matter in the universe, which corresponds to a Hubble constant of 64.5 ± 0.7 km s^{-1} Mpc^{-1}. This value of the Hubble constant was obtained from the redshift and luminosity measurements of type 1a supernova for very small redshifts ($z < 0.1$).

The gravitational field stress-energy source in the right member of the field equations removed the need for the "dark energy" that was represented by a cosmological constant and also eliminated any need for non-baryonic dark matter. Recall also that treating the gravitational field as an energy-momentum source also removed event horizons. This is pretty impressive for such a minor change of viewpoint about what constitutes the source terms of the right member of the field equations.

As outlined above, it is a fairly simple matter to show that gravitational waves propagating through space will carry field energy and momentum in accord with Eqs. 10. These give us some concrete tools for describing how they can transport energy to the masses that are in our gravitational wave detectors. We do not necessarily have to invoke warps propagating in space to describe them. But some considerable work remains to be done for understanding the generation of gravitational waves by accelerating masses. It is worthy of mention that initial calculations using the Yilmaz theory indicate that the compact objects masses that have been reported for the LIGO detections of gravitational waves would be reduced by somewhere between about one third and one half, leaving masses nearer 15 - 20 $M_\odot$. Masses in this range are more like those found in present-day stellar mass black hole candidates and would not require such incredibly massive progenitor stars to produce them.

To this point this discussion has been made as simple as possible; relying on physical arguments and correspondences with special relativity and Newtonian gravity as limiting cases. The results accord with all of the weak-field tests of gravity theory and also encompass the accretion disks of relativistic astrophysics. But it leaves a need for a new explanation of the gravitationally compact objects that are thought to be black holes. If event horizons do not exist, then these must be some other kind of exotic compact object, such as the Eddington balanced MECO (Robertson & Leiter 2003, 2006, Robertson 2016).

5 Summary and Conclusions

In Eq. 5 we found a result that is incompatible with the known solutions of the Einstein field equations for static fixed mass gravitational sources. By considering gravitational waves, it could be easily seen that the issue is a conceptional problem: Einstein's exclusion of gravitational field energy densities as sources in the right members of the Einstein field equation is unwarranted. It leaves the theory as correct only to first order in the gravitational potentials.

It was also shown that the addition of the energy density of the gravitational field of a central mass permitted a Newtonian potential to survive as a solution of the field equations. This restored compatibility of the solution with Eq. 5 and had the effect of eliminating the event horizon from the solution for a point mass. This still leaves a point mass singularity as $r \to 0$, for which $g_{00} \to 0$, but this is just a problem with the concept of a point mass and not a pathological feature of the metric. In the well-known solution of the unmodified field equations, g_{00} vanishes at the event horizon before we reach $r = 0$.

In this regard, we can see that the added gravitational field energy density had a negative curvature effect on the spacetime metric. If sources of positive mass-energy densities produce positive curvature of spacetime, gravitational field stress-energy densities inherently produce the opposite effect on curvature. Without this negative effect, spacetime becomes too warped too soon as we approach the point particle and an event horizon forms for $r > 0$.

A similar effect occurs in the case of a cosmological metric. Accommodating the change of redshift of light from supernovae can be accomplished using a cosmological constant, for which it is freely admitted that it represents a gravitational energy density known as "dark energy", Einstein's exclusion notwithstanding. It has the effect of partially countering the gravitational attraction of the "cosmic dust" particles and allowing the model cosmos to expand at an accelerating rate.[6] But it is clear that the supernovae data can be accommodated better (Robertson 2015) by the inclusion of gravitational field stress-energy tensor source. This required only one free parameter, the mass-energy density of "cosmic dust".

The clear implication of these results is that permitting gravitational field energy to serve as a source in the Einstein field equations allows the solutions to be valid to more than first order in the potentials. The modified theory still passes all of the observational tests that have been devised for relativistic gravity theories. Whether the modified theory will remain correct when higher order tests can be devised remains to be seen. In the meantime, the clear message that needs to be understood here is that gravity is a field in its own right. It is NOT merely and entirely and only an effect of the geometry of spacetime.

5.1 What's Next?

If gravity must be regarded as something more than spacetime geometry, then what is it? That remains to be determined by theorists who take up the task in the future. The Yilmaz theory is a good underpinning for the descriptive and geometric aspects that have been discussed here, but it remains to be seen if it can be extended to a matter continuum or anywhere else that a "harmonic coordinate condition" would fail. This would seem to be a promising research area.

As for the physical reality that undergirds gravity, it is probably some sort of quantum field. It could involve quarks and gluons, Higgs bosons or even photons. For example, it has been proposed that gravity might be electromagnetic in origin (e.g., see Puthoff 1989, 1999 and references therein) and gravity somewhat similar to a van der Waals force. But whatever one may think, it will require elements of reality that are more than geometry.

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Appendix I: The Principle of Equivalence and Gravitational Redshift

According to the principle of equivalence, neglecting tidal effects due to nonuniform gravitational fields, the effects of gravitation should be indistinguishable from an acceleration of the same magnitude as that which would occur in free fall in the gravitational field. So consider two elevators, one of which is at rest in a gravitational field and the other in free space accelerating upward with the same equivalent constant acceleration; call it g. At the time the second elevator begins to accelerate, let a photon be emitted from a source at its floor and let it be absorbed later in a detector in its ceiling, a distance L away in the frame of the accelerating elevator. While the photon is in transit, the detector acquires some speed, v. From the position of the detector, it is the same as if the source were receding from it at speed v. Thus if the frequency of the photon emitted at the floor is ν_0, the ceiling detector will detect the Doppler shifted frequency

$$\nu = \frac{\nu_0(1 - v/c)}{\sqrt{1 - v^2/c^2}} \tag{16}$$

We can determine the speed, v, of the ceiling photon detector from the special relativistic relation

$$a_z = \frac{dv}{dt} = \frac{a'_z}{(\gamma^3)(1 + u'_z v/c^2)^3} = \frac{g}{\gamma^3} \tag{17}$$

where $\gamma = 1/\sqrt{(1 - v^2/c^2)}$ and $u'_z = 0$ is the detector speed relative to a comoving and coincident inertial frame at the instant that the photon arrives at the detector. Proper time increments, $d\tau$ in the accelerating elevator are contracted such that $d\tau = dt/\gamma$ Substituting for dt into Eq. 17, integrating and setting $\tau = L/c$, we obtain

$$v/c = \tanh(gL/c^2) \tag{18}$$

Substituting Eq. 18 into Eq. 16, we find

$$\nu = \nu_0 e^{-gL/c^2} \tag{19}$$

By the principle of equivalence, the first elevator which is at rest in a gravitational field (for which the free fall acceleration would be g) would have to produce the same red shift gravitationally. In this elevator, the change of (dimensionless) gravitational potential between floor and ceiling is, of course, $\Delta\phi = gL/c^2$ So the gravitational frequency decrease is again given **EXACTLY** by the exponential function $e^{-\Delta\phi}$. As noted previously, this result was well known by Einstein (H.M. Schwartz, 1977)

Appendix II: Spatial Dependence of the Exponential Metric:

We can write an isotropic metric form as

$$ds^2 = e^{2\phi} c^2 dt^2 - e^{\lambda}(dx^2 + dy^2 + dz^2) \tag{20}$$

with $\lambda(x, y, z)$ a function of spatial coordinates and time yet to be determined. We expect the spacetime described by Eq. 20 to permit the existence of photons. We can represent these as wave packets of the form

$$\psi = \Sigma e^{i(\omega t - \vec{k}\cdot\vec{r})} \tag{21}$$

The component waves should satisfy the generalized d'Alembertian equation

$$\Box\psi = \sqrt{-g}^{\,-1} \partial_k(\sqrt{-g}\; g^{kj}\partial_j)\psi = 0 \tag{22}$$

where $(g) = -e^{(3\lambda+2\phi)}$ is the determinant of the metric.

Substituting for a component of ψ from Eq. 21, expanding Eq. 22 and rearranging yields

$$\omega^2 + i\omega\partial_0(3\lambda/2 - \phi) + c^2 e^{(-\lambda+2\phi)}[k^2 + i\vec{k}\cdot\vec{\nabla}(\lambda/2 + \phi)] = 0 \tag{23}$$

There are several circumstances in which the metric is static; i.e. $\phi = \phi(x, y, z)$ is independent of time. When that is the case the time derivatives and second terms of Eq. 23 are zero. When time dependence would be present these terms could lead to absorption of light. If the last terms of Eq. 23 do not vanish, they would lead to dispersion with wave speeds being dependent on wavelengths and gravity field gradients as well as their direction of motion relative to the field. There might be cases of light propagation in a matter continuum in which gravity might be dispersive, but in matter-free space, this cannot be allowed. Thus it is necessary to have $\vec{\nabla}(\lambda/2 + \phi) = 0$, which requires $\lambda/2 + \phi = Constant$ and the constant must be zero as a boundary condition at infinity. So, for most present circumstances we require.

$$\lambda = -2\phi \tag{24}$$

In this case, the metric is the Yilmaz exponential metric (Yilmaz 1958). It should be noted, however, that it is possible to permit time dependence of the metric while maintaining a requirement that gravity not produce dispersion of photons. Define functions ϕ_0 and ϕ_1 such that

$$4\phi_0 = 3\lambda/2 - \phi \qquad 4\phi_1 = \lambda/2 + \phi \tag{25}$$

and require that $\partial_0\phi_0 = 0$ and $\vec{\nabla}\phi_1 = 0$. Thus ϕ_0 depends only on (x, y, z) while ϕ_1 depends only upon time, t. The metric can then be written as

$$ds^2 = e^{(2\phi_0+6\phi_1)}c^2dt^2 - e^{(-2\phi_0+2\phi_1)}(dx^2 + dy^2 + dz^2) \tag{26}$$

This is the form of the metric that I used for the solution of the Yilmaz field equations for a cosmological spacetime (Robertson 2015). This produced a very good agreement between theory and observations of the redshifts of distant type 1a supernovae without the need of any "dark energy" or "non-baryonic "dark matter".

For the static field of a single central mass Eq. 26 can be used with $\phi_0 = -GM/c^2r$ while $\phi_1 = 0$, In general, whether there is time dependence or not, the condition that waves pass without dispersion and the same speed in all directions is ensured by what is known as the "Harmonic Coordinate Condition",

$$\partial_k(\sqrt{-g}\; g^{jk}) = 0 \tag{27}$$

where $(-g)$ is the positive definite determinant of the metric tensor. This condition produces Eq. 23 for the metric form of Eq. 20.

Appendix III: Notes on Yilmaz Theory

Unlike General Relativity, the Yilmaz theory seems to be a theory of particles and fields. It is not clear how it might be modified to encompass a matter continuum. Yilmaz has presented the theory in a set of gravitational potential tensor equations in some cases and as given here next in others. These are equivalent as long as a harmonic coordinate condition can be maintained. For example, either approach leads to the same set of equations for a metric for a description of a pressureless cosmological "dust", but it can be shown that the harmonic condition cannot hold if there is internal pressure. Thus it is not clear that the theory can be applied to the description of the early moments of the Big Bang.

As considered here, the Yilmaz theory differs from the Einstein theory primarily by the inclusion of a true stress-energy tensor as a source term in the right member of the field equations

$$G_i^j = -(8\pi G/c^4)(T_i^j + t_i^j) \tag{28}$$

where $T_i^j = \sigma u_i u^j$ is the matter tensor (when no non-gravitational forces contribute), and t_i^j is the gravitational field stress-energy tensor. Other requirements are[7]

$$(T_i^j + t_i^j); j = 0 \quad \textit{Bianchi requirement} \tag{29}$$

[7] Einstein's theory would require $T_i^j; j = 0$. As noted by Landau & Lifshitz, (Landau & Lifshitz 1962), this does not express any conservation law whatever. It would not be a statement of energy-momentum conservation, because it would not include the energy-momentum of the gravitational field. Eq. 30 is the energy-momentum conservation law of the Yilmaz theory.

$$\bar{\partial}_j(\sqrt{-K}\ T_i^j) = 0 \quad Freud\ requirement \tag{30}$$

Here the overbar represents a covariant derivative with respect to local Minkowskian coordinates which share the same origin and orientation as those of the metric. $\sqrt{-K} = \sqrt{-g}/\sqrt{-\eta}$, where $\sqrt{-g}$ is the determinant of the metric and $\sqrt{-\eta}$ is the determinant of the metric of the Minkowskian background. In rectangular coordinates (x,y,z,t), $\sqrt{-\eta} = 1$ and all Christoffel symbols vanish, leaving a normal partial derivative.

This elaborate derivative procedure is necessary to eliminate pseudotensors that might otherwise arise. The pseudotensor problem has been discussed in detail (Yilmaz 1992). This procedure eliminates them. Pseudotensor problems can be avoided in two ways. The first simply consists of the use of rectangular coordinates, in which they never appear. The second is to take derivatives as covariant derivatives in local Minkowskian coordinates.

Einstein's gravitational energy expression, t_i^j, has been shown to be a pseudotensor; however, it can be expressed in terms that eliminate pseudotensors and leave a true tensor quantity. First define terms

$$\mathbf{g}^{ij} = \sqrt{-K} g^{ij} \quad \mathbf{g}_{ij} = g_{ij}/\sqrt{-K} \tag{31}$$

and

$$W_i^j = (1/8\sqrt{-K})\mathbf{g}^{jk}[\bar{\partial}_k \mathbf{g}_{ab} \bar{\partial}_i \mathbf{g}^{ab} - 2\bar{\partial}_k \sqrt{-K}\bar{\partial}_i(1/\sqrt{-K}) - 2\bar{\partial}_a \mathbf{g}_{kb} \bar{\partial}_i \mathbf{g}^{ab}] \tag{32}$$

then

$$t_i^j = W_i^j - (1/2)W_k^k \tag{33}$$

These equations are supposed to construct all of the parts of the gravitational field stress-energy tensor in terms of the components of the metric tensor. I have not verified this in general, but they seem to be correct in every special case that I have considered. It is cumbersome, slow, difficult and painful to work through these by hand calculations, but they do provide the correct right members for all of the Ricci tensor equations in the main text here. The solution of Yilmaz Eq. 28 above for a point mass, M, is, indeed, the Newtonian potential, $\phi = -GM/c^2r$ and the field energy density is given correctly and consistently. This is in sharp contrast with results for the unmodified Einstein theory. The Einstein pseudo-tensor for the gravitational field energy density of the Schwarzschild solution is non-zero in spherical coordinates. But Schrodinger showed that it vanishes when transformed into rectangular coordinates. Similarly in curvilinear coordinates, an Einsteinian flat space time will show a spurious gravity field energy.

Yilmaz often used expressions that incorporated a harmonic coordinate condition. The Yilmaz expressions for W_i^j and t_i^j given here were based on Pauli's decomposition of the Einstein tensor with no harmonic coordinate conditions imposed. Yilmaz (1992) also provided a general expression for T_i^j.

$$(1/4\sqrt{-K})\bar{\partial}_a[\bar{\mathbf{g}}^{ak}\bar{\mathbf{g}}^{jb}(\bar{\partial}_b \mathbf{g}_{ik} - \bar{\partial}_k \mathbf{g}_{ib}) + \delta_i^j \bar{\partial}_b \mathbf{g}^{ba} - \delta_i^a \bar{\partial}_b \mathbf{g}^{bj}] = T_i^j \tag{34}$$

Appendix IV: Particle Mechanics

The energy-momentum equation in spherical coordinates is also of particular interest when the potential is that of a central mass, M, and and static with $\phi = \phi_0 = -GM/c^2r$ in the metric form of Eq. 26. In this case it should be noted that g_{00}, as given by Eq. 5, is nonzero for $r > 0$. This metric does not describe an object with an event horizon for any $r > 0$. In this case, the energy-momentum equation is:

$$g^{ij}p_ip_j = e^{-2\phi}E^2/c^2 - e^{2\phi}[p_r^2 + p_\theta^2/r^2 + p_\Phi^2/(r\ sin\theta)^2)] = m_0^2c^2 \tag{35}$$

Here we have used $p_0 = E/c$, $g^{00} = 1/g_{00} = e^{-2\phi}$, $g^{11} = 1/g_{11} = -e^{2\phi}$, $g^{22} = 1/g_{22} = -1/(r^2 e^{-2\phi})$ and $g^{33} = 1/g_{33} = -1/[(r\ sin\theta)^2 e^{-2\phi}]$. For a particle in an equatorial plane, $\theta = \pi,\ d\theta = 0,\ p_\theta = 0$, this becomes

$$e^{-2\phi} E^2/c^2 - e^{2\phi}[p_r^2 + p_\Phi^2/r^2] = m_0^2 c^2 \tag{36}$$

This equation easily accounts for all of the classical weak-field tests of general relativity that have been performed in the solar system. These include the perihelion shift of the planet Mercury, for which the orbital axis precesses by 43 seconds of arc per century, the deflection of star light that grazes the sun, and the time delay of radar echoes from Venus when their path grazes the sun. (For photons of light or radar, $m_0 = 0$ in Eqs. 35 and 36.) In each of these tests, it is only necessary to include in Eq. 36 the terms in the metric that are of low order in ϕ, such that $e^{2\phi} \approx 1 + 2\phi + 2\phi^2$ and $e^{-2\phi} \approx 1 - 2\phi + 2\phi^2$. The largest value of ϕ available in the solar system is $\phi = 2 \times 10^{-6}$ at the surface of the sun. At earth's surface $\phi = 7 \times 10^{-10}$ due to the earth's gravity. The potential at the location of the earth due to the presence of the sun is 10^{-8}, hence larger still than that of earth.

There is an interesting astrophysical application of Eq. 36 consisting of the description of particles in accretion disks in x-ray binary star systems. Consider a particle in orbit around a central mass, M, for which $\phi = -GM/c^2 r$. Assuming that angular momentum, p_Φ, is conserved, and rearranging Eq. 36 we obtain.

$$e^{4\phi}(p_r/m_0 c)^2 = (E/m_0 c^2)^2 - e^{2\phi}(1 + a^2\phi^2 e^{2\phi}) \tag{37}$$

Here $a = p_\Phi c/(GMm_0)$ is now a dimensionless conserved angular momentum parameter. Eq. 37 is similar to the energy equation of classical mechanics with the last terms at the right taking the role of an effective potential for the radial motion; i.e. $U(r) = e^{2\phi}(1 + a^2\phi^2 e^{2\phi})$. Bound orbits can occur for suitably low energies. Circular orbits can occur for $p_r = 0$. Their radii can be located by setting the derivative of the effective potential with respect to ϕ to zero. Circular orbits occur for $dU/dr = 0$, for which we find

$$a^2 = -e^{-2\phi}/(\phi + 2\phi^2) \tag{38}$$

with particle energies of

$$E = m_0 c^2 \exp(\phi)\sqrt{\frac{1+\phi}{1+2\phi}} \tag{39}$$

Orbits are stable if the second derivatives are positive at turning points. They are unstable otherwise. There is an innermost (marginally) stable orbit that can be found by setting the first two derivatives of the effective potential with respect to ϕ to zero. This produces coupled equations which have the solution, $\phi = -(3 - \sqrt{5})/4 \approx$ -0.191 for which $a^2 = 12.4$. For this innermost marginally stable circular orbit, the particle energy is reduced below its rest mass value, $m_0 c^2$ by 5.5% in spite of its motion at 34.5% of light speed. This is a result of the reduction of mass that occurs for a particle that is lower in the gravity field of the central mass for $\phi < 0$, in accord with Eq. 2.

A Bit of Astrophysics:

In relativistic astrophysical accretion disks, such as those found in low mass x-ray binary systems, orbiting fluid particles lose energy due to friction while fluid viscosity transports angular momentum outward. This allows the fluid to spiral inward to a central compact star or black hole candidate. In the process, accreting plasma can become hot enough to allow energy to be radiated away as soft x-rays. Apparently 5.5% of the rest mass energy can be converted to radiation for particles that would reach an innermost marginally stable orbit.[8] This provides luminosities far in excess of what can be obtained from nuclear interactions. Nuclear processes typically result in energy losses of less than about one percent. Note also that it would take an extremely compact object to exist inside the innermost marginally stable orbit. A mass as large as that of the sun would have a radius of only 7.8 km if it were all contained inside its innermost stable orbit. Thus it is not surprising that the x-ray binary systems have very compact neutron stars or black hole candidates as their central massive objects.

According to Eq. 39, it would appear to require infinite energy for a particle to exist in a circular orbit for $\phi = -1/2$. But there are simply no stable orbits with $\phi < -0.191$. As can be shown by working through the solutions for photon trajectories (take $m_0 = 0$ in Eq. 36), there is an unstable circular orbit for photons for $\phi = -1/2$. [9] For particles with nonzero rest mass, there are no stable elliptical orbits that pass inside the photon orbit.

Appendix V: Mach's Principle and and Exponential Metric

Assuming that the (appropriately retarded) gravitational potentials are additive, the potential of a collection of masses, $(M_1,\ M_2,\ M_3,\ .\ .\ .)$ would be

$$\phi = \Sigma_i \phi_i = -(\frac{GM_1}{c^2 r_1} + \frac{GM_2}{c^2 r_2} + \frac{GM_3}{c^2 r_3} + ...) \qquad (40)$$

and the metric would retain the form

$$ds^2 = e^{2\phi} c^2 dt^2 - e^{-2\phi}(dx^2 + dy^2 + dz^2) \qquad (41)$$

Now suppose that one is relatively near to one of the masses, say M_1, such that only its potential would vary significantly over a region of interest. Then one can redefine time and distance scales such that

$$dt' = dte^{-(\frac{GM_2}{c^2 r_2} + \frac{GM_3}{c^2 r_3} + ...)} \qquad (42)$$

and

$$dx' = dxe^{(\frac{GM_2}{c^2 r_2} + \frac{GM_3}{c^2 r_3} + ...)}, \quad dy' = dye^{(\frac{GM_2}{c^2 r_2} + \frac{GM_3}{c^2 r_3} + ...)} .... \ r' = r_1 e^{(\frac{GM_2}{c^2 r_2} + \frac{GM_3}{c^2 r_3} + ...)} \qquad (43)$$

Thus

$$ds^2 = e^{2\phi_1} c^2 dt'^2 - e^{-2\phi_1}(dx'^2 + dy'^2 + dz'^2) \qquad (44)$$

This amounts to letting the collective effects of all distant masses determine the local measures of distance and time. To complete the removal of all vestiges of our former coordinates, we can also redefine

$$M' = M_1 e^{\phi_2 + \phi_3 + ...} \qquad (45)$$

[8]The innermost marginally stable orbit of the Schwarzschild metric of General Relativity has a radius of $6GM/c^2$ and the energy lost in an accretion disk would be 5.7% of the accreting particles rest mass energy. In the isotropic metric of General Relativity, the marginally stable orbit radius is $4.96GM/c^2$ and the accretion disk energy loss is 5.7%. General Relativity thus encompasses mass changes similar to those of Eqs. 2 and 39.

[9]It is meaningless happenstance that the photon orbit here has $\phi = -1/2$ which corresponds to the event horizon in Schwarzschild coordinates. In isotropic coordinates for General Relativity, $g_{00} = (1 - GM/2c^2 r)^2/(1 + GM/2c^2 r)^2$. The event horizon occurs for $\phi = -2$ and $r = GM/2c^2$.

which is just what we would expect from special relativity in accord with Eq. 2. Substituting from Eq. 45 for M_1 and from Eq. 43 for r_1 in ϕ_1, and then dropping subscripts and primes, the local metric near a dominating mass M is

$$ds^2 = e^{2\phi}c^2dt^2 - e^{-2\phi}(dx^2 + dy^2 + dz^2) \tag{46}$$

where $\phi = -GM/c^2r$ for the mass M.

In this way, we can regard our local measures of mass, length and time to have been determined by the distant masses of the universe, as was suggested by Mach. This factoring of the metric is unique to the exponential metric and is a powerful argument in its favor.

Appendix VI: The GR Isotropic Metric For A Point Mass

The definition of a potential and special relativity were the only things used to obtain Eq. 5 of the text, which is repeated here:

$$g_{00} = e^{2\phi} \tag{47}$$

But as noted there, this is not part of the solution of the unmodified field equations of the Einstein theory for the point particle metric in isotropic coordinates.[10] In isotropic coordinates Einstein's field equations yield

$$g_{00} = (1 - GM/2c^2r)^2/(1 + GM/2c^2r)^2 \tag{48}$$

There is an obvious metric singularity for $r \to GM/2c^2r$, which is the location of the event horizon for this metric[11].

Unless one wants to seriously question special relativity, it seems that Eq. 5 must hold in any case and the gravitational potential in the Einstein theory must be

$$\phi = ln(1 - GM/2c^2r) - ln(1 + GM/2c^2r) \tag{49}$$

In this case, the gravitational three-acceleration of a test particle of mass, m, would be

$$\mathbf{a} = \mathbf{F}/m = -c^2\nabla\phi = \frac{-(GM/r^2)\hat{\mathbf{r}}}{(1 - (GM/2c^2r)^2)} \tag{50}$$

As long as the proper speed of the particle remains below the speed of light Eq. 58 (Appendix VII) and Eq. 49 show that the gravitational mass of the particle would approach zero at the event horizon. Nevertheless, its gravitational acceleration would be divergent as shown by Eq. 50. Similar infinite accelerations occur in Hilbert's version of the Schwarzschild solution for a point mass. It is quite amazing to me that otherwise serious physicists take these seriously. For example, read Kip Thorne's fanciful tale of approaching the event horizon of a monstrously massive black hole named Gargantua in his book *Black Holes & Time Warps*. As the event horizon was approached, it required dramatically increasing rocket engine power to counter the gravitational force of Gargantua, but no consideration was given to how the thrust that would be generated by rocket exhaust of vanishing mass.

For the metric of Eq. 20, (Appendix II) the GR isotropic solution has

$$\lambda = 4\ ln(1 + GM/2c^2r) \tag{51}$$

[10]The isotropic metric form is that of Eq. 20. of Appendix II.

[11]The fact that the radius of the event horizon is four times larger in Schwarzschild coordinates suggests that it might not exist as a part of physical reality.

Thus the condition that led to appendix Eq. 24 is not satisfied and the speed of wave packet components for photons would be dependent on wavelength and direction of motion in this metric. It is therefore not clear that a wave-particle duality for photons can always be maintained in solutions of the unmodified Einstein field equations for a point particle metric. This may be another level of apparent disagreement between General Relativity and quantum mechanics. If so, it is also resolved by the inclusion of gravitational field energy density as a source term in the field equations, which then leaves $\lambda = -2\phi$.

Appendix VII: Mass changes in gravitational fields

It should be well known to the reader from the special theory of relativity that an increase of mass occurs when objects are set in motion. The result of Eq. 2 of a mass depending on its location in a gravitational potential field may be a new idea to some readers, but it lies at the foundation of general relativity. From special relativity we learned of mass increase, length contraction and time dilation. Our measures of mass length and time are dependent on states of relative motion and we now see that mass is dependent on position in a gravitational field. It should be no surprise that our measures of length and time are also affected by gravitational potentials.

In the special theory of relativity, the quantity

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2), \tag{52}$$

where dt is the increment of time in which a particle moves distance $dl = \sqrt{(dx^2 + dy^2 + dz^2)}$ and c is the speed of light, is invariant; i.e., it will have the same value as computed in any frame of reference (set of coordinates) that moves at constant speed along the direction of the particle's motion. In a frame that moves with a particle, there is no incremental change of a space coordinate in an increment of time. In this rest frame of the particle, increments of mass, length or time are called "proper". If we denote a proper time increment as $d\tau$, then it is apparent that $ds^2 = c^2 d\tau^2$.

If our reference frame is located in a gravitational field, it is necessary to account for the additional changes of our measures of length and time that are caused by gravity. This can be accomplished by introducing correction factors, g_{ij}, known as metric coefficients, to the coordinates in Eq. 52. For example, in one special case of rectangular coordinates $(t, x, y, z) = (x^0, x^1, x^2, x^3)$ we can write

$$ds^2 = g_{00} c^2 dt^2 + g_{11} dx^2 + g_{22} dy^2 + g_{33} dz^2 = c^2 d\tau^2 \tag{53}$$

Note $x^0 = ct$. It should be noted that the relativistic invariant, ds^2, in this case still corresponds to the proper time increment $d\tau$ even for a particle in a gravitational potential, but the metric coefficients g_{ij} will soon be seen to be dependent on the gravitational potential. If no gravitational potentials are present, the metric coefficients in Eq. 53 are those of a Minkowskian reference frame with $g_{00} = 1,\ g_{11} = g_{22} = g_{33} = -1$.

Comparing Eqs 52 and 53, it appears that the correspondences between coordinate increments and proper time and length increments would be $d\tau \leftrightarrow \sqrt{g_{00}}dt$ and $dl \leftrightarrow \sqrt{-(g_{11}dx^2 + g_{22}dy^2 + g_{33}dz^2)}$. Thus, we can regard the ratio $v = \sqrt{-(g_{11}dx^2 + g_{22}dy^2 + g_{33}dz^2)}/\sqrt{g_{00}}dt$ as the "proper speed" of a particle. Note that for photons, $ds = 0$ and $v = c$. Thus in a proper frame, co-moving with a particle, the proper speed of light is always c. Someone moving along with the particle could never find variations in the speed of light to suggest the presence of a gravitational field.

The energy-momentum equation of a particle in special relativity can be written as

$$E^2/c^2 - (p_x^2 + p_y^2 + p_z^2) = m_0^2 c^2 \tag{54}$$

As modified in the presence of gravitational potentials, the particle energy-momentum equation can be written as:

$$g_{ij}p^ip^j = g^{ij}p_ip_j = m_0^2c^2 \tag{55}$$

where $i, j = 0, 1, 2, 3$ and any repeated index is to be summed over all four values. Here g_{ij} are covariant components of the metric tensor and p_k the covariant components of energy-momentum. Superscripted quantities are known as contravariant. g^{ij} is the inverse of the matrix g_{ij}.

For the metric form of Eq. 53, the energy-momentum equation of a particle can be written as

$$g^{00}p_o^2 + g^{11}p_1^2 + g^{22}p_2^2 + g^{33}p_3^2 = m_0^2c^2 \tag{56}$$

Since the metric of the Minkowskian space of special relativity has $g_{00} = g^{00} = 1$, and $g_{11} = g^{11} = g_{22} = g^{22} = g_{33} = g^{33} = -1$, Eq. 56 reduces to Eq. 54 when no gravitational potentials are present. Since this matrix of metric coefficients is diagonal, the algebraic inverse of each coefficient is the corresponding inverse matric element and there is no reason to distinguish between contravariant and covariant quantities. But when gravitational potentials are considered some distinctions are necessary. A simple way to keep them straight is to consider sets of familiar coordinates such as cartesian, cylindrical or polar as contravariant; e.g., $ct = x^0,\ x = x^1,\ y = x^2$ or $r = x^1,\ \theta = x^2,\ \Phi = x^3$. Corresponding contravariant momentum component forms would be $p^0,\ p^1 = mdx^1/dt,\ p^2 = mdx^2/dt = md\theta/dt$, etc; however, the conserved quantities that are familiar from Newtonian mechanics are the covariant quantities such as $p_0 = E/c,\ p_\theta, p_\Phi$, etc.

Contravariant and covariant quantities can be manipulated and related by using the metric coefficients to raise or lower indexes. For example, we state without proof that if A^i and A_i are contravariant and covariant vectors, respectively, then $A_i = g_{ij}A^j$ and $A^i = g^{ik}A_k$. Note again that repeated indexes are to be summed.

For a particle at rest, $p_1 = p_2 = p_3 = 0$, $p_0 = E/c$ and $E = mc^2$. Eq. 56 becomes $\sqrt{g^{00}}E/c = m_oc$. Note that $g^{00} = 1/g_{00}$, (assuming all other $g_{0j} = 0$) and substituting for m from Eq. 2 then leads immediately back to

$$g_{00} = e^{2\phi} \tag{57}$$

Using the "proper speed", v, as defined above, and substituting Eq. 57 into Eq. 56 and rearranging provides a relation that shows that the combined effects of both gravitational potentials and motion on particle mass are exactly as should be expected:

$$m = m_0e^{\phi}/\sqrt{1 - v^2/c^2} \tag{58}$$

Eq. 57 shows that in the presence of a gravitational potential, proper time increments would be given by $e^{\phi}dt$. It should not be surprising to find that proper lengths should also be dependent on gravitational potentials. For many practical purposes we can use a form of the metric for which the speed of light is the same in all coordinate directions. This is known as an isotropic metric and is characterized by the conditions $g_{11} = g_{22} = g_{33}$. Further, as shown in Appendix II, requiring the speed of light to be independent of the direction of the gravitational field and independent of wavelength in regions free of any matter (except a small test particle), leads to the condition $g_{11} = g_{22} = g_{33} = -e^{-2\phi}$. A proper length increment would then be given by $e^{-\phi}dl$. The metric forms of interest for now would be

$$ds^2 = e^{2\phi}c^2dt^2 - e^{-2\phi}(dx^2 + dy^2 + dz^2) \tag{59}$$

or

$$ds^2 = e^{2\phi}c^2dt^2 - e^{-2\phi}(dr^2 + r^2d\theta^2 + r^2sin^2\theta d\Phi^2) \tag{60}$$

for spherical coordinates. (Note Φ is the azimuthal angle coordinate and $\Phi \neq \phi$.) These are the most familiar forms of the Yilmaz exponential metric.