The Löb axiom and sub-conjecture □⊥>⊥ as contra-examples to Gödel incompleteness theorem

Abstract: We show the Löb axiom □(□⊥>⊥)>□⊥ is *not* tautologous, and the conjecture □⊥>⊥ is *not* contradictory. These serve as two contra-examples to the Gödel incompleteness theorem, hence refuting it.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated *proof* value, **F** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). Results are a 16-valued truth table in row-major and horizontal, or repeating fragments of 128-tables for more variables.

LET p, q: p, q; ~ Not; > Imply, greater than; = Equivalent; @ Not Equivalent;
(p=p) `T`autology; (q@q) **F** contradiction, ⊥;
% possibility, ◇, for one or some, ∃; # necessity, □, for every or all, ∀.

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The Löb axiom is supposed to define transitivity and second-order converse well-foundedness as:

□(□p>p)>□p (1.1)

#(#p>p)>#p ; (1.2)

We decompose the variables in Eq. 1.2 to show the table results at each step.

p=(p=p) ;	**F**T**F**T **F**T**F**T **F**T**F**T **F**T**F**T	(1.2.1.2)
#p=(p=p) ;	**F**N**F**N **F**N**F**N **F**N**F**N **F**N**F**N	(1.2.2.2)
#p>p ;	TTTT TTTT TTTT TTTT	(1.2.3.2)
#(#p>p)=(p=p) ;	NNNN NNNN NNNN NNNN	(1.2.4.2)
#(#p>p)>#p ;	CTCT CTCT CTCT CTCT	(1.2.5.2)

We replace the variable p with the symbol for contradiction ⊥:

□(□⊥>⊥)>□⊥ (2.1)

#(#(q@q)>(q@q))>#(q@q) ; CCCC CCCC CCCC CCCC (2.2)

We decompose the variables in Eq. 2.2 to show the table results at each step.

(q@q)=(p=p) ;	CCCC CCCC CCCC CCCC	(2.2.1.2)
#(q@q)=(p=p) ;	**FFFF FFFF FFFF FFFF**	(2.2.2.2)
#(q@q)>(q@q) ;	TTTT TTTT TTTT TTTT	(2.2.3.2)
#(#(q@q)>(q@q))=(p=p) ;	NNNN NNNN NNNN NNNN	(2.2.4.2)
#(#(q@q)>(q@q))>#(q@q) ;	CCCC CCCC CCCC CCCC	(2.2.5.2)

Eqs. 1.2 and 2.2 are *not* tautologous, hence refuting the Löb axiom.

The simpler conjectures of □p>p or □⊥>⊥, as rendered in Eqs. 1.2.3.2 or 2.2.3.2, are tautologous. However according to the Gödel incompleteness theorem, these should be *not* tautologous. Similarly the conjecture of the Löb axiom should be tautologous, but it is not. Consequently these serve as contra-examples to the Gödel incompleteness theorem, hence refuting it.