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Some Fixed Point Results for Contractive Type Conditions in Cone b-Metric Spaces and Applications

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Abstract: In this paper, we establish some fixed point results for contractive type conditions in the framework of complete cone b-metric spaces and give some applications of our results. The results presented in this paper generalize, extend and unify several well-known comparable results in the existing literature.

Key Words: Fixed point, contractive type condition, cone b-metric space, cone.

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§1. Introduction

Fixed point theory plays a basic role in applications of many branches of mathematics. Finding fixed point of contractive mappings becomes the center of strong research activity. Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle [2] in 1922.

In [3], Bakhtin introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous contraction principle in metric spaces. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces (see [4, 5, 11] and references therein). In recent investigation, the fixed point in non-convex analysis, especially in an ordered normed space, occupies a prominent place in many aspects (see [14, 15, 17, 20]). The authors define an ordering by using a cone, which naturally induces a partial ordering in Banach spaces.

In 2007, Huang and Zhang [14] introduced the concept of cone metric spaces as a generalization of metric spaces and establish some fixed point theorems for contractive mappings in normal cone metric spaces. Subsequently, several other authors [1, 16, 20, 23] studied the existence of fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric space. In 2008, Rezapour and Hamlbarani [20] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space.

Recently, Hussain and Shah [15] introduced the concept of cone b-metric space as a general-

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ization of b-metric space and cone metric spaces. They established some topological properties in such spaces and improved some recent results about KKM mappings in the setting of a cone b-metric space. In this paper, we give some examples in cone b-metric spaces, then obtain some fixed theorems for contractive type conditions in the setting of cone b-metric spaces.

Definition 1.1([14]) *Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold:)*

(c_1) P is closed, nonempty and $P \neq \{0\}$;
(c_2) $a, b \in R$, $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;
(c_3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in P^0$, where P^0 stands for the interior of P. If $P^0 \neq \emptyset$ then P is called a solid cone (see [22]).

There exist two kinds of cones- normal (with the normal constant K) and non-normal ones ([12]).

Let E be a real Banach space, $P \subset E$ a cone and $\leq$ partial ordering defined by P. Then P is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \text{ imply } \|x\| \leq K\|y\|, \tag{1.1}$$

or equivalently, if $(\forall n)$ $x_n \leq y_n \leq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x. \tag{1.2}$$

The least positive number K satisfying (1.1) is called the normal constant of P.

Example 1.2([22]) Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E : x(t) \geq 0\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $0 \leq x_n \leq y_n$, and $\lim_{n\to\infty} y_n = 0$, but $\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence x_n does not converge to zero. It follows by (1.2) that P is a non-normal cone.

Definition 1.3([14,24]) *Let X be a nonempty set. Suppose that the mapping $d \colon X \times X \to E$ satisfies:*

(d_1) *$0 \leq d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = 0 \iff x = y$;*
(d_2) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
(d_3) *$d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

Then d is called a cone metric on X and (X, d) is called a cone metric space [14].

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 1.4([14]) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ and $d \colon X \times X \to E$

defined by $d(x, y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space with normal cone P where $K = 1$.

Example 1.5([19]) Let $E = \ell^2$, $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0, \text{for all } n\}$, (X, ρ) a metric space, and $d\colon X \times X \to E$ defined by $d(x, y) = \{\rho(x, y)/2^n\}_{n\geq 1}$. Then (X, d) is a cone metric space.

Clearly, the above examples show that class of cone metric spaces contains the class of metric spaces.

Definition 1.6([15]) *Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d\colon X \times X \to E$ is said to be cone b-metric if and only if, for all x, y, $z \in X$, the following conditions are satisfied:*

(b_1) $0 \leq d(x, y)$ *with* $x \neq y$ *and* $d(x, y) = 0 \Leftrightarrow x = y$*;*
(b_2) $d(x, y) = d(y, x)$*;*
(b_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a cone b-metric space.

Remark 1.7 The class of cone b-metric spaces is larger than the class of cone metric space since any cone metric space must be a cone b-metric space. Therefore, it is obvious that cone b-metric spaces generalize b-metric spaces and cone metric spaces.

We give some examples, which show that introducing a cone b-metric space instead of a cone metric space is meaningful since there exist cone b-metric space which are not cone metric space.

Example 1.8([13]) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x \geq 0, y \geq 0\} \subset E$, $X = \mathbb{R}$ and $d\colon X \times X \to E$ defined by $d(x, y) = (|x-y|^p, \alpha|x-y|^p)$, where $\alpha \geq 0$ and $p > 1$ are two constants. Then (X, d) is a cone b-metric space with the coefficient $s = 2^p > 1$, but not a cone metric space.

Example 1.9([13]) Let $X = \ell^p$ with $0 < p < 1$, where $\ell^p = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Let $d\colon X \times X \to \mathbb{R}_+$ defined by $d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}$, where $x = \{x_n\}$, $y = \{y_n\} \in \ell^p$. Then (X, d) is a cone b-metric space with the coefficient $s = 2^{1/p} > 1$, but not a cone metric space.

Example 1.10([13]) Let $X = \{1, 2, 3, 4\}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x \geq 0, y \geq 0\}$. Define $d\colon X \times X \to E$ by

$$d(x, y) = \begin{cases} (|x-y|^{-1}, |x-y|^{-1}) & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then (X, d) is a cone b-metric space with the coefficient $s = \frac{6}{5} > 1$. But it is not a cone metric space since the triangle inequality is not satisfied,

$$d(1, 2) > d(1, 4) + d(4, 2), \quad d(3, 4) > d(3, 1) + d(1, 4).$$

Definition 1.11([15]) *Let (X, d) be a cone b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then*

• $\{x_n\}$ *is a Cauchy sequence whenever, if for every* $c \in E$ *with* $0 \ll c$*, then there is a natural number* N *such that for all* $n, m \geq N$, $d(x_n, x_m) \ll c$*;*

• $\{x_n\}$ *converges to* x *whenever, for every* $c \in E$ *with* $0 \ll c$*, then there is a natural number* N *such that for all* $n \geq N$, $d(x_n, x) \ll c$*. We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$ *as* $n \to \infty$.

• (X, d) *is a complete cone b-metric space if every Cauchy sequence is convergent.*

In the following (X, d) will stands for a cone b-metric space with respect to a cone P with $P^0 \neq \emptyset$ in a real Banach space E and $\leq$ is partial ordering in E with respect to P.

Definition 1.12([10]) *Let* (X, d) *be a metric space. A self mapping* $T: X \to X$ *is called quasi contraction if it satisfies the following condition:*

$$d(Tx, Ty) \leq h \max\Big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\Big\}$$

for all $x, y \in X$ *and* $h \in (0, 1)$ *is a constant.*

Definition 1.13([10]) *Let* (X, d) *be a metric space. A self mapping* $T: X \to X$ *is called Ciric quasi-contraction if it satisfies the following condition:*

$$d(Tx, Ty) \leq h \max\Big\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\Big\}$$

for all $x, y \in X$ *and* $h \in (0, 1)$ *is a constant.*

The following lemmas are often used (in particular when dealing with cone metric spaces in which the cone need not be normal).

Lemma 1.14([17]) *Let* P *be a cone and* $\{a_n\}$ *be a sequence in* E*. If* $c \in int\, P$ *and* $0 \leq a_n \to 0$ *as* $n \to \infty$*, then there exists* N *such that for all* $n > N$*, we have* $a_n \ll c$*.*

Lemma 1.15([17]) *Let* x*,* y*,* $z \in E$*, if* $x \leq y$ *and* $y \ll z$*, then* $x \ll z$*.*

Lemma 1.16([15]) *Let* P *be a cone and* $0 \leq u \ll c$ *for each* $c \in int\, P$*, then* $u = 0$*.*

Lemma 1.17([8]) *Let* P *be a cone, if* $u \in P$ *and* $u \leq k\,u$ *for some* $0 \leq k < 1$*, then* $u = 0$*.*

Lemma 1.18([17]) *Let* P *be a cone and* $a \leq b + c$ *for each* $c \in int\, P$*, then* $a \leq b$*.*

§2. Main Results

In this section we shall prove some fixed point theorems of contractive type conditions in the framework of cone b-metric spaces.

Theorem 2.1 *Let* (X, d) *be a complete cone b-metric space with the coefficient* $s \geq 1$*. Suppose* *that the mapping $T\colon X \to X$ satisfies the contractive type condition:*

$$\begin{aligned} d(Tx,Ty) &\leq \alpha d(x,y)+\beta d(x,Tx)+\gamma d(y,Ty) \\ &\quad +\mu[d(x,Ty)+d(y,Tx)] \end{aligned} \tag{2.1}$$

for all $x,y \in X$, where $\alpha,\beta,\gamma,\mu \geq 0$ are constants such that $s\alpha+\beta+s\gamma+(s^2+s)\mu < 1$. Then T has a unique fixed point in X.

Proof Choose $x_0 \in X$. We construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \geq 1$, that is, $x_{n+1} = Tx_n = T^{n+1}x_0$. From (2.1), we have

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \\ &\leq \alpha d(x_n,x_{n-1})+\beta d(x_n,Tx_n)+\gamma d(x_{n-1},Tx_{n-1}) \\ &\quad +\mu[d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)] \\ &= \alpha d(x_n,x_{n-1})+\beta d(x_n,x_{n+1})+\gamma d(x_{n-1},x_n) \\ &\quad +\mu[d(x_n,x_n)+d(x_{n-1},x_{n+1})] \\ &= \alpha d(x_n,x_{n-1})+\beta d(x_n,x_{n+1})+\gamma d(x_{n-1},x_n) \\ &\quad +\mu d(x_{n-1},x_{n+1}) \\ &\leq \alpha d(x_n,x_{n-1})+\beta d(x_n,x_{n+1})+\gamma d(x_{n-1},x_n) \\ &\quad +s\mu[d(x_{n-1},x_n)+d(x_n,x_{n+1})] \\ &= (\alpha+\gamma+s\mu)d(x_n,x_{n-1}) \\ &\quad +(\beta+s\mu)d(x_n,x_{n+1}). \end{aligned} \tag{2.2}$$

This implies that

$$\begin{aligned} d(x_{n+1},x_n) &\leq \Big(\frac{\alpha+\gamma+s\mu}{1-\beta-s\mu}\Big)d(x_n,x_{n-1}) \\ &= \lambda\, d(x_n,x_{n-1}) \end{aligned} \tag{2.3}$$

where

$$\lambda = \Big(\frac{\alpha+\gamma+s\mu}{1-\beta-s\mu}\Big).$$

As $s\alpha+\beta+s\gamma+(s^2+s)\mu < 1$, it is clear that $\lambda < 1/s$.

Similarly, we obtain

$$d(x_{n-1},x_n) \leq \lambda\, d(x_{n-2},x_{n-1}). \tag{2.4}$$

Using (2.4) in (2.3), we get

$$d(x_{n+1},x_n) \leq \lambda^2\, d(x_{n-1},x_{n-2}). \tag{2.5}$$

Continuing this process, we obtain

$$d(x_{n+1},x_n) \leq \lambda^n\, d(x_1,x_0). \tag{2.6}$$

Let $m \geq 1$, $p \geq 1$, we have

$$\begin{aligned}
d(x_m, x_{m+p}) &\leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) \\
&\quad + \cdots + s^{p-1} d(x_{m+p-1}, x_{m+p}) \\
&\leq s\lambda^m d(x_1, x_0) + s^2 \lambda^{m+1} d(x_1, x_0) + s^3 \lambda^{m+2} d(x_1, x_0) \\
&\quad + \cdots + s^p k^{m+p-1} d(x_1, x_0) \\
&= s\lambda^m [1 + s\lambda + s^2\lambda^2 + s^3\lambda^3 + \cdots + (s\lambda)^{p-1}] d(x_1, x_0) \\
&\leq \Big[\frac{s\lambda^m}{1 - s\lambda}\Big] d(x_1, x_0).
\end{aligned}$$

Let $0 \ll \varepsilon$ be given. Notice that $\Big[\frac{s\lambda^m}{1-s\lambda}\Big] d(x_1, x_0) \to 0$ as $m \to \infty$ for any p since $0 < s\lambda < 1$. Making full use of Lemma 1.14, we find $m_0 \in \mathbb{N}$ such that

$$\Big[\frac{s\lambda^m}{1 - s\lambda}\Big] d(x_1, x_0) \ll \varepsilon$$

for each $m > m_0$. Thus

$$d(x_m, x_{m+p}) \leq \Big[\frac{s\lambda^m}{1 - s\lambda}\Big] d(x_1, x_0) \ll \varepsilon$$

for all $m \geq 1$, $p \geq 1$. So, by Lemma 1.15, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone b-metric space, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Take $n_0 \in \mathbb{N}$ such that $d(x_n, u) \ll \frac{\varepsilon(1-s(\beta+\mu))}{s(\alpha+\mu+1)}$ for all $n > n_0$. Hence,

$$\begin{aligned}
d(Tu, u) &\leq s[d(Tu, Tx_n) + d(Tx_n, u)] \\
&= sd(Tu, Tx_n) + sd(Tx_n, u) \\
&\leq s\Big\{\alpha d(u, x_n) + \beta d(u, Tu) + \gamma d(x_n, Tx_n) \\
&\quad + \mu[d(u, Tx_n) + d(x_n, Tu)]\Big\} + sd(x_{n+1}, u) \\
&= s\Big\{\alpha d(u, x_n) + \beta d(u, Tu) + \gamma d(x_n, x_{n+1}) \\
&\quad + \mu[d(u, x_{n+1}) + d(x_n, Tu)]\Big\} + sd(x_{n+1}, u) \\
&= s(\alpha + \mu + 1) d(x_n, u) + s(\beta + \mu) d(Tu, u). \qquad (2.7)
\end{aligned}$$

This implies that

$$d(Tu, u) \leq \Big(\frac{s(\alpha + \mu + 1)}{1 - s(\beta + \mu)}\Big) \ll \varepsilon,$$

for each $n > n_0$. Then, by Lemma 1.16, we deduce that $d(Tu, u) = 0$, that is, $Tu = u$. Thus u is a fixed point of T.

Now, we show that the fixed point is unique. If there is another fixed point u^* of T such that $Tu^* = u^*$, then from (2.1), we have

$$\begin{aligned} d(u,u^*) &= d(Tu,Tu^*) \\ &\leq \alpha d(u,u^*) + \beta d(u,Tu) + \gamma d(u^*,Tu^*) \\ &\quad +\mu[d(u,Tu^*) + d(u^*,Tu)] \\ &\leq \alpha d(u,u^*) + \beta d(u,u) + \gamma d(u^*,u^*) \\ &\quad +\mu[d(u,u^*) + d(u^*,u)] \\ &= (\alpha + 2\mu)d(u,u^*) \\ &\leq (s\alpha + \beta + s\gamma + (s^2+s)\mu)d(u,u^*). \end{aligned}$$

By Lemma 1.17, we have $u = u^*$. This completes the proof. □

Remark 2.2 Theorem 2.1 extends Theorem 2.1 of Huang and Xu in [13] to the case of weaker contractive condition considered in this paper.

From Theorem 2.1, we obtain the following result as corollaries.

Corollary 2.3 *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies the contractive condition:*

$$d(Tx,Ty) \leq \alpha d(x,y)$$

for all $x,y \in X$, where $\alpha \in \left[0, \frac{1}{s}\right)$ is a constant. Then T has a unique fixed point in X.

Proof The proof of Corollary 2.3 is immediately follows from Theorem 2.1 by taking $\beta = \gamma = \mu = 0$. This completes the proof. □

Corollary 2.4 *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies the contractive condition:*

$$d(Tx,Ty) \leq \beta[d(x,Tx) + d(y,Ty)]$$

for all $x,y \in X$, where $\beta \in \left[0, \frac{1}{1+s}\right)$ is a constant. Then T has a unique fixed point in X.

Proof The proof of Corollary 2.4 is immediately follows from Theorem 2.1 by taking $\alpha = \mu = 0$ and $\beta = \gamma$. This completes the proof. □

Corollary 2.5 *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies the contractive condition:*

$$d(Tx,Ty) \leq \mu[d(x,Ty) + d(y,Tx)]$$

for all $x,y \in X$, where $\mu \in \left[0, \frac{1}{s+s^2}\right)$ is a constant. Then T has a unique fixed point in X.

Proof The proof of Corollary 2.5 is immediately follows from Theorem 2.1 by taking

$\alpha = \beta = \gamma = 0$. This completes the proof. □

Remark 2.6 Corollaries 2.3, 2.4 and 2.5 extend Theorem 1, 3 and 4 of Huang and Zhang [14] to the case of cone b-metric space without normal constant considered in this paper.

Remark 2.7 Corollary 2.3 also extends the well known Banach contraction principle [2] to that in the setting of cone b-metric spaces.

Remark 2.8 Corollary 2.4 also extends the Kannan contraction [18] to that in the setting of cone b-metric spaces.

Remark 2.9 Corollary 2.5 also extends the Chatterjea contraction [7] to that in the setting of cone b-metric spaces.

Remark 2.10 Theorem 2.1 also extends several results from the existing literature to the case of weaker contractive condition considered in this paper in the setting of cone b-metric spaces.

Theorem 2.11 *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies the contractive type condition:*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Ty) + \gamma d(y, Tx) \tag{2.8}$$

for all $x, y \in X$ and $\alpha, \beta, \gamma \geq 0$ are constants such that $s\alpha + s(1 + s)\gamma < 1$. Then T has a unique fixed point in X.

Proof Choose $x_0 \in X$. We construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \geq 1$, that is, $x_{n+1} = Tx_n = T^{n+1}x_0$. From (2.8), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \alpha d(x_n, x_{n-1}) + \beta d(x_n, Tx_{n-1}) + \gamma d(x_{n-1}, Tx_n) \\ &= \alpha d(x_n, x_{n-1}) + \beta d(x_n, x_n) + \gamma d(x_{n-1}, x_{n+1}) \\ &= \alpha d(x_n, x_{n-1}) + \gamma d(x_{n-1}, x_{n+1}) \\ &\leq \alpha d(x_n, x_{n-1}) + s\gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &= (\alpha + s\gamma)d(x_n, x_{n-1}) + s\gamma d(x_n, x_{n+1}). \end{aligned} \tag{2.9}$$

This implies that

$$d(x_{n+1}, x_n) \leq \Big(\frac{\alpha + s\gamma}{1 - s\gamma}\Big) d(x_n, x_{n-1}) = \rho\, d(x_n, x_{n-1}), \tag{2.11}$$

where

$$\rho = \Big(\frac{\alpha + s\gamma}{1 - s\gamma}\Big).$$

As $s\alpha + s(s + 1)\gamma < 1$, it is clear that $\rho < 1/s$.

Similarly, we obtain

$$d(x_{n-1}, x_n) \leq \rho\, d(x_{n-2}, x_{n-1}). \tag{2.11}$$

Using (2.11) in (2.10), we get

$$d(x_{n+1}, x_n) \leq \rho^2\, d(x_{n-1}, x_{n-2}). \tag{2.12}$$

Continuing this process, we obtain

$$d(x_{n+1}, x_n) \leq \rho^n\, d(x_1, x_0). \tag{2.13}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\ &\quad + \cdots + s^{n+m-1} d(x_{n+m-1}, x_m) \\ &\leq s\rho^n d(x_1, x_0) + s^2 \rho^{n+1} d(x_1, x_0) + s^3 \rho^{n+2} d(x_1, x_0) \\ &\quad + \cdots + s^m \rho^{n+m-1} d(x_1, x_0) \\ &= s\rho^n [1 + s\rho + s^2\rho^2 + s^3\rho^3 + \cdots + (s\rho)^{m-1}] d(x_1, x_0) \\ &\leq \left[\frac{s\rho^n}{1 - s\rho}\right] d(x_1, x_0). \end{aligned}$$

Let $0 \ll \varepsilon_1$ be given. Notice that $\left[\frac{s\rho^n}{1-s\rho}\right] d(x_1, x_0) \to 0$ as $n \to \infty$ since $0 < s\rho < 1$. Making full use of Lemma 1.14, we find $n_0 \in \mathbb{N}$ such that

$$\left[\frac{s\rho^n}{1 - s\rho}\right] d(x_1, x_0) \ll \varepsilon_1$$

for each $n > n_0$. Thus

$$d(x_n, x_m) \leq \left[\frac{s\rho^n}{1 - s\rho}\right] d(x_1, x_0) \ll \varepsilon_1$$

for all $n, m \geq 1$. So, by Lemma 1.15, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone b-metric space, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. Take $n_1 \in \mathbb{N}$ such that $d(x_n, v) \ll \frac{\varepsilon_1(1-s\gamma)}{s(\alpha+1)}$ for all $n > n_1$. Hence,

$$\begin{aligned} d(Tv, v) &\leq s[d(Tv, Tx_n) + d(Tx_n, v)] \\ &= sd(Tv, Tx_n) + sd(Tx_n, v) \\ &\leq s[\alpha d(v, x_n) + \beta d(v, Tx_n) + \gamma d(x_n, Tv)] + sd(x_{n+1}, v) \\ &= s[\alpha d(v, x_n) + \beta d(v, x_{n+1}) + \gamma d(x_n, Tv)] + sd(x_{n+1}, v) \\ &= s(\alpha + 1) d(v, x_n) + s\gamma d(Tv, v). \end{aligned} \tag{2.14}$$

This implies that

$$d(Tv,v) \leq \Big(\frac{s(\alpha+1)}{1-s\gamma}\Big) d(x_n,v) \ll \varepsilon_1,$$

for each $n > n_1$. Then, by Lemma 1.16, we deduce that $d(Tv,v) = 0$, that is, $Tv = v$. Thus v is a fixed point of T.

Now, we show that the fixed point is unique. If there is another fixed point v^* of T such that $Tv^* = v^*$, then from (2.8), we have

$$\begin{aligned} d(v,v^*) &= d(Tv,Tv^*) \\ &\leq \alpha d(v,v^*) + \beta d(v,Tv^*) + \gamma d(v^*,Tv) \\ &= \alpha d(v,v^*) + \beta d(v,v^*) + \gamma d(v^*,v) \\ &= (\alpha+\beta+\gamma) d(v,v^*) \\ &\leq (s\alpha + s(1+s)\gamma) d(v,v^*). \end{aligned}$$

By Lemma 1.17, we have $v = v^*$. This completes the proof. □

Theorem 2.12 *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies the following contractive condition: there exists*

$$u(x,y) \in \Big\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2s}, \frac{d(x,Ty)+d(y,Tx)}{2s}\Big\}$$

such that

$$d(Tx,Ty) \leq k\, u(x,y), \tag{2.15}$$

for all $x,y \in X$, where $k \in [0,1)$ is a constant with $ks < 1$. Then T has a unique fixed point in X.

Proof Choose $x_0 \in X$. We construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \geq 1$, that is, $x_{n+1} = Tx_n = T^{n+1}x_0$. From (2.15), we have

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \\ &\leq k\, u(x_n,x_{n-1}) \leq \cdots \leq k^n\, u(x_1,x_0). \end{aligned} \tag{2.16}$$

Let $m,n \geq 1$ and $m > n$, we have

$$\begin{aligned} d(x_n,x_m) &\leq s[d(x_n,x_{n+1}) + d(x_{n+1},x_m)] \\ &= sd(x_n,x_{n+1}) + sd(x_{n+1},x_m) \\ &\leq sd(x_n,x_{n+1}) + s^2[d(x_{n+1},x_{n+2}) + d(x_{n+2},x_m)] \\ &= sd(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + s^2 d(x_{n+2},x_m) \\ &\leq sd(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + s^3 d(x_{n+2},x_{n+3}) \\ &\quad + \cdots + s^{n+m-1} d(x_{n+m-1},x_m) \end{aligned}$$

$$\begin{aligned} &\leq sk^n u(x_1,x_0) + s^2k^{n+1}u(x_1,x_0) + s^3k^{n+2}u(x_1,x_0) \\ &\quad + \cdots + s^m k^{n+m-1}u(x_1,x_0) \\ &= sk^n[1 + sk + s^2k^2 + s^3k^3 + \cdots + (sk)^{m-1}]u(x_1,x_0) \\ &\leq \Big[\frac{sk^n}{1-sk}\Big]u(x_1,x_0). \end{aligned}$$

Let $0 \ll r$ be given. Notice that

$$\Big[\frac{sk^n}{1-sk}\Big]u(x_1,x_0) \to 0$$

as $n \to \infty$ since $0 < k < 1$. Making full use of Lemma 1.14, we find $n_0 \in \mathbb{N}$ such that

$$\Big[\frac{sk^n}{1-sk}\Big]u(x_1,x_0) \ll r$$

for each $n > n_0$. Thus

$$d(x_n,x_m) \leq \Big[\frac{sk^n}{1-sk}\Big]u(x_1,x_0) \ll r$$

for all $n,m \geq 1$. So, by Lemma 1.15, $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is a complete cone b-metric space, there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. Take $n_1 \in \mathbb{N}$ such that $d(x_n,p) \ll \frac{r}{s(k+1)}$ for all $n > n_1$. Hence,

$$\begin{aligned} d(Tp,p) &\leq s[d(Tp,Tx_n) + d(Tx_n,p)] \\ &= sd(Tp,Tx_n) + sd(Tx_n,p) \\ &\leq sk\,u(p,x_n) + s\,d(x_{n+1},p) \\ &\leq sk\,d(p,x_n) + s\,d(x_n,p) \\ &= s(k+1)\,d(x_n,p). \end{aligned}$$

This implies that

$$d(Tp,p) \ll r,$$

for each $n > n_1$. Then, by Lemma 1.16, we deduce that $d(Tp,p) = 0$, that is, $Tp = p$. Thus p is a fixed point of T.

Now, we show that the fixed point is unique. If there is another fixed point q of T such that $Tq = q$, then by the given condition (2.15), we have

$$d(p,q) = d(Tp,Tq) \leq k\,u(p,q) = k\,d(p,q).$$

By Lemma 1.17, we have $p = q$. This completes the proof. □

Theorem 2.13 *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies the following contractive condition:*

$$d(Tx,Ty) \leq h\ \max\{d(x,y), d(x,Tx), d(y,Ty)\} \tag{2.17}$$

for all $x, y \in X$, where $h \in [0,1)$ is a constant with $sh < 1$. Then T has a unique fixed point in X.

Proof Choose $x_0 \in X$. We construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \geq 1$, that is, $x_{n+1} = Tx_n = T^{n+1}x_0$. From (2.17), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq h \max\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\} \\ &= h \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\ &\leq h\, d(x_n, x_{n-1}). \end{aligned} \tag{2.18}$$

Similarly, we obtain

$$d(x_{n-1}, x_n) \leq h\, d(x_{n-2}, x_{n-1}). \tag{2.19}$$

Using (2.19) in (2.18), we get

$$d(x_{n+1}, x_n) \leq h^2\, d(x_{n-1}, x_{n-2}). \tag{2.20}$$

Continuing this process, we obtain

$$d(x_{n+1}, x_n) \leq h^n\, d(x_1, x_0). \tag{2.21}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &= sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^2d(x_{n+2}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^3d(x_{n+2}, x_{n+3}) \\ &\quad + \cdots + s^{n+m-1}d(x_{n+m-1}, x_m) \\ &\leq sh^n d(x_1, x_0) + s^2h^{n+1}d(x_1, x_0) + s^3h^{n+2}d(x_1, x_0) \\ &\quad + \cdots + s^m h^{n+m-1}d(x_1, x_0) \\ &= sh^n[1 + sh + s^2h^2 + s^3h^3 + \cdots + (sh)^{m-1}]d(x_1, x_0) \\ &\leq \Big[\frac{sh^n}{1 - sh}\Big]d(x_1, x_0). \end{aligned}$$

Let $0 \ll c$ be given. Notice that

$$\Big[\frac{sh^n}{1 - sh}\Big]d(x_1, x_0) \to 0$$

as $n \to \infty$ since $0 < h < 1$. Making full use of Lemma 1.14, we find $N_0 \in \mathbb{N}$ such that

$$\Big[\frac{sh^n}{1 - sh}\Big]d(x_1, x_0) \ll c$$

for each $n > N_0$. Thus

$$d(x_n, x_m) \le \Big[\frac{sh^n}{1-sh}\Big] d(x_1, x_0) \ll c$$

for all $n, m \ge 1$. So, by Lemma 1.15, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone b-metric space, there exists $q \in X$ such that $x_n \to q$ as $n \to \infty$. Take $N_1 \in \mathbb{N}$ such that $d(x_n, q) \ll \frac{c}{s(h+1)}$ for all $n > N_1$. Hence,

$$\begin{aligned}
d(Tq, q) &\le s[d(Tq, Tx_n) + d(Tx_n, q)] \\
&= sd(Tq, Tx_n) + sd(Tx_n, q) \\
&\le sh\max\{d(q, x_n), d(x_n, Tx_n), d(q, Tq)\} + s\,d(x_{n+1}, q) \\
&= sh\max\{d(q, x_n), d(x_n, x_{n+1}), d(q, Tq)\} + s\,d(x_{n+1}, q) \\
&\le sh\,d(q, x_n) + s\,d(x_n, q) \\
&= s(h+1)\,d(x_n, q).
\end{aligned}$$

This implies that

$$d(Tq, q) \ll c,$$

for each $n > N_1$. Then, by Lemma 1.16, we deduce that $d(Tq, q) = 0$, that is, $Tq = q$. Thus q is a fixed point of T.

Now, we show that the fixed point is unique. If there is another fixed point q' of T such that $Tq' = q'$, then by the given condition (2.17), we have

$$\begin{aligned}
d(q, q') &= d(Tq, Tq') \\
&\le h\max\{d(q, q'), d(q, Tq), d(q', Tq')\} \\
&= h\max\{d(q, q'), d(q, q), d(q', q')\} \\
&= h\max\{d(q, q'), 0, 0\} \\
&\le h\,d(q, q')
\end{aligned}$$

By Lemma 1.17, we have $q = q'$. This completes the proof. □

Example 2.14([13]) Let $X = [0, 1]$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x \ge 0, y \ge 0\} \subset E$ and $d\colon X \times X \to E$ defined by $d(x, y) = (|x-y|^p, |x-y|^p)$ for all $x, y \in X$ where $p > 1$ is a constant. Then (X, d) is a complete cone b-metric space. Let us define $T\colon X \to X$ as $T(x) = \frac{1}{2}x - \frac{1}{4}x^2$ for all $x \in X$. Therefore,

$$\begin{aligned}
d(Tx, Ty) &= (|Tx - Ty|^p, |Tx - Ty|^p) \\
&= \Big(\Big|\frac{1}{2}(x-y) - \frac{1}{4}(x-y)(x+y)\Big|^p, \Big|\frac{1}{2}(x-y) - \frac{1}{4}(x-y)(x+y)\Big|^p\Big) \\
&= \Big(|x-y|^p.\Big|\frac{1}{2} - \frac{1}{4}(x+y)\Big|^p, |x-y|^p.\Big|\frac{1}{2} - \frac{1}{4}(x+y)\Big|^p\Big) \\
&\le \frac{1}{2^p}(|x-y|^p, |x-y|^p) = \frac{1}{2^p}d(x, y).
\end{aligned}$$

Hence $0 \in X$ is the unique fixed point of T.

Other consequence of our result for the mapping involving contraction of integral type is the following.

Denote Λ the set of functions $\varphi\colon [0,\infty) \to [0,\infty)$ satisfying the following hypothesis:

(h_1) φ is a Lebesgue-integrable mapping on each compact subset of $[0,\infty)$;

(h_2) for any $\varepsilon > 0$ we have $\int_0^\infty \varphi(t)dt > 0$.

Theorem 2.15 *Let (X,d) be a complete cone b-metric space (CCbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$\int_0^{d(Tx,Ty)} \psi(t)dt \quad \leq \quad \beta \int_0^{d(x,y)} \psi(t)dt$$

for all $x, y \in X$, where $\beta \in [0,1)$ is a constant with $s\beta < 1$ and $\psi \in \Lambda$. Then T has a unique fixed point in X.

Remark 2.16 Theorem 2.15 extends Theorem 2.1 of Branciari [6] from complete metric space to that setting of complete cone b-metric space considered in this paper.

§3. Applications

In this section we shall apply Theorem 2.1 to the first order differential equation.

Example 3.1 $X = C([1,3],\mathbb{R})$, $E = \mathbb{R}^2$, $\alpha > 0$ and

$$d(x,y) = \Big(\sup_{t\in[1,3]} |x(t)-y(t)|^2,\ \alpha \sup_{t\in[1,3]} |x(t)-y(t)|^2\Big)$$

for every $x, y \in X$, and $P = \big\{(u,v) \in \mathbb{R}^2 : u,\, v \geq 0\big\}$. Then (X,d) is a cone b-metric space. Define $T\colon X \to X$ by

$$T(x(t)) = 4 + \int_1^t \Big(x(u) + u^2\Big)e^{u-5}du.$$

For $x, y \in X$,

$$\begin{aligned} d(Tx,Ty) &= \Big(\sup_{t\in[1,3]} |T(x(t)) - T(y(t))|^2,\ \alpha \sup_{t\in[1,3]} |T(x(t)) - T(y(t))|^2\Big) \\ &\leq \Big(\int_1^3 |(x(u)-y(u))|^2 e^{-2}du,\ \alpha \int_1^3 |(x(u)-y(u))|^2 e^{-2}du\Big) \\ &= 2e^{-2}d(x,y) \\ &\leq 2e^{-1}d(x,y). \end{aligned}$$

Thus for $\alpha = \frac{2}{e} < 1$, $\beta = \gamma = \mu = 0$, all conditions of Theorem 2.1 are satisfied and so T

has a unique fixed point, which is the unique solution of the integral equation:

$$x(t) = 4 + \int_1^t \Big(x(u) + u^2\Big) e^{u-5} du,$$

or the differential equation:

$$x'(t) = \Big(x(t) + t^2\Big) e^{t-5},\ t \in [1, 3],\ x(1) = 4.$$

Hence, the use of Theorem 2.1 is a delightful way of showing the existence and uniqueness of solutions for the following class of integral equations:

$$q + \int_p^t K(x(u), u) du = x(t) \in C\big([p, q],\, \mathbb{R}^n\big).$$

Now, we shall apply Corollary 2.3 to the first order periodic boundary problem

$$\begin{cases} \dfrac{dx}{dt} = F(t, x(t)), \\ x(0) = \mu, \end{cases} \tag{3.1}$$

where $F\colon [-h, h] \times [\mu - \theta, \mu + \theta]$ is a continuous function.

Example 3.2([13]) Consider the boundary problem (3.1) with the continuous function F and suppose $F(x, y)$ satisfies the local Lipschitz condition, i.e., if $|x| \le h,\ y_1, y_2 \in [\mu - \theta, \mu + \theta]$, it induces

$$|F(x, y_1) - F(x, y_2)| \le L\, |y_1 - y_2|.$$

Set $M = \max_{[-h,h]\times[\mu-\theta,\mu+\theta]} |F(x, y)|$ such that $h^2 < \min\{\theta/M^2, 1/L^2\}$, then there exists a unique solution of (3.1).

Proof Let $X = E = C([-h, h])$ and $P = \{u \in E : u \ge 0\}$. Put $d\colon X \times X \to E$ as $d(x, y) = f(t) \max_{-h \le t \le h} |x(t) - y(t)|^2$ with $f\colon [-h, h] \to \mathbb{R}$ such that $f(t) = e^t$. It is clear that (X, d) is a complete cone b-metric space.

Note that (3.1) is equivalent to the integral equation

$$x(t) = \mu + \int_0^t F(u, x(u)) du.$$

Define a mapping $T\colon C([-h, h]) \to \mathbb{R}$ by $x(t) = \mu + \int_0^t F(u, x(u)) du$. If

$$x(t), y(t) \in B(\mu, f\,\theta) = \{\varphi(t) \in C([-h, h]) : d(\mu, \varphi) \le f\,\theta\},$$

then from

$$\begin{aligned} d(Tx,Ty) &= f(t)\max_{-h\le t\le h}\Big|\int_0^t F(u,x(u))du - \int_0^t F(u,y(u))du\Big|^2 \\ &= f(t)\max_{-h\le t\le h}\Big|\int_0^t [F(u,x(u)) - F(u,y(u))]du\Big|^2 \\ &\le h^2 f(t)\max_{-h\le t\le h}|F(u,x(u)) - F(u,y(u))|^2 \\ &\le h^2L^2 f(t)\max_{-h\le t\le h}|x(u)-y(u)|^2 = h^2L^2 d(x,y), \end{aligned}$$

and

$$\begin{aligned} d(Tx,\mu) &= f(t)\max_{-h\le t\le h}\Big|\int_0^t F(u,x(u))du\Big|^2 \\ &\le h^2 f\max_{-h\le t\le h}|F(u,x(u))|^2 \le h^2M^2 f \le f\theta, \end{aligned}$$

we speculate $T\colon B(\mu, f\theta) \to B(\mu, f\theta)$ is a contraction mapping.

Lastly, we prove that $(B(\mu, f\theta), d)$ is complete. In fact, suppose $\{x_n\}$ is a Cauchy sequence in $B(\mu, f\theta)$. Then $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, there is $q \in X$ such that $x_n \to q$ $(n \to \infty)$, So, for each $c \in int\, P$, there exists N, whenever $n > N$, we obtain $d(x_n, q) \ll c$. Thus, it follows from

$$d(\mu,q) \le d(x_n,\mu) + d(\mu,q) \le f\theta + c$$

and Lemma 1.18 that $d(\mu, q) \le f\theta$, which means $q \in B(\mu, f\theta)$, that is, $(B(\mu, f\theta), d)$ is complete. Thus, from the above statement, all the conditions of Corollary 2.3 are satisfied. Hence T has a unique fixed point $x(t) \in B(\mu, f\theta)$ or we say that, there exists a unique solution of (3.1). □

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