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Ricci Soliton and Conformal Ricci Soliton in Lorentzian β-Kenmotsu Manifold

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Abstract: In this paper we have studied quasi conformal curvature tensor, Ricci tensor, projective curvature tensor, pseudo projective curvature tensor in Lorentzian β-Kenmotsu manifold admitting Ricci soliton and conformal Ricci soliton.

Key Words: Trans-Sasakian manifold, β-Kenmotsu manifold, Lorentzian β-Kenmotsu manifold, Ricci soliton, conformal Ricci flow.

AMS(2010): 53C25, 35K65, 53C44, 53D10, 53D15.

§1. Introduction

Hamilton started the study of Ricci flow [12] in 1982 and proved its existence. This concept was developed to answer Thurston's geometric conjecture which says that each closed three manifold admits a geometric decomposition. Hamilton also [11]classified all compact manifolds with positive curvature operator in dimension four. Since then, the Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman also did an excellent work on Ricci flow [15], [16].

The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S \tag{1.1}$$

on a compact Riemannian manifold M with Riemannian metric g. A solution to the Ricci flow is called a Ricci soliton if it moves only by a one-parameter group of diffeomorphism and scaling.Ramesh Sharma [18], M. M. Tripathi [19], Bejan, Crasmareanu [4]studied Ricci soliton in contact metric manifolds also. The Ricci soliton equation is given by

$$\pounds_X g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $\pounds_X$ is the Lie derivative, S is Ricci tensor, g is Riemannian metric, X is a vector field and λ is a scalar. The Ricci soliton is said to be shrinking, steady and expanding according as

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λ is negative, zero and positive respectively.

In 2005, A.E. Fischer [10] introduced the concept of conformal Ricci flow which is a variation of the classical Ricci flow equation. In classical Ricci flow equation the unit volume constraint plays an important role but in conformal Ricci flow equation scalar curvature R is considered as constraint. Since the conformal geometry plays an important role to constrain the scalar curvature and the equations are the vector field sum of a conformal flow equation and a Ricci flow equation, the resulting equations are named as the conformal Ricci flow equations. The conformal Ricci flow equation on M where M is considered as a smooth closed connected oriented n-manifold($n > 3$), is defined by the equation [10]

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg \tag{1.3}$$

and $r = -1$, where p is a scalar non-dynamical field(time dependent scalar field), r is the scalar curvature of the manifold and n is the dimension of manifold.

In 2015, N. Basu and A. Bhattacharyya [3] introduced the notion of conformal Ricci soliton and the equation is as follows

$$\pounds_X g + 2S = [2\lambda - (p + \frac{2}{n})]g. \tag{1.4}$$

The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

An almost contact metric structure (ϕ, ξ, η, g) on a manifold M is called a trans-Sasakian structure [14] if the product manifold belongs to the class W_4 where W_4 is a class of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [6]. A trans-Sasakian structure of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic [5], $\beta-$Kenmotsu [13], and $\alpha-$Sasakian [13], respectively.

§2. Preliminaries

A differentiable manifold of dimension n is called Lorentzian Kenmotsu manifold [2] if it admits a $(1, 1)$ tensor field ϕ, a covarient vector field ξ, a 1-form η and Lorentzian metric g which satisfy on M respectively such that

$$\phi^2 X = X + \eta(X)\xi, g(X, \xi) = \eta(X), \tag{2.1}$$

$$\eta(\xi) = -1, \eta(\phi X) = 0, \phi\xi = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

for all $X, Y \in \chi(M)$.

If Lorentzian Kenmotsu manifold M satisfies

$$\nabla_X \xi = \beta[X - \eta(X)\xi], (\nabla_X \phi)Y = \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.4}$$

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y), \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. Then the manifold M is called Lorentzian $\beta-$Kenmotsu manifold.

Furthermore, on an Lorentzian β-Kenmotsu manifold M the following relations hold [1], [17]:

$$\eta(R(X,Y)Z) = \beta^2[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)], \tag{2.6}$$

$$R(\xi, X)Y = \beta^2[\eta(Y)X - g(X,Y)\xi], \tag{2.7}$$

$$R(X,Y)\xi = \beta^2[\eta(X)Y - \eta(Y)X], \tag{2.8}$$

$$S(X,\xi) = -(n-1)\beta^2\eta(X), \tag{2.9}$$

$$Q\xi = -(n-1)\beta^2\xi, \tag{2.10}$$

$$S(\xi,\xi) = (n-1)\beta^2, \tag{2.11}$$

where β is some constant, R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$ for all $X, Y \in \chi(M)$.

Now from definition of Lie derivative we have

$$\begin{aligned}(\pounds_\xi g)(X,Y) &= (\nabla_\xi g)(X,Y) + g(\beta[X - \eta(X)\xi], Y) + g(X, \beta[Y - \eta(Y)\xi]) \\ &= 2\beta g(X,Y) - 2\beta\eta(X)\eta(Y). \end{aligned} \tag{2.12}$$

Applying Ricci soliton equation (1.2) in (2.12) we get

$$\begin{aligned} S(X,Y) &= \frac{1}{2}[-2\lambda g(X,Y)] - \frac{1}{2}[2\beta g(X,Y) - 2\beta\eta(X)\eta(Y)] \\ &= -\lambda g(X,Y) - \beta g(X,Y) + \beta\eta(X)\eta(Y) \\ &= \acute{A} g(X,Y) + \beta\eta(X)\eta(Y), \end{aligned} \tag{2.13}$$

where $\acute{A} = (-\lambda - \beta)$, which shows that the manifold is η-Einstein.

Also

$$QX = \acute{A}X + \beta\eta(X)\xi, \tag{2.14}$$

$$S(X,\xi) = (\acute{A} + \beta)\eta(X) = A\eta(X). \tag{2.15}$$

If we put $X = Y = e_i$ in (2.13) where $\{e_i\}$ is the orthonormal basis of the tangent space TM where TM is a tangent bundle of M and summing over i, we get

$$R(g) = \acute{A}n + \beta.$$

Proposition 2.1 *A Lorentzian β-Kenmotsu manifold admitting Ricci soliton is η-Einstein.*

Again applying conformal Ricci soliton (1.4) in (2.12) we get

$$\begin{aligned} S(X,Y) &= \frac{1}{2}[2\lambda - (p+\frac{2}{n})]g(X,Y) - \frac{1}{2}[2\beta g(X,Y) - 2\beta\eta(X)\eta(Y)] \\ &= \acute{B}g(X,Y) + \beta\eta(X)\eta(Y), \end{aligned} \tag{2.16}$$

where

$$\acute{B} = \frac{1}{2}[2\lambda - (p+\frac{2}{n})] - \beta, \tag{2.17}$$

which also shows that the manifold is η-Einstein.

Also

$$QX = \acute{B}X + \beta\eta(X)\xi, \tag{2.18}$$

$$S(X,\xi) = (\acute{B}+\beta)\eta(X) = B\eta(X). \tag{2.19}$$

If we put $X = Y = e_i$ in (2.16) where $\{e_i\}$ is the orthonormal basis of the tangent space TM where TM is a tangent bundle of M and summing over i, we get

$$r = \acute{B}n + \beta.$$

For conformal Ricci soliton $r(g) = -1$. So

$$-1 = \acute{B}n + \beta$$

which gives $B = \frac{1}{n}(-\beta - 1)$.

Comparing the values of B from (2.17) with the above equation we get

$$\lambda = \frac{1}{n}(\beta(n-1)-1) + \frac{1}{2}(p+\frac{2}{n})$$

Proposition 2.2 *A Lorentzian β-Kenmotsu manifold admitting conformal Ricci soliton is η-Einstein and the value of the scalar*

$$\lambda = \frac{1}{n}(\beta(n-1)-1) + \frac{1}{2}(p+\frac{2}{n}).$$

§3. Lorentzian β-Kenmotsu Manifold Admitting Ricci Soliton, Conformal Ricci Soliton and $R(\xi,X).\tilde{C} = 0$

Let M be a n dimensional Lorentzian β-Kenmotsu manifold admitting Ricci soliton (g,V,λ). Quasi conformal curvature tensor $\tilde{C}$ on M is defined by

$$\begin{aligned} \tilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad -[\frac{r}{2n+1}][\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.1}$$

where r is scalar curvature.

Putting $Z = \xi$ in (3.1) we have

$$\begin{aligned}\tilde{C}(X,Y)\xi &= aR(X,Y)\xi + b[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY] \\ &\quad -[\frac{r}{2n+1}][\frac{a}{2n} + 2b][g(Y,\xi)X - g(X,\xi)Y]. \end{aligned} \tag{3.2}$$

Using (2.1), (2.8), (2.14), (2.15) in (3.2) we get

$$\tilde{C}(X,Y)\xi = [-a\beta^2 + Ab + \acute{A}b - [\frac{r}{2n+1}][\frac{a}{2n} + 2b]](\eta(Y)X - \eta(X)Y).$$

Let

$$D = -a\beta^2 + Ab + \acute{A}b - [\frac{r}{2n+1}][\frac{a}{2n} + 2b],$$

so we have

$$\tilde{C}(X,Y)\xi = D(\eta(Y)X - \eta(X)Y). \tag{3.3}$$

Taking inner product with Z in (3.3) we get

$$-\eta(\tilde{C}(X,Y)Z) = D[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]. \tag{3.4}$$

Now we consider that the Lorentzian β-Kenmotsu manifold M which admits Ricci soliton is quasi conformally semi symmetric i.e. $R(\xi,X).\tilde{C} = 0$ holds in M, which implies

$$R(\xi,X)(\tilde{C}(Y,Z)W) - \tilde{C}(R(\xi,X)Y,Z)W - \tilde{C}(Y,R(\xi,X)Z)W - \tilde{C}(Y,Z)R(\xi,X)W = 0, \tag{3.5}$$

for all vector fields X, Y, Z, W on M.

Using (2.7) in (3.5) and putting $W = \xi$ we get

$$\begin{aligned}&\eta(\tilde{C}(Y,Z)\xi)X - g(X,\tilde{C}(Y,Z)\xi)\xi - \eta(Y)\tilde{C}(X,Z)\xi + g(X,Y)\tilde{C}(\xi,Z)\xi \\ &-\eta(Z)\tilde{C}(Y,X)\xi + g(X,Z)\tilde{C}(Y,\xi)\xi - \eta(\xi)\tilde{C}(Y,Z)X + g(X,\xi)\tilde{C}(Y,Z)\xi = 0. \end{aligned} \tag{3.6}$$

Taking inner product with ξ in (3.6) and using (2.2), (3.3) we obtain

$$g(X,\tilde{C}(Y,Z)\xi) + \eta(\tilde{C}(Y,Z)X) = 0. \tag{3.7}$$

Putting $Z = \xi$ in (3.7) and using (3.3) we get

$$-Dg(X,Y) - D\eta(X)\eta(Y) + \eta(\tilde{C}(Y,Z)X) = 0. \tag{3.8}$$

Now from (3.1) we can write

$$\begin{aligned}\tilde{C}(Y,\xi)X &= aR(Y,\xi)X + b[S(\xi,X)Y - S(Y,X)\xi + g(\xi,X)QY - g(Y,X)Q\xi] \\ &\quad -[\frac{r}{2n+1}][\frac{a}{2n} + 2b][g(\xi,X)Y - g(Y,X)\xi]. \end{aligned} \tag{3.9}$$

Taking inner product with ξ and using $(2.2), (2.7), (2.9), (2.10)$ in (3.9) we get

$$\begin{aligned}\eta(\tilde{C}(Y,\xi)X) &= a\eta(\beta^2(g(X,Y)\xi - \eta(X)Y)) + b[A\eta(X)\eta(Y) + S(X,Y) + \eta(X)(\acute{A}\eta(Y) \\ &\quad -\beta\eta(Y)) - g(X,Y)(-\acute{A}+\beta)] - [\frac{r}{2n+1}][\frac{a}{2n} + 2b][\eta(X)\eta(Y) + g(X,Y)].\end{aligned}$$

After a long simplification we have

$$\begin{aligned}\eta(\tilde{C}(Y,\xi)X) &= g(X,Y)[\acute{A}b - b\beta - a\beta^2 - [\frac{r}{2n+1}][\frac{a}{2n} + 2b]] \\ &\quad +\eta(X)\eta(Y)[2\acute{A}b - a\beta^2 - [\frac{r}{2n+1}][\frac{a}{2n} + 2b]] + bS(X,Y). \qquad (3.10)\end{aligned}$$

Putting (3.10) in (3.5) we get

$$\rho g(X,Y) + \sigma\eta(X)\eta(Y) = S(X,Y), \qquad (3.11)$$

where

$$\rho = \frac{1}{b}[D + b\beta + a\beta^2 - \acute{A}b + [\frac{r}{2n+1}][\frac{a}{2n} + 2b]]$$

and

$$\sigma = \frac{1}{b}[D + a\beta^2 - 2\acute{A}b + [\frac{r}{2n+1}][\frac{a}{2n} + 2b]].$$

So from (3.11) we conclude that the manifold becomes η-Einstein manifold. Thus we can write the following theorem:

Theorem 3.1 *If a Lorentzian β-Kenmotsu manifold admits Ricci soliton and is quasi conformally semi symmetric i.e. $R(\xi,X).\tilde{C} = 0$, then the manifold is η-Einstein manifold where $\tilde{C}$ is quasi conformal curvature tensor and $R(\xi,X)$ is derivation of tensor algebra of the tangent space of the manifold.*

If a Lorentzian β-Kenmotsu manifold admits conformal Ricci soliton then after a brief calculation we can also establish that the manifold becomes η-Einstein, only the values of constants ρ, σ will be changed which would not hamper our main result.

Hence we can state the following theorem:

Theorem 3.2 *A Lorentzian β-Kenmotsu manifold admitting conformal Ricci soliton and is quasi conformally semi symmetric i.e. $R(\xi,X).\tilde{C} = 0$, then the manifold is η-Einstein manifold where $\tilde{C}$ is quasi conformal curvature tensor and $R(\xi,X)$ is derivation of tensor algebra of the tangent space of the manifold.*

§4. Lorentzian β-Kenmotsu Manifold Admitting Ricci Soliton, Conformal Ricci Soliton and $R(\xi,X).S = 0$

Let M be a n dimensional Lorentzian β-Kenmotsu manifold admitting Ricci soliton (g,V,λ). Now we consider that the tensor derivative of S by $R(\xi,X)$ is zero i.e. $R(\xi,X).S = 0$. Then the

Lorentzian β-Kenmotsu manifold admitting Ricci soliton is Ricci semi symmetric which implies

$$S(R(\xi,X)Y,Z)+S(Y,R(\xi,X)Z)=0. \tag{4.1}$$

Using (2.13) in (4.1) we get

$$\acute{A}g(R(\xi,X)Y,Z)+\beta\eta(R(\xi,X)Y)\eta(Z)+\acute{A}g(Y,R(\xi,X)Z)+\beta\eta(Y)\eta(R(\xi,X)Z)=0. \tag{4.2}$$

Using (2.7) in (4.2) we get

$$\acute{A}g(\beta^2[\eta(Y)X-g(X,Y)\xi],Z)+\acute{A}g(Y,\beta^2[\eta(Z)X-g(X,Z)\xi])+\beta\eta(\beta^2[\eta(Y)X-$$
$$g(X,Y)\xi])\eta(Z)+\beta\eta(Y)\eta(\beta^2[\eta(Z)X-g(X,Z)\xi])=0. \tag{4.3}$$

Using (2.2) in (4.3) we have

$$\acute{A}\beta^2\eta(Y)g(X,Z)-\acute{A}\beta^2\eta(Z)g(X,Y)+\acute{A}\beta^2\eta(Z)g(X,Y)-\acute{A}\beta^2\eta(Y)g(X,Z)$$
$$+\beta^3\eta(Y)\eta(X)\eta(Z)+\beta^3g(X,Y)\eta(Z)+\beta^3\eta(Y)\eta(X)\eta(Z)+\beta^3g(X,Z)\eta(Y)=0. \tag{4.4}$$

Putting $Z=\xi$ in (4.4) and using (2.2) we obtain

$$g(X,Y)=-\eta(X)\eta(Y).$$

Hence we can state the following theorem:

Theorem 4.1 *If a Lorentzian β-Kenmotsu manifold admits Ricci soliton and is Ricci semi symmetric i.e. $R(\xi,X).S=0$, then $g(X,Y)=-\eta(X)\eta(Y)$ where S is Ricci tensor and $R(\xi,X)$ is derivation of tensor algebra of the tangent space of the manifold.*

If a Lorentzian β-Kenmotsu manifold admits conformal Ricci soliton then by similar calculation we can obtain the same result. Hence we can state the following theorem:

Theorem 4.2 *A Lorentzian β-Kenmotsu manifold admitting conformal Ricci soliton and is Ricci semi symmetric i.e. $R(\xi,X).S=0$, then $g(X,Y)=-\eta(X)\eta(Y)$ where S is Ricci tensor and $R(\xi,X)$ is derivation of tensor algebra of the tangent space of the manifold.*

§5. Lorentzian β-Kenmotsu Manifold Admitting Ricci Soliton, Conformal Ricci Soliton and $R(\xi,X).P=0$

Let M be a n dimensional Lorentzian β-Kenmotsu manifold admitting Ricci soliton (g,V,λ). The projective curvature tensor P on M is defined by

$$P(X,Y)Z=R(X,Y)Z-\frac{1}{2n}[S(Y,Z)X-S(X,Z)Y]. \tag{5.1}$$

Here we consider that the manifold is projectively semi symmetric i.e. $R(\xi,X).P=0$ holds.

So

$$R(\xi, X)(P(Y,Z)W) - P(R(\xi, X)Y, Z)W - P(Y, R(\xi, X)Z)W - P(Y,Z)R(\xi, X)W = 0, \quad (5.2)$$

for all vector fields X, Y, Z, W on M.

Using (2.7) and putting $Z = \xi$ in (5.2) we have

$$\eta(P(Y,\xi)W)X - g(X, P(Y,\xi)W)\xi - \eta(Y)P(X,\xi)W + g(X,Y)P(\xi,\xi)W$$
$$-\eta(\xi)P(Y,X)W + g(X,\xi)P(Y,\xi)W - \eta(W)P(Y,\xi)X + g(X,W)P(Y,\xi)\xi = 0. \quad (5.3)$$

Now from (5.1) we can write

$$P(X,\xi)Z = R(X,\xi)Z - \frac{1}{n-1}[S(\xi,Z)X - S(X,Z)\xi]. \quad (5.4)$$

Using (2.7), (2.15) in (5.4) we get

$$P(X,\xi)Z = \beta^2 g(X,Z)\xi + \frac{1}{n-1}S(X,Z)\xi + (\frac{A}{n-1} - \beta^2)\eta(Z)X. \quad (5.5)$$

Putting (5.5) and $W = \xi$ in (5.3) and after a long calculation we get

$$\frac{1}{n-1}S(X,Y)\xi + (\frac{A}{n-1} + \beta^2)\eta(X)Y - \frac{A}{n-1}g(X,Y)\xi$$
$$-(\frac{A}{n-1} + \beta^2)\eta(Y)X = 0. \quad (5.6)$$

Taking inner product with ξ in (5.6) we obtain

$$S(X,Y) = -Ag(X,Y),$$

which clearly shows that the manifold in an Einstein manifold.

Thus we can conclude the following theorem:

Theorem 5.1 *If a Lorentzian β-Kenmotsu manifold admits Ricci soliton and is projectively semi symmetric i.e. $R(\xi, X).P = 0$ holds, then the manifold is an Einstein manifold where P is projective curvature tensor and $R(\xi, X)$ is derivation of tensor algebra of the tangent space of the manifold.*

If a Lorentzian β-Kenmotsu manifold admits conformal Ricci soliton then using the same calculation we can obtain similar result, only the value of constant A will be changed which would not hamper our main result. Hence we can state the following theorem:

Theorem 5.2 *A Lorentzian β-Kenmotsu manifold admitting conformal Ricci soliton and is projectively semi symmetric i.e. $R(\xi, X).P = 0$ holds, then the manifold is an Einstein manifold where P is projective curvature tensor and $R(\xi, X)$ is derivation of tensor algebra of the tangent space of the manifold.*

§6. Lorentzian β-Kenmotsu Manifold Admitting Ricci Soliton, Conformal Ricci Soliton and $R(\xi, X).\tilde{P} = 0$

Let M be a n dimensional Lorentzian β-Kenmotsu manifold admitting Ricci soliton (g, V, λ). The pseudo projective curvature tensor $\tilde{P}$ on M is defined by

$$\tilde{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}[\frac{a}{n-1} + b][g(Y,Z)X - g(X,Z)Y]. \tag{6.1}$$

Here we consider that the manifold is pseudo projectively semi symmetric i.e. $R(\xi, X).\tilde{P} = 0$ holds.

So

$$R(\xi,X)(\tilde{P}(Y,Z)W) - \tilde{P}(R(\xi,X)Y,Z)W - \tilde{P}(Y,R(\xi,X)Z)W - \tilde{P}(Y,Z)R(\xi,X)W = 0, \tag{6.2}$$

for all vector fields X, Y, Z, W on M.

Using (2.7) and putting $W = \xi$ in (6.2) we have

$$\eta(\tilde{P}(Y,Z)\xi)X - g(X,\tilde{P}(Y,Z)\xi)\xi - \eta(Y)\tilde{P}(X,Z)\xi + g(X,Y)\tilde{P}(\xi,Z)\xi - \eta(Z)\tilde{P}(Y,X)\xi + g(X,Z)\tilde{P}(Y,\xi)\xi - \eta(\xi)\tilde{P}(Y,Z)X + \eta(X)\tilde{P}(Y,Z)\xi = 0. \tag{6.3}$$

Now from (6.1) we can write

$$\tilde{P}(X,Y)\xi = aR(X,Y)\xi + b[S(Y,\xi)X - S(X,\xi)Y] + \frac{r}{n}[\frac{a}{n-1} + b][g(Y,\xi)X - g(X,\xi)Y]. \tag{6.4}$$

Using $(2.1), (2.8), (2.15)$ in (6.4) and after a long calculation we get

$$\tilde{P}(X,Y)\xi = \varphi(\eta(X)Y - \theta(Y)X), \tag{6.5}$$

where $\varphi = (a\beta^2 - Ab - \frac{r}{n}[\frac{a}{n-1} + b])$.

Using (6.5) and putting $Z = \xi$ in (6.3) we obtain

$$\tilde{P}(Y,\xi)X + \varphi\eta(X)Y - \varphi g(X,Y)\xi = 0. \tag{6.6}$$

Taking inner product with ξ in (6.6) we get

$$\eta(\tilde{P}(Y,\xi)X) + \varphi\eta(X)\eta(Y) - \varphi g(X,Y) = 0. \tag{6.7}$$

Again from (6.1) we can write

$$\tilde{P}(X,\xi)Z = a(X,\xi)Z + b[S(\xi,Z)X - S(X,Z)\xi] + \frac{r}{n}[\frac{a}{n-1} + b][g(\xi,Z)X - g(X,Z)\xi]. \tag{6.8}$$

Using $(2.1), (2.7), (2.15)$ in (6.8) we get

$$\tilde{P}(X,\xi)Z = a\beta^2[g(X,Z)\xi - \eta(Z)X] + b[A\eta(Z)X - S(X,Z)\xi]$$
$$+\frac{r}{n}[\frac{a}{n-1} + b][g(\xi,Z)X - g(X,Z)\xi]. \quad (6.9)$$

Taking inner product with ξ and replacing X by Y, Z by X in (6.9) we have

$$\eta(\tilde{P}(Y,\xi)X) = a\beta^2[-g(X,Y) - \eta(X)\eta(Y)] + b[A\eta(X)\eta(Y) + S(X,Y)]+$$
$$\frac{r}{n}[\frac{a}{n-1} + b][\eta(X)\eta(Y) - g(X,Y)]. \quad (6.10)$$

Using (6.10) in (6.7) and after a brief simplification we obtain

$$S(X,Y) = Tg(X,Y) + U\eta(X)\eta(Y), \quad (6.11)$$

where $T = -\frac{1}{b}[-a\beta^2 - \frac{r}{n}[\frac{a}{n-1} + b] - \varphi]$ and $U = -\frac{1}{b}[\varphi + \frac{r}{n}[\frac{a}{n-1} + b] + Ab - a\beta^2]$.

From (6.11) we can conclude that the manifold is η-Einstein. Thus we have the following theorem:

Theorem 6.1 *If a Lorentzian β-Kenmotsu manifold admits Ricci soliton and is pseudo projectively semi symmetric i.e. $R(\xi,X).\tilde{P} = 0$ holds, then the manifold is η Einstein manifold where $\tilde{P}$ is pseudo projective curvature tensor and $R(\xi,X)$ is derivation of tensor algebra of the tangent space of the manifold.*

If a Lorentzian β-Kenmotsu manifold admits conformal Ricci soliton then by following the same calculation we would obtain the same result, only the constant value of T and U will be changed. Hence we can state the following theorem:

Theorem 6.2 *A Lorentzian β-Kenmotsu manifold admitting conformal Ricci soliton and is pseudo projectively semi symmetric i.e. $R(\xi,X).\tilde{P} = 0$ holds, then the manifold is η Einstein manifold where $\tilde{P}$ is pseudo projective curvature tensor and $R(\xi,X)$ is derivation of tensor algebra of the tangent space of the manifold.*

§7. An Example of a 3-Dimensional Lorentzian β-Kenmotsu Manifold

In this section we construct an example of a 3-dimensional Lorentzian β-kenmotsu manifold.To construct this, we consider the three dimensional manifold $M = \{(x,y,z) \in R^3 : z \neq 0\}$ where (x,y,z) are the standard coordinates in R^3. The vector fields

$$e_1 = e^{-z}\frac{\partial}{\partial x}, e_2 = e^{-z}\frac{\partial}{\partial y}, e_3 = e^{-z}\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Lorentzian metric defined by

$$g(e_1, e_1) = 1, g(e_2, e_2) = 1, g(e_3, e_3) = -1,$$

$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

Let η be the 1-form which satisfies the relation

$$\eta(e_3) = -1.$$

Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = -e_2, \phi(e_2) = -e_1, \phi(e_3) = 0$. Then we have

$$\phi^2(Z) = Z + \eta(Z)e_3,$$
$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W)$$

for any $Z, W \in \chi(M^3)$. Thus for $e_3 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on M. Now, after calculating we have

$$[e_1, e_3] = e^{-z}e_1, [e_1, e_2] = 0, [e_2, e_3] = e^{-z}e_2.$$

The Riemannian connection ∇ of the metric is given by the Koszul's formula which is

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Y g(Z, X) - Zg(X, Y)$$
$$-g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \quad (7.1)$$

By Koszul's formula we get

$$\nabla_{e_1} e_1 = e^{-z}e_3, \nabla_{e_2} e_1 = 0, \nabla_{e_3} e_1 = 0,$$

$$\nabla_{e_1} e_2 = 0, \nabla_{e_2} e_2 = `e^{-z}e_3, \nabla_{e_3} e_2 = 0,$$

$$\nabla_{e_1} e_3 = e^{-z}e_1, \nabla_{e_2} e_3 = e^{-z}e_2, \nabla_{e_3} e_3 = 0.$$

From the above we have found that $\beta = e^{-z}$ and it can be easily shown that $M^3(\phi, \xi, \eta, g)$ is a Lorentzian β-kenmotsu manifold. The results established in this note can be verified on this manifold.

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