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On the Leap Zagreb Indices of Generalized xyz-Point-Line Transformation Graphs $T^{xyz}(G)$ when $z = 1$

B. Basavanagoud and Chitra E.

(Department of Mathematics, Karnatak University, Dharwad - 580 003, Karnataka, India)

E-mail: b.basavanagoud@gmail.com, chitrachandra79.cecc@gmail.com

Abstract: For a graph G, the first, second and third leap Zagreb indices are the sum of squares of 2-distance degree of vertices of G; the sum of product of 2-distance degree of end vertices of edges in G and the sum of product of 1-distance degree and 2-distance degrees of vertices of G, respectively. In this paper, we obtain the expressions for these three leap Zagreb indices of generalized xyz point line transformation graphs $T^{xyz}(G)$ when $z = 1$.

Key Words: Distance, degree, diameter, Zagreb index, leap Zagreb index, reformulated Zagreb index.

AMS(2010): 05C90, 05C35, 05C12, 05C07.

§1. Introduction

Let $G = (V, E)$ be a simple graph of order n and size m. The *k-distance degree of a vertex* $v \in V(G)$, denoted by $d_k(v/G) = |N_k(v/G)|$ where $N_k(v/G) = \{u \in V(G) : d(u, v) = k\}$ [17] in which $d(u, v)$ is the distance between the vertices u and v in G that is the length of the shortest path joining u and v in G. The *degree of a vertex* v in a graph G is the number of edges incident to it in G and is denoted by $d_G(v)$. Here $N_1(v/G)$ is nothing but $N_G(v)$ and $d_1(v/G)$ is same as $d_G(v)$. If u and v are two adjacent vertices of G, then the edge connecting them will be denoted by uv. The *degree of an edge* $e = uv$ in G, denoted by $d_1(e/G)$ (or $d_G(e)$), is defined by $d_1(e/G) = d_1(u/G) + d_1(v/G) - 2$.

The *complement* of a graph G is denoted by $\overline{G}$ whose vertex set is $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are nonadjacent in G. $\overline{G}$ has n vertices and $\frac{n(n-1)}{2} - m$ edges. The *line graph* $L(G)$ of a graph G with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G have a vertex incident in common. The *complement of line graph* $\overline{L(G)}$ *or jump graph* $J(G)$ of a graph G is a graph with vertex set as the edge set of G and two vertices of $J(G)$ are adjacent whenever the corresponding edges in G have no vertex incident in common. The *subdivision graph* $S(G)$ of a graph G whose vertex set is $V(G) \bigcup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is

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an edge of G incident with it. The *partial complement of subdivision graph* $\overline{S}(G)$ of a graph G whose vertex set is $V(G)\bigcup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and the other is an edge of G non incident with it.

We follow [11] and [13] for unexplained graph theoretic terminologies and notations.

The first and second Zagreb indices [9] of a graph G are defined as follows:

$$M_1(G) = \sum_{v\in V(G)} d_G(v)^2 \quad \text{and} \quad M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v),$$

respectively. These are widely studied degree based topological indices due to their applications in chemistry. For details see the papers [5, 7, 8, 10, 18]. The first Zagreb index [15] can also be expressed as

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)]$$

Ashrafi et al. [1] defined the first and second Zagreb coindices as

$$\overline{M_1}(G) = \sum_{uv\notin E(G)} [d_G(u) + d_G(v)] \quad \text{and} \quad \overline{M_2}(G) = \sum_{uv\notin E(G)} [d_G(u)d_G(v)],$$

respectively.

In 2004, Milićević et al. [14] reformulated the Zagreb indices in terms of edge-degrees instead of vertex-degrees. The first and second reformulated Zagreb indices are defined, respectively, as

$$EM_1(G) = \sum_{e\in E(G)} d_G(e)^2 \quad \text{and} \quad EM_2(G) = \sum_{e\sim f} [d_G(e)d_G(f)]$$

In [12], Hosamani and Trinajstić defined the first and second reformulated Zagreb coindices respectively as

$$\begin{aligned}\overline{EM_1}(G) &= \sum_{e\sim f} [d_G(e) + d_G(f)],\\ \overline{EM_2}(G) &= \sum_{e\nsim f} [d_G(e) + d_G(f)].\end{aligned}$$

In 2017, Naji et al. [16] introduced the leap Zagreb indices. For a graph G, the first, second, and third leap Zagreb indices [16] are denoted and defined respectively as:

$$\begin{aligned}LM_1(G) &= \sum_{v\in V(G)} d_2(v/G)^2,\\ LM_2(G) &= \sum_{uv\in E(G)} d_2(u/G)d_2(v/G),\\ LM_3(G) &= \sum_{v\in V(G)} d_1(v/G)d_2(v/G).\end{aligned}$$

Throughout this paper, in our results we write the notations $d_1(v)$ and $d_1(e)$ respectively for degree of a vertex v and degree of an edge e of a graph.

§2. Generalized xyz-Point-Line Transformation Graph $T^{xyz}(G)$

The procedure of obtaining a new graph from a given graph by using incidence (or nonincidence) relation between vertex and an edge and an adjacency (or nonadjacency) relation between two vertices or two edges of a graph is known as *graph transformation* and the graph obtained by doing so is called a transformation graph. For a graph $G = (V, E)$, let G^0 be the graph with $V(G^0) = V(G)$ and with no edges, G^1 the complete graph with $V(G^1) = V(G)$, $G^+ = G$, and $G^- = \overline{G}$. Let $\mathcal{G}$ denotes the set of simple graphs. The following graph operations depending on $x, y, z \in \{0, 1, +, -\}$ induce functions $T^{xyz} : \mathcal{G} \to \mathcal{G}$. These operations are introduced by Deng et al. in [6]. They called these resulting graphs as xyz-transformations of G, denoted by $T^{xyz}(G) = G^{xyz}$ and studied the Laplacian characteristic polynomials and some other Laplacian parameters of xyz-transformations of an r-regular graph G. In [2], Wu Bayoindureng et al. introduced the total transformation graphs and studied the basic properties of total transformation graphs. Motivated by this, Basavanagoud [3] studied the basic properties of the xyz-transformation graphs by calling them xyz-point-line transformation graphs by changing the notion of xyz-transformations of a graph G as $T^{xyz}(G)$ to avoid confusion between parent graph G and its xyz-transformations.

Definition 2.1([6]) *Given a graph G with vertex set $V(G)$ and edge set $E(G)$ and three variables $x, y, z \in \{0, 1, +, -\}$, the xyz-point-line transformation graph $T^{xyz}(G)$ of G is the graph with vertex set $V(T^{xyz}(G)) = V(G) \cup E(G)$ and the edge set $E(T^{xyz}(G)) = E((G)^x) \cup E((L(G))^y) \cup E(W)$ where $W = S(G)$ if $z = +$, $W = \overline{S}(G)$ if $z = -$, W is the graph with $V(W) = V(G) \cup E(G)$ and with no edges if $z = 0$ and W is the complete bipartite graph with parts $V(G)$ and $E(G)$ if $z = 1$.*

Since there are 64 distinct 3 - permutations of $\{0, 1, +, -\}$. Thus obtained 64 kinds of generalized xyz-point-line transformation graphs. There are 16 different graphs for each case when $z = 0$, $z = 1$, $z = +$, $z = -$.

In this paper, we consider the xyz-point-line transformation graphs $T^{xyz}(G)$ when $z = 1$.

Example 2.1 Let $G = K_2 \cdot K_3$ be a graph. Then G^0 be the graph with $V(G^0) = V(G)$ and with no edges, G^1 the complete graph with $V(G^1) = V(G)$, $G^+ = G$, and $G^- = \overline{G}$ which are depicted in the following Figure 1.

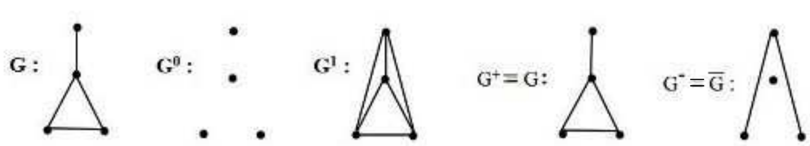

Figure 1

The self-explanatory examples of the path P_4 and its *xyz-point-line transformation graphs* $T^{xy1}(P_4)$ are depicted in Figure 2.

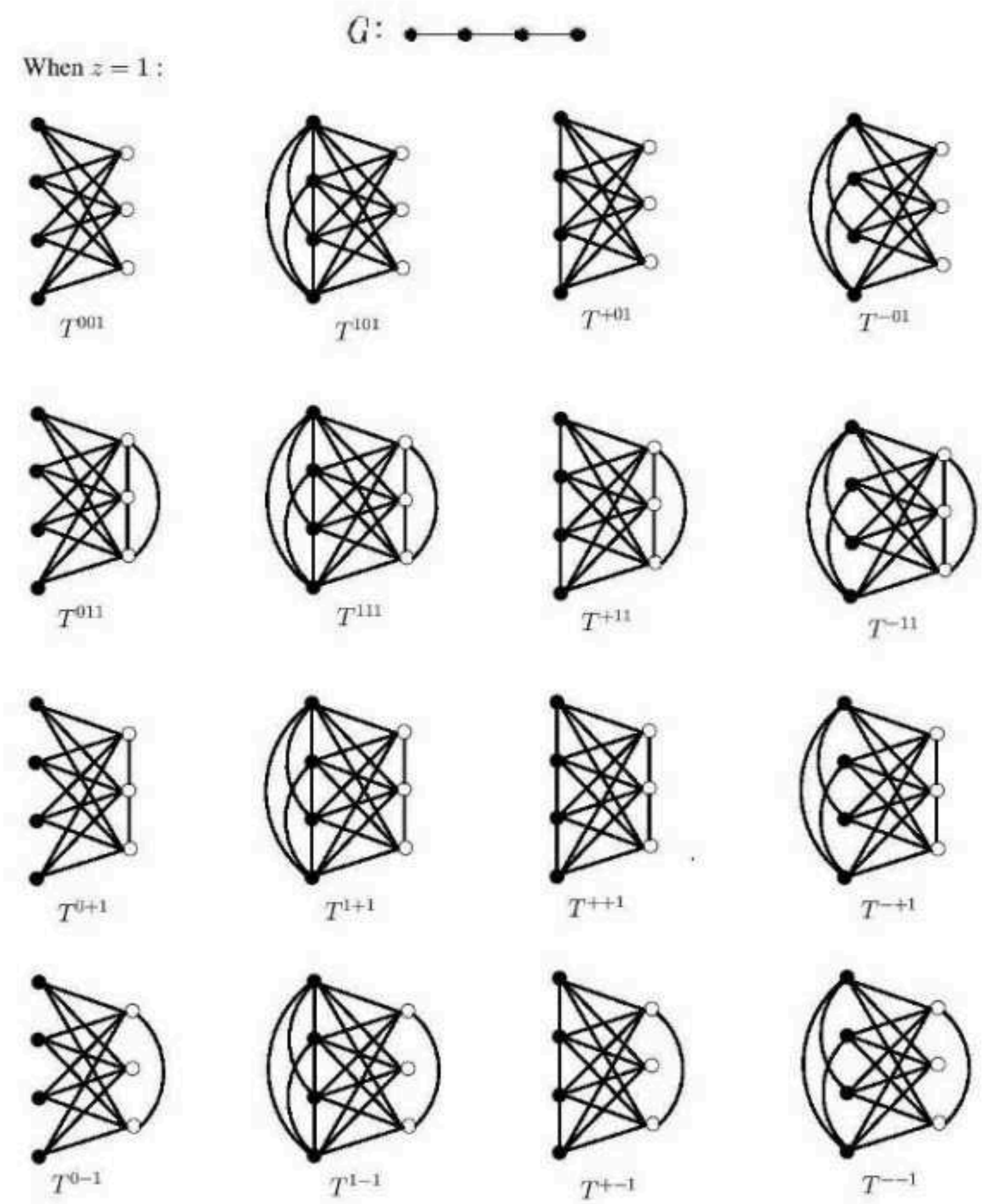
G:
When z = 1 :
T001
T101
T+01
T−01
T011
T111
T+11
T−11
T0+1
T1+1
T++1
T−+1
T0−1
T1−1
T+−1
T−−1

Figure 2

§3. Leap Zagreb Indices of $T^{xy1}(G)$

Theorem 3.1([3]) *Let G be a graph of order n and size m. Then*

(1) $|V(T^{xyz}(G))| = n+m$;

(2) $|E(T^{xyz}(G))| = |E(G^x)| + |E(L(G)^y)| + |E(W)|$, *where*

$$|E(G^x)|=\begin{cases} 0 & if\ x=0. \\ \binom{n}{2} & if\ x=1. \\ m & if\ x=+. \\ \binom{n}{2}-m & if\ x=-. \end{cases}$$

$$|E(L(G)^y)|=\begin{cases} 0 & if\ y=0. \\ \binom{m}{2} & if\ y=1. \\ -m+\frac{1}{2}M_1 & if\ y=+. \\ \binom{m+1}{2}-\frac{1}{2}M_1 & if\ y=-. \end{cases}$$

$$|E(W)|=\begin{cases} 0 & if\ z=0. \\ mn & if\ z=1. \\ m & if\ z=+. \\ m(n-2) & if\ z=-. \end{cases}$$

The following Propositions are useful for calculating $d_2(T^{xy1}(G))$ in Observation 3.4.

Proposition 3.2([4]) *Let G be a graph of order n and size m. Let v be a vertex of G. Then*

$$d_{T^{xy1}(G)}(v)=\begin{cases} m & if\ x=0, y\in\{0,1,+,-\} \\ n+m-1 & if\ x=1, y\in\{0,1,+,-\} \\ m+d_G(v) & if\ x=+, y\in\{0,1,+,-\} \\ n+m-1-d_G(v) & if\ x=-, y\in\{0,1,+,-\} \end{cases}$$

Proposition 3.3([4]) *Let G be a graph of order n and size m. Let e be an edge of G. Then*

$$d_{T^{xy1}(G)}(e)=\begin{cases} n & if\ y=0, x\in\{0,1,+,-\} \\ n+m-1 & if\ y=1, x\in\{0,1,+,-\} \\ n+d_G(e) & if\ y=+, x\in\{0,1,+,-\} \\ n+m-1-d_G(e) & if\ y=-, x\in\{0,1,+,-\} \end{cases}$$

Observation 3.4 Let G be a connected (n,m) graph. Then

(1) $d_2(v/T^{001})(G)=\begin{cases} (n-1) & if\ v\in V(G) \\ (m-1) & if\ v=e\in E(G) \end{cases}$

(2) $d_2(v/T^{101})(G)=\begin{cases} 0 & if\ v\in V(G) \\ (m-1) & if\ v=e\in E(G) \end{cases}$

(3) $d_2(v/T^{+01})(G)=\begin{cases} n-1-d_1(v/G) & if\ v\in V(G) \\ (m-1) & if\ v=e\in E(G) \end{cases}$

(4) $d_2(v/T^{-01})(G)=\begin{cases} d_1(v/G) & if\ v\in V(G) \\ (m-1) & if\ v=e\in E(G) \end{cases}$

(5) $d_2(v/T^{011})(G)=\begin{cases} n-1 & if\ v\in V(G) \\ 0 & if\ v=e\in E(G) \end{cases}$

(6) $d_2(v/T^{111})(G)=\begin{cases} 0 & if\ v\in V(G) \\ 0 & if\ v=e\in E(G) \end{cases}$

(7) $d_2(v/T^{+11})(G)=\begin{cases} n-1-d_1(v/G) & if\ v\in V(G) \\ 0 & if\ v=e\in E(G) \end{cases}$

(8) $d_2(v/T^{-11})(G)=\begin{cases} d_1(v/G) & if\ v\in V(G) \\ 0 & if\ v=e\in E(G) \end{cases}$

(9) $d_2(v/T^{0+1})(G)=\begin{cases} n-1 & if\ v\in V(G) \\ m-1-d_1(e/G) & if\ v=e\in E(G) \end{cases}$

(10) $d_2(v/T^{1+1})(G)=\begin{cases} 0 & if\ v\in V(G) \\ m-1-d_1(e/G) & if\ v=e\in E(G) \end{cases}$

(11) $d_2(v/T^{++1})(G)=\begin{cases} n-1-d_1(v/G) & if\ v\in V(G) \\ m-1-d_1(e/G) & if\ v=e\in E(G) \end{cases}$

(12) $d_2(v/T^{-+1})(G)=\begin{cases} d_1(v/G) & if\ v\in V(G) \\ m-1-d_1(e/G) & if\ v=e\in E(G) \end{cases}$

(13) $d_2(v/T^{0-1})(G)=\begin{cases} n-1 & if\ v\in V(G) \\ d_1(e/G) & if\ v=e\in E(G) \end{cases}$

(14) $d_2(v/T^{1-1})(G)=\begin{cases} 0 & if\ v\in V(G) \\ d_1(e/G) & if\ v=e\in E(G) \end{cases}$

(15) $d_2(v/T^{+-1})(G)=\begin{cases} n-1-d_1(v/G) & if\ v\in V(G) \\ d_1(e/G) & if\ v=e\in E(G) \end{cases}$

(16) $d_2(v/T^{+-1})(G)=\begin{cases} d_1(v/G) & if\ v\in V(G) \\ d_1(e/G) & if\ v=e\in E(G) \end{cases}$

The above Observation 3.4 is useful for computing leap Zagreb indices of transformation graphs $T^{xy1}(G)$ in the forthcoming theorems.

Theorem 3.5 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{001}(G)) = n(n-1)^2 + m(m-1)^2$;

(2) $LM_2(T^{001}(G)) = mn(m-1)(n-1)$;

(3) $LM_3(T^{001}(G)) = mn(m+n-2)$.

Proof The graph $T^{001}(G)$ has $n+m$ vertices and mn edges, refer Theorem 3.1. By definitions of the first, second and the third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4 we get the following.

$$\begin{aligned} LM_1(T^{001}(G)) &= \sum_{v\in V(T^{001}(G))} d_2(v/T^{001}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{001}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{001}(G))^2 \\ &= n(n-1)^2 + m(m-1)^2. \end{aligned}$$

$$\begin{aligned} LM_2(T^{001}(G)) &= \sum_{uv\in E(T^{001}(G))} \left[d_2(u/T^{001}(G))\right]\left[d_2(v/T^{001}(G))\right] \\ &= \sum_{uv\in E(S(G))} \left[d_2(u/T^{001}(G))\right]\left[d_2(v/T^{001}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{001}(G))\right]\left[d_2(v/T^{001}(G))\right] \\ &= (n-1)(m-1)2m + (n-1)(m-1)(mn-2m) = mn(n-1)(m-1). \end{aligned}$$

$$\begin{aligned} LM_3(T^{001}(G)) &= \sum_{v\in V(T^{001}(G))} \left[d_1(v/T^{001}(G))\right]\left[d_2(v/T^{001}(G))\right] \\ &= \sum_{v\in V(G)} \left[d_1(v/T^{001}(G))\right]\left[d_2(v/T^{001}(G))\right] \\ &\quad + \sum_{e\in E(G)} \left[d_1(e/T^{001}(G))\right]\left[d_2(e/T^{001}(G))\right] \\ &= mn(n-1) + mn(m-1) = mn(m+n-2). \end{aligned}$$ □

Theorem 3.6 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{101}(G)) = m(m-1)^2$;

(2) $LM_2(T^{101}(G)) = 0$;

(3) $LM_3(T^{101}(G)) = mn(m-1)$.

Proof Notice that the graph $T^{101}(G)$ has $n+m$ vertices and $mn + \frac{n(n-1)}{2}$ edges by Theorem 3.1. According to the definitions of first, second and third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4, calculation shows the following.

$$\begin{aligned} LM_1(T^{101}(G)) &= \sum_{v\in V(T^{101}(G))} d_2(v/T^{101}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{101}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{101}(G))^2 \\ &= m(m-1)^2. \end{aligned}$$

$$\begin{aligned} LM_2(T^{101}(G)) &= \sum_{uv\in E(T^{101}(G))} \left[d_2(u/T^{101}(G))\right]\left[d_2(v/T^{101}(G))\right] \\ &= \sum_{uv\in E(G)} \left[d_2(u/T^{101}(G))\right]\left[d_2(v/T^{101}(G))\right] \\ &\quad + \sum_{uv\notin E(G)} \left[d_2(u/T^{101}(G))\right]\left[d_2(v/T^{101}(G))\right] \\ &\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{101}(G))\right]\left[d_2(v/T^{101}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{101}(G))\right]\left[d_2(v/T^{101}(G))\right] = 0. \end{aligned}$$

$$\begin{aligned} LM_3(T^{101}(G)) &= \sum_{v\in V(T^{101}(G))} \left[d_1(v/T^{101}(G))\right]\left[d_2(v/T^{101}(G))\right] \\ &= \sum_{v\in V(G)} \left[d_1(v/T^{101}(G))\right]\left[d_2(v/T^{101}(G))\right] \\ &\quad + \sum_{e\in E(G)} \left[d_1(e/T^{101}(G))\right]\left[d_2(e/T^{101}(G))\right] = mn(m-1). \qquad \square \end{aligned}$$

Theorem 3.7 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{+01}(G)) = n(n-1)^2 + m(m-1)^2 + M_1(G) - 4m(n-1)$;

(2) $LM_2(T^{+01}(G)) = M_2(G) - (n-1)M_1(G) + m[(n-1)^2 + (m-1)(n^2-n-2m)]$;

(3) $LM_3(T^{+01}(G)) = m[n(n+m) - 2(m+1)] - M_1(G)$.

Proof By Theorem 3.1, we know that the graph $T^{+01}(G)$ has $n+m$ vertices and $m(n+1)$ edges. By using the definitions of first, second and third leap Zagreb indices and applying

Propositions 3.2, 3.3 and Observation 3.4 we get the following.

$$\begin{aligned}
LM_1(T^{+01}(G)) &= \sum_{v\in V(T^{+01}(G))} d_2(v/T^{+01}(G))^2\\
&= \sum_{v\in V(G)} d_2(v/T^{+01}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{+01}(G))^2\\
&= \sum_{v\in V(G)} (n-1-d_1(v/G))^2 + \sum_{e\in E(G)} (m-1)^2\\
&= \sum_{v\in V(G)} \left[(n-1)^2 + d_1(v/G)^2 - 2(n-1)d_1(v/G)\right] + \sum_{e\in E(G)} (m-1)^2\\
&= n(n-1)^2 + m(m-1)^2 + M_1(G) - 4m(n-1).
\end{aligned}$$

$$\begin{aligned}
LM_2(T^{+01}(G)) &= \sum_{uv\in E(T^{+01}(G))} \left[d_2(u/T^{+01}(G))\right]\left[d_2(v/T^{+01}(G))\right]\\
&= \sum_{uv\in E(G)} \left[d_2(u/T^{+01}(G))\right]\left[d_2(v/T^{+01}(G))\right]\\
&\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{+01}(G))\right]\left[d_2(v/T^{+01}(G))\right]\\
&\quad + \sum_{uv\in E(\overline{S}(G))} [d_2(u/T^{+01}(G))][d_2(v/T^{+01}(G))]\\
&= \sum_{uv\in E(G)} \left[(n-1)^2 - (n-1)(d_1(u/G)+d_1(v/G)) + d_1(u/G)\cdot d_1(v/G)\right]\\
&\quad + \sum_{uv\in E(S(G))} (m-1)(n-1-d_1(u/(G))\\
&\quad + \sum_{uv\in E(\overline{S}(G))} (m-1)(n-1-d_1(u/(G))\\
&= M_2(G) - (n-1)M_1(G) + m[(n-1)^2 + (m-1)(n^2-n-2m)].
\end{aligned}$$

$$\begin{aligned}
LM_3(T^{+01}(G)) &= \sum_{v\in V(T^{+01}(G))} \left[d_1(v/T^{+01}(G))\right]\left[d_2(v/T^{+01}(G))\right]\\
&= \sum_{v\in V(G)} \left[d_1(v/T^{+01}(G))\right]\left[d_2(v/T^{+01}(G))\right]\\
&\quad + \sum_{e\in E(G)} \left[d_1(e/T^{+01}(G))\right]\left[d_2(e/T^{+01}(G))\right]\\
&= \sum_{v\in V(G)} [(m+d_1(v/G))(n-1-d_1(v/G))] + \sum_{e\in E(G)} n(m-1)\\
&= m[n(n+m) - 2(m+1)] - M_1(G). \qquad \square
\end{aligned}$$

Theorem 3.8 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{-01}(G)) = M_1(G) + m(m-1)^2$;

(2) $LM_2(T^{-01}(G)) = \overline{M_2}(G) + 2m^2(m-1)$;

(3) $LM_3(T^{-01}(G)) = m[n(m+1) + 2(m-1)] - M_1(G)$.

Proof We know the graph $T^{-01}(G)$ has $n+m$ vertices and $(n-1)(\frac{n}{2}+m)$ edges, refer Theorem 3.1. By definitions of the first, second and third leap Zagreb indices and applying Propositions 3.2, 3.3 and Observation 3.4 we have the following.

$$\begin{aligned} LM_1(T^{-01}(G)) &= \sum_{v\in V(T^{-01}(G))} d_2(v/T^{-01}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{-01}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{-01}(G))^2 \\ &= M_1(G) + m(m-1)^2. \end{aligned}$$

$$\begin{aligned} LM_2(T^{-01}(G)) &= \sum_{uv\in E(T^{-01}(G))} \left[d_2(u/T^{-01}(G))\right]\left[d_2(v/T^{-01}(G))\right] \\ &= \sum_{uv\notin E(G)} \left[d_2(u/T^{-01}(G))\right]\left[d_2(v/T^{-01}(G))\right] \\ &\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{-01}(G))\right]\left[d_2(v/T^{-01}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{-01}(G))\right]\left[d_2(v/T^{-01}(G))\right] \\ &= \sum_{uv\notin E(G)} [d_1(u/G)]\,[d_1(v/G)] + \sum_{uv\in E(S(G))} (m-1)d_1(u/G) \\ &\quad + \sum_{uv\in E(\overline{S}(G))} (m-1)d_1(u/G) \\ &= \overline{M_2}(G) + 2m^2(m-1). \end{aligned}$$

$$\begin{aligned} LM_3(T^{-01}(G)) &= \sum_{v\in V(T^{-01}(G))} \left[d_1(v/T^{-01}(G))\right]\left[d_2(v/T^{-01}(G))\right] \\ &= \sum_{v\in V(G)} \left[d_1(v/T^{-01}(G))\right]\left[d_2(v/T^{-01}(G))\right] \\ &\quad + \sum_{e\in E(G)} \left[d_1(e/T^{-01}(G))\right]\left[d_2(e/T^{-01}(G))\right] \\ &= m[n(m+1) + 2(m-1)] - M_1(G). \end{aligned}$$

□

Theorem 3.9 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{011}(G)) = n(n-1)^2$;

(2) $LM_2(T^{011}(G)) = 0$;

(3) $LM_3(T^{011}(G)) = mn(n-1)$.

Proof We are easily know that the graph $T^{011}(G)$ has $n+m$ vertices and $m(\frac{m-1}{2}+n)$

edges by Theorem 3.1. By definitions of the first, second and the third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4 we know the following.

$$\begin{aligned} LM_1(T^{011}(G)) &= \sum_{v\in V(T^{011}(G))} d_2(v/T^{011}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{011}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{011}(G))^2 \\ &= n(n-1)^2. \end{aligned}$$

$$\begin{aligned} LM_2(T^{011}(G)) &= \sum_{uv\in E(T^{011}(G))} \left[d_2(u/T^{011}(G))\right]\left[d_2(v/T^{011}(G))\right] \\ &= \sum_{uv\in E(L(G))} \left[d_2(u/T^{011}(G))\right]\left[d_2(v/T^{011}(G))\right] \\ &\quad + \sum_{uv\notin E(L(G))} \left[d_2(u/T^{011}(G))\right]\left[d_2(v/T^{011}(G))\right] \\ &\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{011}(G))\right]\left[d_2(v/T^{011}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{011}(G))\right]\left[d_2(v/T^{011}(G))\right] = 0. \end{aligned}$$

$$\begin{aligned} LM_3(T^{011}(G)) &= \sum_{v\in V(T^{011}(G))} \left[d_1(v/T^{011}(G))\right]\left[d_2(v/T^{011}(G))\right] \\ &= \sum_{v\in V(G)} \left[d_1(v/T^{011}(G))\right]\left[d_2(v/T^{011}(G))\right] \\ &\quad + \sum_{e\in E(G)} \left[d_1(e/T^{011}(G))\right]\left[d_2(e/T^{011}(G))\right] = mn(n-1). \end{aligned}$$ □

Theorem 3.10 *Let G be (n,m) graph. Then*

$$LM_1(T^{111}(G)) = LM_2(T^{111}(G)) = LM_3(T^{111}(G)) = 0.$$

Proof Notice that the graph $T^{111}(G)$ has $n+m$ vertices and $\frac{n(n-1)}{2} + \frac{m(m-1)}{2} + mn$ edges by Theorem 3.1. By definitions of the first, second and third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4, we get similarly the desired result as the proof of above theorems. □

Theorem 3.11 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{+11}(G)) = (n-1)(n^2-n-4m) + M_1(G)$;
(2) $LM_2(T^{+11}(G)) = m(n-1)^2 - (n-1)M_1(G) + M_2(G)$;
(3) $LM_3(T^{+11}(G)) = m[(n-1)(n+2) - 2m] - M_1(G)$.

Proof Clearly, the graph $T^{+11}(G)$ has $n+m$ vertices and $\frac{m(m+1)}{2} + mn$ edges by Theorem

3.1. By definitions of the first, second and the third leap Zagreb indices, we get the following by applying Propositions 3.2, 3.3 and Observation 3.4.

$$\begin{aligned} LM_1(T^{+11}(G)) &= \sum_{v\in V(T^{+11}(G))} d_2(v/T^{+11}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{+11}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{+11}(G))^2 \\ &= \sum_{v\in V(G)} [(n-1)^2 + d_1(v/G)^2 - 2(n-1)d_1(v/G)] \\ &= (n-1)(n^2-n-4m) + M_1(G). \end{aligned}$$

$$\begin{aligned} LM_2(T^{+11}(G)) &= \sum_{uv\in E(T^{+11}(G))} \left[d_2(u/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\ &= \sum_{uv\in E(G)} \left[d_2(u/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\ &\quad + \sum_{uv\in E(L(G))} \left[d_2(u/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\ &\quad + \sum_{uv\notin E(L(G))} \left[d_2(u/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\ &\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\ &= m(n-1)^2 - (n-1)M_1(G) + M_2(G). \end{aligned}$$

$$\begin{aligned} LM_3(T^{+11}(G)) &= \sum_{v\in V(T^{+11}(G))} \left[d_1(v/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\ &= \sum_{v\in V(G)} \left[d_1(v/T^{+11}(G))\right]\left[d_2(v/T^{+11}(G))\right] \\ &\quad + \sum_{e\in E(G)} \left[d_1(e/T^{+11}(G))\right]\left[d_2(e/T^{+11}(G))\right] \\ &= m[(n-1)(n+2) - 2m] - M_1(G). \end{aligned}$$ □

Theorem 3.12 *Let G be (n, m) graph. Then*

(1) $LM_1(T^{-11}(G)) = M_1(G)$;

(2) $LM_2(T^{-11}(G)) = \overline{M_2}(G)$;

(3) $LM_3(T^{-11}(G)) = 2m(n + m - 1) - M_1(G)$.

Proof Obviously, the graph $T^{-11}(G)$ has $n+m$ vertices and $\frac{n(n-1)}{2} + \frac{m(m-3)}{2} + mn$ edges, refer Theorem 3.1. Similarly, by definitions of the first, second and the third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4 we know the following.

$$\begin{aligned}
LM_1(T^{-11}(G)) &= \sum_{v\in V(T^{-11}(G))} d_2(v/T^{-11}(G))^2 \\
&= \sum_{v\in V(G)} d_2(v/T^{-11}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{-11}(G))^2 \\
&= M_1(G).
\end{aligned}$$

$$\begin{aligned}
LM_2(T^{-11}(G)) &= \sum_{uv\in E(T^{-11}(G))} \left[d_2(u/T^{-11}(G))\right]\left[d_2(v/T^{-11}(G))\right] \\
&= \sum_{uv\notin E(G)} \left[d_2(u/T^{-11}(G))\right]\left[d_2(v/T^{-11}(G))\right] \\
&\quad + \sum_{uv\in E(L(G))} \left[d_2(u/T^{-11}(G))\right]\left[d_2(v/T^{-11}(G))\right] \\
&\quad + \sum_{uv\notin E(L(G))} \left[d_2(u/T^{-11}(G))\right]\left[d_2(v/T^{-11}(G))\right] \\
&\quad + \sum_{uv\in E(S(G))} [d_2(u/T^{-11}(G))][d_2(v/T^{-11}(G))] \\
&\quad + \sum_{uv\in E(\overline{S}(G))} [d_2(u/T^{-11}(G))][d_2(v/T^{-11}(G))] \\
&= \overline{M_2}(G).
\end{aligned}$$

$$\begin{aligned}
LM_3(T^{-11}(G)) &= \sum_{v\in V(T^{-11}(G))} \left[d_1(v/T^{-11}(G))\right]\left[d_2(v/T^{-11}(G))\right] \\
&= \sum_{v\in V(G)} \left[d_1(v/T^{-11}(G))\right]\left[d_2(v/T^{-11}(G))\right] \\
&\quad + \sum_{e\in E(G)} \left[d_1(e/T^{-11}(G))\right]\left[d_2(e/T^{-11}(G))\right] \\
&= 2m(n+m-1) - M_1(G).
\end{aligned}$$

□

Theorem 3.13 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{0+1}(G)) = n(n-1)^2 + m(m-1)(m+3) - 2(m-1)M_1(G) + EM_1(G)$;

(2) $LM_2(T^{0+1}(G)) = [\frac{(m-1)^2}{2} - n(n-1)]M_1(G) - (m-1)EM_1(G) + EM_2(G)$
$+ m(m-1)[n(n-1) - (m-1)] + 2mn(n-1)$;

(3) $LM_3(T^{0+1}(G)) = (m+n-1)M_1(G) - EM_1(G) + m[n(n+m) - 2(m-1)]$.

Proof Notice that the graph $T^{0+1}(G)$ has $n+m$ vertices and $m(n-1) + \frac{M_1(G)}{2}$ edges by Theorem 3.1. By definitions of the first, second and the third leap Zagreb indices we get the following by applying Propositions 3.2, 3.3 and Observation 3.4.

$$\begin{aligned}
LM_1(T^{0+1}(G)) &= \sum_{v\in V(T^{0+1}(G))} d_2(v/T^{0+1}(G))^2 \\
&= \sum_{v\in V(G)} d_2(v/T^{0+1}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{0+1}(G))^2 \\
&= \sum_{v\in V(G)} (n-1)^2 + \sum_{e\in E(G)} [(m-1)^2 + d_1(e/G)^2 - 2(m-1)d_1(e/G)] \\
&= n(n-1)^2 + m(m-1)(m+3) - 2(m-1)M_1(G) + EM_1(G).
\end{aligned}$$

$$\begin{aligned}
LM_2(T^{0+1}(G)) &= \sum_{uv\in E(T^{0+1}(G))} \left[d_2(u/T^{0+1}(G))\right]\left[d_2(v/T^{0+1}(G))\right] \\
&= \sum_{uv\in E(L(G))} \left[d_2(u/T^{0+1}(G))\right]\left[d_2(v/T^{0+1}(G))\right] \\
&\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{0+1}(G))\right]\left[d_2(v/T^{0+1}(G))\right] \\
&\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{0+1}(G))\right]\left[d_2(v/T^{0+1}(G))\right] \\
&= \sum_{uv\in E(L(G))} \left[(m-1)^2 - (m-1)(d_1(u/G)+d_1(v/G)) + d_1(u/G)\cdot d_1(v/G)\right] \\
&\quad + \sum_{uv\in E(S(G))} [(n-1)(m-1-d_1(v/G))] \\
&\quad + \sum_{uv\in E(\overline{S}(G))} [(n-1)(m-1-d_1(v/G))] \\
&= [\frac{(m-1)^2}{2} - n(n-1)]M_1(G) - (m-1)EM_1(G) + EM_2(G) \\
&\quad + m(m-1)[n(n-1) - (m-1)] + 2mn(n-1).
\end{aligned}$$

$$\begin{aligned}
LM_3(T^{0+1}(G)) &= \sum_{v\in V(T^{0+1}(G))} \left[d_1(v/T^{0+1}(G))\right]\left[d_2(v/T^{0+1}(G))\right] \\
&= \sum_{v\in V(G)} \left[d_1(v/T^{0+1}(G))\right]\left[d_2(v/T^{0+1}(G))\right] \\
&\quad + \sum_{e\in E(G)} \left[d_1(e/T^{0+1}(G))\right]\left[d_2(e/T^{0+1}(G))\right] \\
&= \sum_{v\in V(G)} [m(n-1)] + \sum_{e\in E(G)} [(n+d_1(e/G))(m-1-d_1(e/G))] \\
&= (m+n-1)M_1(G) - EM_1(G) + m[n(n+m) - 2(m-1)].
\end{aligned}$$

□

Theorem 3.14 *Let G be (n, m) graph. Then*

(1) $LM_1(T^{1+1}(G)) = m(m-1)(m+3) - 2(m-1)M_1(G) + EM_1(G)$;
(2) $LM_2(T^{1+1}(G)) = \frac{(m-1)^2}{2}M_1(G) - (m-1)EM_1(G) + EM_2(G) - m(m-1)^2$;

(3) $LM_3(T^{1+1}(G)) = (m-n-1)M_1(G) - EM_1(G) + m[n(m+1) - 2(m-1)]$.

Proof Clearly, the graph $T^{1+1}(G)$ has $n+m$ vertices and $(n-1)(\frac{n}{2}+m)+\frac{M_1(G)}{2}$ edges by Theorem 3.1. By definitions of the first, second and the third leap Zagreb indices we therefore get the following by Propositions 3.2, 3.3 and Observation 3.4.

$$\begin{aligned} LM_1(T^{1+1}(G)) &= \sum_{v\in V(T^{1+1}(G))} d_2(v/T^{1+1}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{1+1}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{1+1}(G))^2 \\ &= \sum_{e\in E(G)} \left[(m-1)^2 + d_1(e/G)^2 - 2(m-1)d_1(e/G)\right] \\ &= m(m-1)(m+3) - 2(m-1)M_1(G) + EM_1(G). \end{aligned}$$

$$\begin{aligned} LM_2(T^{1+1}(G)) &= \sum_{uv\in E(T^{1+1}(G))} \left[d_2(u/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\ &= \sum_{uv\in E(G)} \left[d_2(u/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\ &\quad + \sum_{uv\notin E(G)} \left[d_2(u/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\ &\quad + \sum_{uv\in E(L(G))} \left[d_2(u/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\ &\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\ &= \sum_{uv\in E(L(G))} \left[(m-1)^2 - (m-1)(d_1(u/G)+d_1(v/G)) + d_1(u/G)\cdot d_1(v/G)\right] \\ &= \frac{(m-1)^2}{2}M_1(G) - (m-1)EM_1(G) + EM_2(G) - m(m-1)^2. \end{aligned}$$

$$\begin{aligned} LM_3(T^{1+1}(G)) &= \sum_{v\in V(T^{1+1}(G))} \left[d_1(v/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\ &= \sum_{v\in V(G)} \left[d_1(v/T^{1+1}(G))\right]\left[d_2(v/T^{1+1}(G))\right] \\ &\quad + \sum_{e\in E(G)} \left[d_1(e/T^{1+1}(G))\right]\left[d_2(e/T^{1+1}(G))\right] \\ &= (m-n-1)M_1(G) - EM_1(G) + m[n(m+1) - 2(m-1)]. \qquad \square \end{aligned}$$

Theorem 3.15 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{++1}(G)) = (n-1)[n(n-1) - 4m] + m(m-1)(m+3)$

$$- (2m-3)M_1(G) + EM_1(G);$$

(2) $$LM_2(T^{++1}(G)) = \Big[\tfrac{(m-1)^2}{2} - (n-1)(n+1)\Big)M_1(G) + M_2(G) - (m-1)EM_1(G) \\ +EM_2(G) + m[n(2n-3) - m(3m-4) + mn(n-1)] \\ + \sum_{u\in V(G), v\in E(G), u\sim v} d_2(u/G)d_2(v/G) \\ + \sum_{u\in V(G), v\in E(G), u\nsim v} d_2(u/G)d_2(v/G);$$

(3) $LM_3(T^{++1}(G)) = mn(m+n-2) + (m-n-2)M_1(G) - EM_1(G)$.

Proof Clearly, the graph $T^{++1}(G)$ has $n+m$ vertices and $mn+\frac{M_1(G)}{2}$ edges by Theorem 3.1. Now by definitions of the first, second and the third leap Zagreb indices, applying Propositions $3.2, 3.3$ and Observation 3.4 we have the following.

$$\begin{aligned} LM_1(T^{++1}(G)) &= \sum_{v\in V(T^{++1}(G))} d_2(v/T^{++1}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{++1}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{++1}(G))^2 \\ &= \sum_{v\in V(G)} [n-1-d_1(v/G)]^2 + \sum_{e\in E(G)} [m-1-d_1(e/G)]^2 \\ &= (n-1)[n(n-1)-4m] + m(m-1)(m+3) - (2m-3)M_1(G) \\ &\quad +EM_1(G). \end{aligned}$$

$$\begin{aligned} LM_2(T^{++1}(G)) &= \sum_{uv\in E(T^{++1}(G))} \left[d_2(u/T^{++1}(G))\right]\left[d_2(v/T^{++1}(G))\right] \\ &= \sum_{uv\in E(G)} \left[d_2(u/T^{++1}(G))\right]\left[d_2(v/T^{++1}(G))\right] \\ &\quad + \sum_{uv\in E(L(G))} \left[d_2(u/T^{++1}(G))\right]\left[d_2(v/T^{++1}(G))\right] \\ &\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{++1}(G))\right]\left[d_2(v/T^{++1}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{++1}(G))\right]\left[d_2(v/T^{++1}(G))\right] \\ &= \left(\frac{(m-1)^2}{2} - (n-1)(n+1)\right) M_1(G) + M_2(G) - (m-1)EM_1(G) \\ &\quad +EM_2(G) + m[n(2n-3) - m(3m-4) + mn(n-1)] \\ &\quad + \sum_{u\in V(G), v\in E(G), u\sim v} d_2(u/G)d_2(v/G) \\ &\quad + \sum_{u\in V(G), v\in E(G), u\nsim v} d_2(u/G)d_2(v/G). \end{aligned}$$

$$
\begin{aligned}
LM_3(T^{++1}(G)) &= \sum_{v\in V(T^{++1}(G))} \left[d_1(v/T^{++1}(G))\right]\left[d_2(v/T^{++1}(G))\right] \\
&= \sum_{v\in V(G)} \left[d_1(v/T^{++1}(G))\right]\left[d_2(v/T^{++1}(G))\right] \\
&\quad + \sum_{e\in E(G)} \left[d_1(e/T^{++1}(G))\right]\left[d_2(e/T^{++1}(G))\right] \\
&= mn(m+n-2) + (m-n-2)M_1(G) - EM_1(G). \qquad \square
\end{aligned}
$$

Theorem 3.16 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{-+1}(G)) = m(m-1)(m+3) - (2m-3)M_1(G) + EM_1(G)$;

(2) $LM_2(T^{-+1}(G)) = \frac{(m-1)^2}{2}M_1(G) + \overline{M_2}(G) - (m-1)EM_1(G) + EM_2(G) + m(m-1)(m+1)$
$-\left[\sum_{u\in V(G),v\in E(G),u\sim v} d_2(u/G)d_2(v/G) + \sum_{u\in V(G),v\in E(G),u\nsim v} d_2(u/G)d_2(v/G)\right]$;

(3) $LM_3(T^{-+1}(G)) = (m-n-2)M_1(G) - EM_1(G) + mn(m+3)$.

Proof Notice that the graph $T^{-+1}(G)$ has $n+m$ vertices and $\frac{n(n-1)}{2} + m(n-2) + \frac{M_1(G)}{2}$ edges, refer Theorem 3.1. We are easily get the following by definitions of the first, second and the third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4.

$$
\begin{aligned}
LM_1(T^{-+1}(G)) &= \sum_{v\in V(T^{-+1}(G))} d_2(v/T^{-+1}(G))^2 \\
&= \sum_{v\in V(G)} d_2(v/T^{-+1}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{-+1}(G))^2 \\
&= m(m-1)(m+3) - (2m-3)M_1(G) + EM_1(G).
\end{aligned}
$$

$$
\begin{aligned}
LM_2(T^{-+1}(G)) &= \sum_{uv\in E(T^{-+1}(G))} \left[d_2(u/T^{-+1}(G))\right]\left[d_2(v/T^{-+1}(G))\right] \\
&= \sum_{uv\notin E(G)} \left[d_2(u/T^{-+1}(G))\right]\left[d_2(v/T^{-+1}(G))\right] \\
&\quad + \sum_{uv\in E(L(G))} \left[d_2(u/T^{-+1}(G))\right]\left[d_2(v/T^{-+1}(G))\right] \\
&\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{-+1}(G))\right]\left[d_2(v/T^{-+1}(G))\right] \\
&\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{-+1}(G))\right]\left[d_2(v/T^{-+1}(G))\right] \\
&= \frac{(m-1)^2}{2}M_1(G) + \overline{M_2}(G) - (m-1)EM_1(G) \\
&\quad + EM_2(G) + m(m-1)(m+1) \\
&\quad - \left[\sum_{u\in V(G),v\in E(G),u\sim v} d_2(u/G)d_2(v/G) + \sum_{u\in V(G),v\in E(G),u\nsim v} d_2(u/G)d_2(v/G)\right].
\end{aligned}
$$

$$\begin{aligned}
LM_3(T^{-+1}(G)) &= \sum_{v\in V(T^{-+1}(G))} \left[d_1(v/T^{-+1}(G))\right]\left[d_2(v/T^{-+1}(G))\right] \\
&= \sum_{v\in V(G)} \left[d_1(v/T^{-+1}(G))\right]\left[d_2(v/T^{-+1}(G))\right] \\
&\quad + \sum_{e\in E(G)} \left[d_1(e/T^{-+1}(G))\right]\left[d_2(e/T^{-+1}(G))\right] \\
&= (m-n-2)M_1(G) - EM_1(G) + mn(m+3). \qquad \Box
\end{aligned}$$

Theorem 3.17 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{0-1}(G)) = n(n-1)^2 + EM_1(G)$;

(2) $LM_2(T^{0-1}(G)) = \overline{EM_2}(G) + n(n-1)M_1(G) - 2mn(n-1)$;

(3) $LM_3(T^{0-1}(G)) = (n+m-1)M_1(G) - EM_1(G) + m(n^2-3n-2m+2)$.

Proof Notice that the graph $T^{0-1}(G)$ has $n+m$ vertices and $m(\frac{m+1}{2}+n) - \frac{M_1(G)}{2}$ edges, refer Theorem 3.1. By definitions of the first, second and the third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4 we get the following.

$$\begin{aligned}
LM_1(T^{0-1}(G)) &= \sum_{v\in V(T^{0-1}(G))} d_2(v/T^{0-1}(G))^2 \\
&= \sum_{v\in V(G)} d_2(v/T^{0-1}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{0-1}(G))^2 \\
&= n(n-1)^2 + EM_1(G).
\end{aligned}$$

$$\begin{aligned}
LM_2(T^{0-1}(G)) &= \sum_{uv\in E(T^{0-1}(G))} \left[d_2(u/T^{0-1}(G))\right]\left[d_2(v/T^{0-1}(G))\right] \\
&= \sum_{uv\notin E(L(G))} \left[d_2(u/T^{0-1}(G))\right]\left[d_2(v/T^{0-1}(G))\right] \\
&\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{0-1}(G))\right]\left[d_2(v/T^{0-1}(G))\right] \\
&\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{0-1}(G))\right]\left[d_2(v/T^{0-1}(G))\right] \\
&= \sum_{uv\notin E(L(G))} \left[d_1(u/G)\cdot d_1(v/G)\right] + \sum_{uv\in E(S(G))} (n-1)d_1(v/G) \\
&\quad + \sum_{uv\in E(\overline{S}(G))} (n-1)d_1(v/G) \\
&= \overline{EM_2}(G) + n(n-1)M_1(G) - 2mn(n-1).
\end{aligned}$$

$$
\begin{aligned}
LM_3(T^{0-1}(G)) &= \sum_{v\in V(T^{0-1}(G))} \left[d_1(v/T^{0-1}(G))\right]\left[d_2(v/T^{0-1}(G))\right] \\
&= \sum_{v\in V(G)} \left[d_1(v/T^{0-1}(G))\right]\left[d_2(v/T^{0-1}(G))\right] \\
&\quad + \sum_{e\in E(G)} \left[d_1(e/T^{0-1}(G))\right]\left[d_2(e/T^{0-1}(G))\right] \\
&= (n+m-1)M_1(G) - EM_1(G) + m(n^2-3n-2m+2). \qquad \square
\end{aligned}
$$

Theorem 3.18 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{1-1}(G)) = EM_1(G)$;

(2) $LM_2(T^{1-1}(G)) = \overline{EM_2}(G)$;

(3) $LM_3(T^{1-1}(G)) = (n+m-1)M_1(G) - EM_1(G) - 2m(n+m-1)$.

Proof Clearly, the graph $T^{1-1}(G)$ has $n+m$ vertices and $\frac{n(n-1)}{2} + m(\frac{m+1}{2}+n) - \frac{M_1(G)}{2}$ edges by Theorem 3.1. Whence, by definitions of the first, second and the third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4 we get the following.

$$
\begin{aligned}
LM_1(T^{1-1}(G)) &= \sum_{v\in V(T^{1-1}(G))} d_2(v/T^{1-1}(G))^2 \\
&= \sum_{v\in V(G)} d_2(v/T^{1-1}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{1-1}(G))^2 \\
&= EM_1(G).
\end{aligned}
$$

$$
\begin{aligned}
LM_2(T^{1-1}(G)) &= \sum_{uv\in E(T^{1-1}(G))} \left[d_2(u/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\
&= \sum_{uv\in E(G)} \left[d_2(u/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\
&\quad + \sum_{uv\notin E(G)} \left[d_2(u/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\
&\quad + \sum_{uv\notin E(L(G))} \left[d_2(u/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\
&\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\
&\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\
&= \overline{EM_2}(G).
\end{aligned}
$$

$$\begin{aligned} LM_3(T^{1-1}(G)) &= \sum_{v\in V(T^{1-1}(G))} \left[d_1(v/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\ &= \sum_{v\in V(G)} \left[d_1(v/T^{1-1}(G))\right]\left[d_2(v/T^{1-1}(G))\right] \\ &\quad + \sum_{e\in E(G)} \left[d_1(e/T^{1-1}(G))\right]\left[d_2(e/T^{1-1}(G))\right] \\ &= (n+m-1)M_1(G) - EM_1(G) - 2m(n+m-1). \end{aligned}$$ □

Theorem 3.19 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{+-1}(G)) = M_1(G) + EM_1(G) + (n-1)[n(n-1)-4m]$;

(2) $LM_2(T^{+-1}(G)) = (n-1)^2M_1(G) + M_2(G) + \overline{EM_2}(G) - m(n-1)(n+1)$
$-\left[\sum_{u\in V(G),v\in E(G),u\sim v} d_2(u/G)d_2(v/G) + \sum_{u\in V(G),v\in E(G),u\nsim v} d_2(u/G)d_2(v/G)\right]$;

(3) $LM_3(T^{+-1}(G)) = (n+m-2)M_1(G) - EM_1(G) + m[(n-1)(n+2) - 2(2m+n-1)]$.

Proof Clearly, the graph $T^{+-1}(G)$ has $n+m$ vertices and $m(\frac{m+3}{2}+n) - \frac{M_1(G)}{2}$edges by Theorem 3.1. By definitions of the first, second and the third leap Zagreb indices along with Propositions 3.2, 3.3 and Observation 3.4 we therefore get the following.

$$\begin{aligned} LM_1(T^{+-1}(G)) &= \sum_{v\in V(T^{+-1}(G))} d_2(v/T^{+-1}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{+-1}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{+-1}(G))^2 \\ &= M_1(G) + EM_1(G) + (n-1)[n(n-1)-4m]. \end{aligned}$$

$$\begin{aligned} LM_2(T^{+-1}(G)) &= \sum_{uv\in E(T^{+-1}(G))} \left[d_2(u/T^{+-1}(G))\right]\left[d_2(v/T^{+-1}(G))\right] \\ &= \sum_{uv\in E(G)} \left[d_2(u/T^{+-1}(G))\right]\left[d_2(v/T^{+-1}(G))\right] \\ &\quad + \sum_{uv\notin E(L(G))} \left[d_2(u/T^{+-1}(G))\right]\left[d_2(v/T^{+-1}(G))\right] \\ &\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{+-1}(G))\right]\left[d_2(v/T^{+-1}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{+-1}(G))\right]\left[d_2(v/T^{+-1}(G))\right] \\ &= (n-1)^2M_1(G) + M_2(G) + \overline{EM_2}(G) - m(n-1)(n+1) \\ &\quad - \left[\sum_{u\in V(G),v\in E(G),u\sim v} d_2(u/G)d_2(v/G) + \sum_{u\in V(G),v\in E(G),u\nsim v} d_2(u/G)d_2(v/G)\right]. \end{aligned}$$

$$\begin{aligned} LM_3(T^{+-1}(G)) &= \sum_{v\in V(T^{+-1}(G))} \left[d_1(v/T^{+-1}(G))\right]\left[d_2(v/T^{+-1}(G))\right] \\ &= \sum_{v\in V(G)} \left[d_1(v/T^{+-1}(G))\right]\left[d_2(v/T^{+-1}(G))\right] \\ &\quad + \sum_{e\in E(G)} \left[d_1(e/T^{+-1}(G))\right]\left[d_2(e/T^{+-1}(G))\right] \\ &= (n+m-2)M_1(G) - EM_1(G) + m[(n-1)(n+2) - 2(2m+n-1)]. \quad \square \end{aligned}$$

Theorem 3.20 *Let G be (n,m) graph. Then*

(1) $LM_1(T^{--1}(G)) = M_1(G) + EM_1(G)$;

(2) $LM_2(T^{--1}(G)) = \overline{M_1}(G) + \overline{EM_2}(G) + \sum_{u\in V(G), v\in E(G), u\sim v} d_2(u/G)d_2(v/G) + \sum_{u\in V(G), v\in E(G), u\nsim v} d_2(u/G)d_2(v/G)$;

(3) $LM_3(T^{--1}(G)) = (n+m-2)M_1(G) - EM_1(G)$.

Proof Notice that the graph $T^{--1}(G)$ has $n+m$ vertices and $\frac{n(n-1)}{2} + m(\frac{m-1}{2}+n) - \frac{M_1(G)}{2}$ edges by Theorem 3.1. By definitions of the first, second and the third leap Zagreb indices, Propositions 3.2, 3.3 and Observation 3.4, we are easily get the following.

$$\begin{aligned} LM_1(T^{--1}(G)) &= \sum_{v\in V(T^{--1}(G))} d_2(v/T^{--1}(G))^2 \\ &= \sum_{v\in V(G)} d_2(v/T^{--1}(G))^2 + \sum_{e\in E(G)} d_2(e/T^{--1}(G))^2 \\ &= M_1(G) + EM_1(G). \end{aligned}$$

$$\begin{aligned} LM_2(T^{--1}(G)) &= \sum_{uv\in E(T^{--1}(G))} \left[d_2(u/T^{--1}(G))\right]\left[d_2(v/T^{--1}(G))\right] \\ &= \sum_{uv\notin E(G)} \left[d_2(u/T^{--1}(G))\right]\left[d_2(v/T^{--1}(G))\right] \\ &\quad + \sum_{uv\notin E(L(G))} \left[d_2(u/T^{--1}(G))\right]\left[d_2(v/T^{--1}(G))\right] \\ &\quad + \sum_{uv\in E(S(G))} \left[d_2(u/T^{--1}(G))\right]\left[d_2(v/T^{--1}(G))\right] \\ &\quad + \sum_{uv\in E(\overline{S}(G))} \left[d_2(u/T^{--1}(G))\right]\left[d_2(v/T^{--1}(G))\right] \\ &= \overline{M_1}(G) + \overline{EM_2}(G) + \sum_{u\in V(G), v\in E(G), u\sim v} d_2(u/G)d_2(v/G) \\ &\quad + \sum_{u\in V(G), v\in E(G), u\nsim v} d_2(u/G)d_2(v/G). \end{aligned}$$

$$
\begin{aligned}
LM_3(T^{--1}(G)) &= \sum_{v \in V(T^{--1}(G))} \left[d_1(v/T^{--1}(G))\right]\left[d_2(v/T^{--1}(G))\right] \\
&= \sum_{v \in V(G)} \left[d_1(v/T^{--1}(G))\right]\left[d_2(v/T^{--1}(G))\right] \\
&\quad + \sum_{e \in E(G)} \left[d_1(e/T^{--1}(G))\right]\left[d_2(e/T^{--1}(G))\right] \\
&= (n+m-2)M_1(G) - EM_1(G). \qquad \square
\end{aligned}
$$

References

[1] A. R. Ashrafi, T. Došlić, A. Hamzeh, The Zagreb coindices of graph operations, *Discrete Appl. Math.*, 158(15), 1571–1578, 2010.

[2] W. Baoyindureng, M. Jixiang, Basic properties of total transformation graphs, *J. Math. Study*, 34(2), 109–116, 2001.

[3] B. Basavanagoud, *Basic properties of generalized xyz-Point-Line transformation graphs*, J. Inf. Optim. Sci., 39(2), 561–580, 2018, DOI: 10.1080/02522667.2017.1395147.

[4] B. Basavanagoud, C. S. Gali, *Computing first and second Zagreb indices of generalized xyz-Point-Line transformation graphs*, J. Global Research Math. Arch., 5(4), 100–122, 2018.

[5] C. M. Da fonseca, D. Stevanović, Further properties of the second Zagreb index, *MATCH Commun. Math. Comput. Chem.*, 72, 655–668, 2014.

[6] A. Deng, A. Kelmans, J. Meng, Laplacian Spectra of regular graph transformations, *Discrete Appl. Math.*, 161, 118–133, 2013.

[7] B. Furtula, I. Gutman, M. Dehmer, On structure-sensitivity of degree-based topological indices, *Appl. Math. Comput.*, 219, 8973–8978, 2013.

[8] M. Goubko, T. Réti, Note on minimizing degree-based topological indices of trees with given number of pendent vertices, *MATCH Commun. Math. Comput. Chem.*, 72, 633–639, 2014.

[9] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals, Total π-electron energy of alternate hydrocarbons, *Chem. Phys. Lett.*, 17, 535–538, 1972.

[10] I. Gutman, K. C. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.*, 50, 83–92, 2004.

[11] F. Harary, *Graph Theory*, Addison-Wesely, Reading Mass, 1969.

[12] S. M. Hosamani, N. Trinajstić, On reformulated Zagreb coindices, *Research Gate*, 2015-05-08 T 09:07:00 UTC.

[13] V. R. Kulli, *College Graph Theory*, Vishwa International Publications, Gulbarga, India, 2012.

[14] A. Milićević, S. Nikolić, N. Trinajstić, On reformulated Zagreb indices, *Mol. Divers.*, 8(4), 393–399, 2004.

[15] S. Nikolić, G. Kovačević, A. Milićević, N. Trinajstić, The Zagreb indices 30 years after, *Croat. Chem. Acta.*, 76(2), 113–124, 2003.

[16] A. M. Naji, N. D. Soner, Ivan Gutman, On Leap Zagreb indices of graphs, *Commun. Comb. Optim.*, 2(2), 99–117, 2017.

[17] N. D. Soner, A. M. Naji, The k-distance neighbourhood polynomial of a graph, *Int. J. Math. Comput. Sci. WASET Conference Proceedings, San Francico, USA, Sep 26-27,* 3(9) Part XV, 2359–2364, 2016.

[18] G. Su, L. Xiong, L. Xu, The Nordhaus-Gaddum-type inequalities for the Zagreb index and coindex of graphs, *Appl. Math. Lett.*, 25(11), 1701–1707, 2012.