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Linear Cyclic Snakes as Super Vertex Mean Graphs

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Abstract: A super vertex mean labeling f of a (p,q) - graph $G(V,E)$ is defined as an injection from E to the set $\{1,2,3,\cdots,p+q\}$ that induces for each vertex v the label defined by the rule $f^v(v) = \text{Round}\left(\frac{\sum_{e\in E_v} f(e)}{d(v)}\right)$, where E_v denotes the set of edges in G that are incident at the vertex v, such that the set of all edge label and the induced vertex labels is $\{1,2,3,\cdots,p+q\}$. All the cycles, C_n, $n \geq 3$ and $n \neq 4$ are super vertex mean graphs. Our attempt in this paper is to show that all the linear cyclic snakes, including kC_4, are also super vertex mean graphs, even though C_4 is not an SVM graph. We also define the term Super Vertex Mean number of graphs.

Key Words: Super vertex mean label, Smarandachely super H-vertex mean labeling, linear cyclic snakes, SVM number.

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§1. Introduction

The graphs considered here will be finite, undirected and simple. The symbols V(G) and E(G) will denote the vertex set and edge set of a graph G respectively. p and q denote the number of vertices and edges of G respectively. A graph of order p and size q is often called a (p,q) - graph.

A labeling of a graph G is an assignment of labels either to the vertices or edges. There are varieties of vertex as well as edge labeling that are already in the literature. Mean labeling was introduced by Somasundaram and Ponraj [14]. Super mean labeling was introduced by D.Ramya et al.[10]. Some results on mean labeling and super mean labeling are given in [5, 6, 9, 10, 11, 13, 14] and [15]. Lourdusamy and Seenivasan [5] introduced vertex mean labeling as an edge analogue of mean labeling.

Continuing on the same line and inspired by the above mentioned concepts, Lourdusamy et al. [7] brought in a new extension of mean labeling, called Super vertex mean labeling of graphs.

§2. Super Vertex Mean Labeling

Definition 2.1 *A super vertex mean labeling f of a (p,q) - graph $G(V,E)$ is defined as an*

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injection from E to the set $\{1, 2, 3, \cdots, p+q\}$ that induces for each vertex v the label defined by the rule $f^v(v) = Round\left(\frac{\sum_{e \in E_v} f(e)}{d(v)}\right)$, where E_v denotes the set of edges in G that are incident at the vertex v, such that the set of all edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, p+q\}$.

Furthermore, a super vertex mean labeling f on G is Smarandachely super H-vertex mean labeling if the induced vertex labels on vertex in H' is $\{1, 2, 3, \cdots, p(H') + q(H')\}$ for each subgraph $H' \cong H$ in G, where $p(H')$ and $q(H')$ are respectively the order and the size of H'.

A graph that accepts super vertex mean labeling is called a super vertex mean (hereafter, SVM) graph. The following results have already been proved in [7] and [8]. We use them for our further study on the super vertex mean behavior of linear cyclic snakes. Before entering into the results, we define the term cyclic snakes.

Definition 2.2 *A kC_n-snake has been defined as a connected graph in which all the blocks are isomorphic to the cycle C_n and the block-cut point graph is a path P, where P is the path of minimum length that contains all the cut vertices of a kC_n-snake. Barrientos [13] has proved that any kC_n-snake is represented by a string $s_1, s_2, s_3, \cdots, s_{k-2}$ of integers of length $k-2$, where the i^{th} integer, s_i on the string is the distance between i^{th} and $i+1^{th}$ cut vertices along the path, P, from one extreme and is taken from $S_n = \{1, 2, 3, \cdots, \lfloor \frac{n}{2} \rfloor\}$. A kC_n-snake is said to be linear if each integer s_i of its string is equal to $\lfloor \frac{n}{2} \rfloor$.*

Remark 2.3 The strings obtained from both the extremes are assumed to be the same. A linear cyclic snake, kC_n is obtained from k copies of C_n by identifying the vertex $v_{i,r+1}$ in the i^{th} copy of C_n at a vertex $v_{i+1,1}$ in the $(i+1)^{th}$ copy of C_n, where $1 \leq i \leq k-1$ and $n = 2r$ or $n = 2r+1$, depending upon whether n is even or odd respectively.

Note 2.4 that $v_{i,r+1} = v_{i+1,1}$ for $1 \leq i \leq k-1$, and we consider this vertex as $v_{i+1,1}$ throughout this paper.

§3. Known Results

We have known results listed in the following.

(1) *All the cycles except C_4 are SVM graphs*([8]);

(2) *All odd cycles, C_n can be SVM labeled as many as $\lfloor \frac{n}{2} \rfloor$ ways and every even cycle, C_n, except C_4 can have $\lfloor \frac{n}{2} \rfloor - 1$ types of SVM labelings*([8]);

(3) *A linear triangular snake, kC_3 with k blocks is an SVM graph*([7]).

§4. Linear Cyclic Snakes of Higher Orders

We proceed to prove that other linear cyclic snakes too are super vertex mean graphs.

Theorem 4.1 *Linear quadrilateral snakes, kC_4 with $k \geq 2$ blocks are SVM graphs.*

Proof Let kC_4 be a linear quadrilateral snake with p vertices and q edges. Then $p = 3k+1$ and $q = 4k$. Define $f : E(kC_4) \rightarrow \{1, 2, 3, \cdots, 7k+1\}$ to be $f(u_i u_{i+1}) = 7i$ if $1 \leq i \leq k-1$ and $7k+1$ $i = k$. When $1 \leq i \leq k-1$ and $k \geq 2$,

$$f(e_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j = 1 \\ 7i-5, & \text{if } 2 \leq i \leq k-1, \text{ and } j = 1 \\ 7i-1, & \text{if } 1 \leq i \leq k-1, \text{ and } j = 2 \\ 7i, & \text{if } 1 \leq i \leq k-1, \text{ and } j = 3 \\ 7i-4, & \text{if } 1 \leq i \leq k-1, \text{ and } j = 4. \end{cases}$$

When $i = k$, $k \geq 2$ and k is even,

$$f(e_{i,j}) = \begin{cases} 7k-6, & \text{if } j = 1, \\ 7k-3, & \text{if } j = 2, \\ 7k-1, & \text{if } j = 3, \\ 7k+1, & \text{if } j = 4. \end{cases}$$

When $i = k$, $k \geq 3$ and k is odd,

$$f(e_{i,j}) = \begin{cases} 7k-5, & \text{if } j = 1, \\ 7k-2, & \text{if } j = 2, \\ 7k+1, & \text{if } j = 3, \\ 7k-4, & \text{if } j = 4. \end{cases}$$

It can be easily verified that f is injective. The induced vertex labels are as follows:

When $1 \leq i \leq k-1$ and $k \geq 2$,

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 7i-6, & \text{if } 2 \leq i \leq k-1, \text{ and } j = 1, \\ 7i-3, & \text{if } 1 \leq i \leq k-1, \text{ and } j = 2, \\ 7i-2, & \text{if } 1 \leq i \leq k-1, \text{ and } j = 4. \end{cases}$$

When $i = k$, $k \geq 2$, and k is even,

$$f^v(v_{i,j}) = \begin{cases} 7k-5, & \text{if } j = 1, \\ 7k-4, & \text{if } j = 2, \\ 7k-2, & \text{if } j = 3, \\ 7k, & \text{if } j = 4. \end{cases}$$

When $i = k$, $k \geq 3$, and k is odd,

$$f^v(v_{i,j}) = \begin{cases} 7k-6, & \text{if } j=1, \\ 7k-3, & \text{if } j=2, \\ 7k, & \text{if } j=3, \\ 7k-1, & \text{if } j=4. \end{cases}$$

It can be easily verified that the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., 7k+1\}$ in two following cases:

Case 1. When k is even,

$$\begin{aligned} f(E) &= \{1, 6, 7, 3, 9, 13, 14, 10, 16, 20, 21, 27, \cdots, 7k-12, \\ &\quad 7k-8, 7k-7, 7k-11, 7k-6, 7k-3, 7k-1, 7k+1\} \end{aligned}$$

and,

$$f^v(V) = \{2, 4, 8, 5, 11, 15, 12, 18, 22, 19, \cdots, 7k-10, 7k-5, 7k-9, 7k-4, 7k-2, 7k\}.$$

Therefore,

$$\begin{aligned} f(E) \cup f^v(V) &= \{1, 2, 3, 4, \cdots, 7k-12, 7k-11, 7k-10, 7k-9, 7k-8, \\ &\quad 7k-7, 7k-6, 7k-5, 17k-4, 7k-3, 7k-2, 7k-2, 7k, 7k+1\}. \end{aligned}$$

Case 2. When k is odd,

$$\begin{aligned} f(E) &= \{1, 6, 7, 3, 9, 13, \cdots, 7k-12, 7k-8, \\ &\quad 7k-7, 7k-11, 7k-5, 7k-2, 7k+1, 7k-4\} \end{aligned}$$

and,

$$f^v(V) = \{2, 4, 8, 5, \cdots, 7k-10, 7k-6, 7k-9, 7k-3, 7k, 7k-1\}.$$

Therefore,

$$\begin{aligned} f(E) \cup f^v(V) &= \{1, 2, 3, 4, \cdots, 7k-12, 7k-11, 7k-10, 7k-9, 7k-8, \\ &\quad 7k-7, 7k-6, 7k-5, 17k-4, 7k-3, 7k-2, 7k-2, 7k, 7k+1\}. \end{aligned}$$

Thus it has been proved that the labeling $f : E(kC_4) \rightarrow \{1, 2, 3, \cdots, 7k+1\}$ is a super vertex mean labeling. □

Example 4.2 The super vertex-mean labelings of two linear quadrilateral snakes with 4 and 3 blocks are shown in figures 1 and 2 respectively.

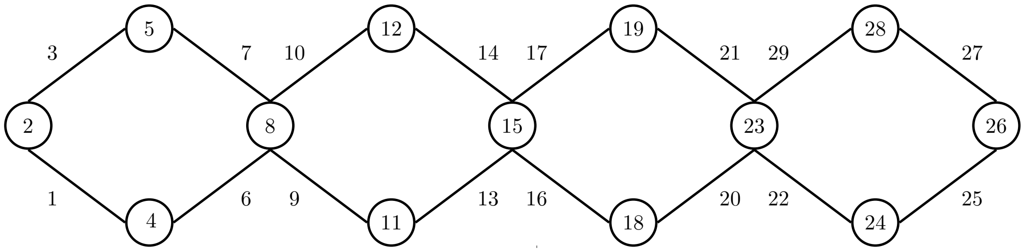

Figure 1 A Super vertex-mean labeling of a linear quadrilateral snake with 4 blocks

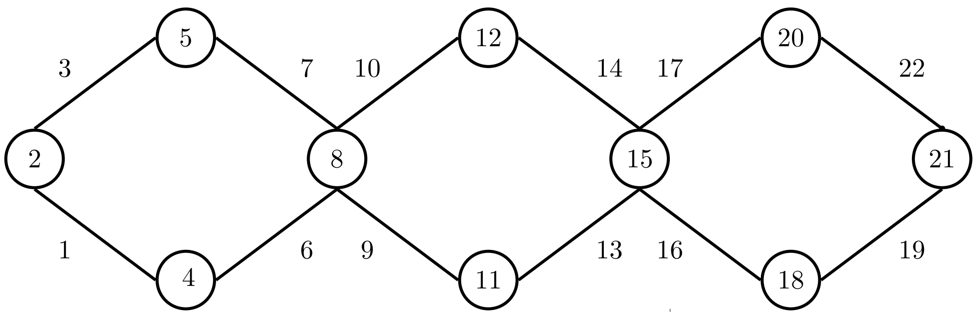

Figure 2 A super vertex-mean labeling of a linear quadrilateral snake with 3 blocks

Theorem 4.3 *All linear pentagonal snakes, kC_5 with $k, k \geq 2$ blocks are SVM graphs.*

Proof Let kC_5 be a linear pentagonal snake with $k, k \geq 2$ blocks of C_5. Here $p = 4k + 1$, $q = 5k$ and $p + q = 9k + 1$. Define $f : E(kC_5) \to \{1, 2, 3, \cdots, 9k + 1\}$ as follows:

When $i = 1$,

$$f(e_{i,j}) = \begin{cases} 5, & \text{if } j = 1, \\ 2j + 4, & \text{if } 2 \leq j \leq 3, \\ 1, & \text{if } j = 4, \\ 3, & \text{if } j = 5. \end{cases}$$

And When $2 \leq i \leq k$,

$$f(e_{i,j}) = \begin{cases} 9i - 9, & \text{if } j = 1, \\ 9i + 2j - 5, & \text{if } 2 \leq j \leq 3, \\ 9i + 3j - 18, & \text{if } 4 \leq j \leq 5. \end{cases}$$

It can be easily verified that f is injective. Then, the induced vertex labels are as follows:

When $i = 1$,

$$f^v(v_{i,j}) = \begin{cases} 4, & \text{if } j = 1, \\ 7, & \text{if } j = 2, \\ 6, & \text{if } j = 4, \\ 2, & \text{if } j = 5. \end{cases}$$

When $2 \leq i \leq k$,

$$f^v(v_{i,j}) = \begin{cases} 9i + 2j - 9, & \text{if } 1 \leq j \leq 2 \\ 9i - 2j + 6, & \text{if } 4 \leq j \leq 5 \\ 9k, & \text{if } i = k, \text{ and } j = 3. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 9k+1\}$. Therefore, all linear pentagonal snakes kC_5 are super vertex mean graphs. □

Example 4.4 A graph given in Figure 3 is an SVM labeling of a linear pentagonal snake with 3 blocks.

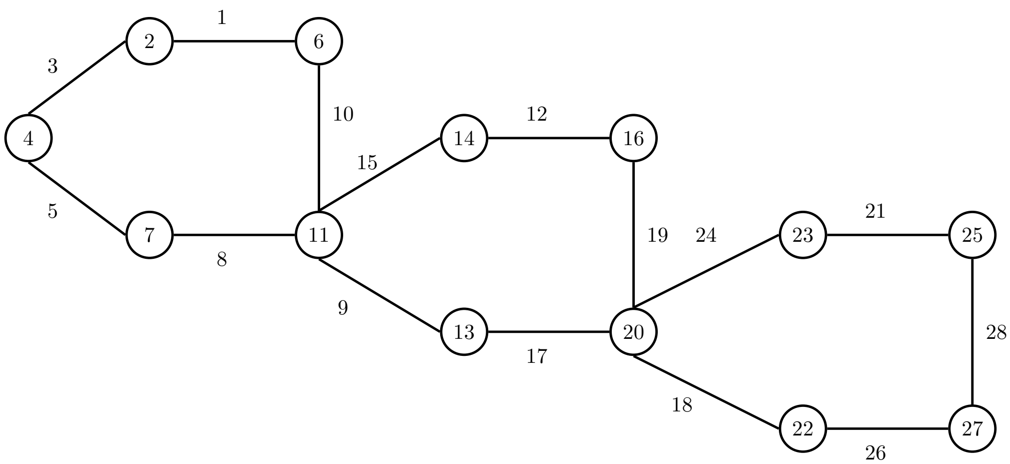

Figure 3 An SVM labeling of a linear pentagonal snake, $3C_5$.

Theorem 4.5 *All linear hexagonal snakes, $kC_6, k \geq 2$ are super vertex mean graphs.*

Proof Let kC_6 be a hexagonal snake with $k, k \geq 2$ blocks of C_6. $p = 5k + 1$ and $q = 6k$ and $p + q = 11k + 1$. Define $f : E(G_n) \to \{1, 2, 3, \cdots, 11k + 1\}$ as follows:

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 4, \\ 7, & \text{if } i = 1, \text{ and } j = 5, \\ 1, & \text{if } i = 1, \text{ and } j = 6, \\ 11i - 4j, & \text{if } 2 \leq i \leq k, \text{ and } 1 \leq j \leq 2, \\ 11i + 3j - 11, & \text{if } 2 \leq i \leq k, \text{ and } 3 \leq j \leq 4, \\ 11i - 8j + 37, & \text{if } 2 \leq i \leq k, \text{ and } 5 \leq j \leq 6. \end{cases}$$

It can be easily verified that f is injective. Then, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 11i+3j-12, & \text{if } 1 \le i \le k, \text{ and } 1 \le j \le 2, \\ 8, & \text{if } i=1, \text{ and } j=3, \\ 11i-5, & \text{if } 2 \le i \le k, \text{ and } j=3, \\ 11k, & \text{if } i=k, \text{ and } j=4, \\ 11i-1, & \text{if } 1 \le i \le k, \text{ and } j=5, \\ 11i-7, & \text{if } 1 \le i \le k, \text{ and } j=6. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 11k+1\}$.

Therefore, linear hexagonal snakes, all kC_6 with k blocks of C_6 are super vertex mean graphs. □

Example 4.6 A graph given in Figure 4 is an SVM labeling of a linear hexagonal snake with 3 blocks.

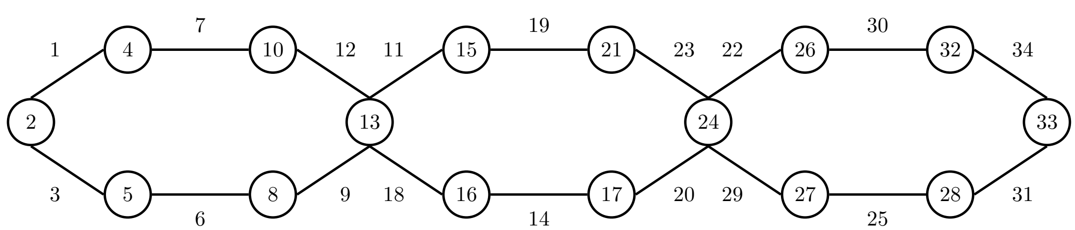

Figure 4 An SVM labeling of a linear hexagonal snake, $3C_6$.

Theorem 4.7 *All linear heptagonal snakes, $kC_7, k \ge 2$ are super vertex mean graphs.*

Proof Let kC_7 be a linear heptagonal snake with $k, k \ge 2$ blocks of C_7. $p = 6k+1$ and $q = 7k$ and $p+q = 13k+1$. Define $f : E(G_n) \to \{1, 2, 3, \cdots, 13k+1\}$ as follows:

$$f(e_{i,j}) = \begin{cases} 4j-1, & \text{if } i=1, \text{ and } 1 \le j \le 3, \\ 30-4j, & \text{if } i=1, \text{ and } 4 \le j \le 6, \\ 1, & \text{if } i=1, \text{ and } j=7, \\ 13i+2j-7, & \text{if } 2 \le i \le k, \text{ and } 1 \le j \le 4, \\ 13i-13, & \text{if } 2 \le i \le k, \text{ and } j=5, \\ 13i+2j-21, & \text{if } 2 \le i \le k, \text{ and } 6 \le j \le 7. \end{cases}$$

It can be easily verified that f is injective. Then, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i=1, \text{ and } j=1, \\ 4j-3, & \text{if } i=1, \text{ and } 2\le j\le 3, \\ 32-4j, & \text{if } i=1, \text{ and } 5\le j\le 7, \\ 13i-10, & \text{if } 2\le i\le k, \text{ and } j=1, \\ 13i+2j-8, & \text{if } 2\le i\le k, \text{ and } 2\le j\le 3, \\ 13i-6, & \text{if } 2\le i\le k, \text{ and } j=5, \\ 13i+3j-29, & \text{if } 2\le i\le k, \text{ and } 6\le j\le 7, \\ 13k, & \text{if } i=k, \text{ and } j=4. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1,2,3,\cdots,13k+1\}$.

Therefore, all linear heptagonal snakes, kC_7 with k blocks of C_7 are super vertex mean graphs. □

Example 4.8 A graph given in Figure 5 is an SVM labeling of a linear heptagonal snake, $3C_7$.

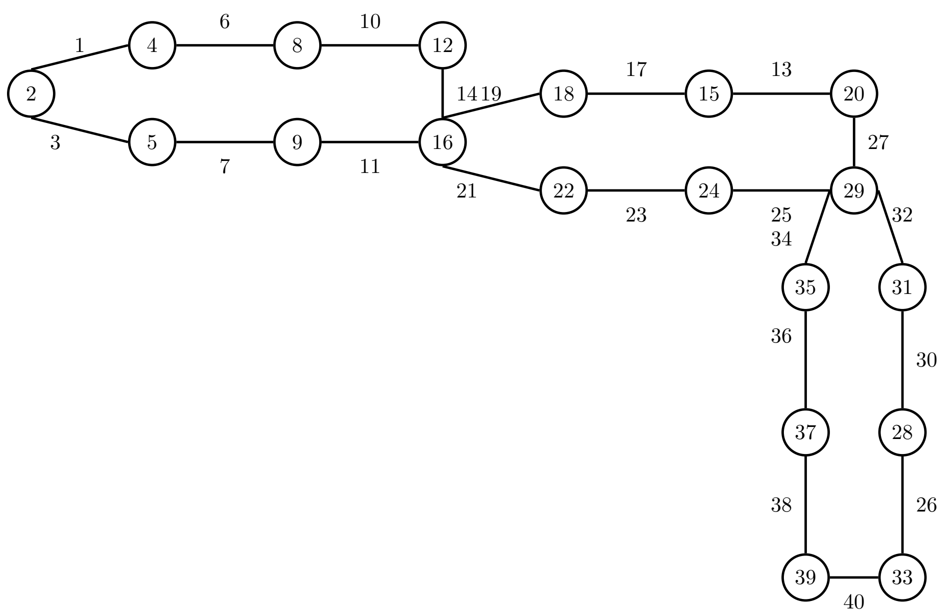

Figure 4 A Super Vertex Mean Labeling of $3C_7$ linear snake.

Theorem 4.9 *Let kC_n be a linear cyclic snake with $k, k\ge 2$ blocks of $C_n, n\ge 8$ and $n\equiv 0(mod\ 2)$. Then kC_n is a super vertex mean graph.*

Proof Let kC_n be a linear cyclic snake with $k, k\ge 2$ blocks of $C_n, n\ge 8$ and $n\equiv 0(mod\ 2)$. Let $n=2r, r\ge 4$. Now, $p=(n-1)k+1$ and $q=nk$ and $p+q=(2n-1)k+1$. Define

$f : E(kC_n) \rightarrow \{1, 2, 3, \cdots, (2n-1)k+1\}$ as follows:

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 4j-3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r, \\ 4n-4j+4, & \text{if } i = 1, \text{ and } r+1 \leq j \leq n-2, \\ 7, & \text{if } i = 1, \text{ and } j = n-1, \\ 1, & \text{if } i = 1, \text{ and } j = n, \\ (2n-1)i-2n+8, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (2n-1)i-2n+4, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (2n-1)i-2n+10, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (2n-1)i-2n+4j-2, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (2n-1)i+2n-4j+5, & \text{if } 2 \leq i \leq k, \text{ and } r+1 \leq j \leq n-1, \\ (2n-1)(i-1), & \text{if } i = k, \text{ and } j = n. \end{cases}$$

And, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j-1, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4, \\ 4j-5, & \text{if } i = 1, \text{ and } 5 \leq j \leq r, \\ 4n-4j+6, & \text{if } i = 1, \text{ and } r+2 \leq j \leq n-1, \\ 4, & \text{if } i = 1, \text{ and } j = n, \\ (2n-1)i-2n+3, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (2n-1)i-2n+6, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (2n-1)i-2n+7, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (2n-1)i-2n+4j-4, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (2n-1)i+2n-4j+7, & \text{if } 2 \leq i \leq k, \text{ and } r+2 \leq j \leq n-1, \\ (2n-1)i-2n+5, & \text{if } 2 \leq i \leq k, \text{ and } j = n, \\ (2n-1)k, & \text{if } i = k, \text{ and } j = r+1. \end{cases}$$

We prove the theorem by mathematical induction on r, where $n = 2r, r \geq 4$. The above edge labeling function $f(e)$ and the induced vertex labeling function $f^v(v)$ are expressed in terms of r as follows:

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 4j-3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r, \\ 8r-4j+4, & \text{if } i = 1, \text{ and } r+1 \leq j \leq 2r-2, \\ 7, & \text{if } i = 1, \text{ and } j = 2r-1, \\ 1, & \text{if } i = 1, \text{ and } j = 2r \end{cases}$$

and

$$f(e_{i,j}) = \begin{cases} (4r-1)i-4r+8, & \text{if } 2 \le i \le k, \text{ and } j=1, \\ (4r-1)i-4r+4, & \text{if } 2 \le i \le k, \text{ and } j=2, \\ (4r-1)i-4r+10, & \text{if } 2 \le i \le k, \text{ and } j=3, \\ (4r-1)i-4r+4j-2, & \text{if } 2 \le i \le k, \text{ and } 4 \le j \le r, \\ (4r-1)i+4r-4j+5, & \text{if } 2 \le i \le k, \text{ and } r+1 \le j \le 2r-1, \\ (4r-1)(i-1), & \text{if } i=k, \text{ and } j=2r. \end{cases}$$

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i=1, \text{ and } j=1, \\ 3j-1, & \text{if } i=1, \text{ and } 2 \le j \le 4, \\ 4j-5, & \text{if } i=1, \text{ and } 5 \le j \le r, \\ 8r-4j+6, & \text{if } i=1, \text{ and } r+2 \le j \le 2r-1, \\ 4, & \text{if } i=1, \text{ and } j=2r, \\ (4r-1)i-4r+3, & \text{if } 2 \le i \le k, \text{ and } j=1, \\ (4r-1)i-4r+6, & \text{if } 2 \le i \le k, \text{ and } j=2, \\ (4r-1)i-4r+7, & \text{if } 2 \le i \le k, \text{ and } j=3, \\ (4r-1)i-4r+4j-4, & \text{if } 2 \le i \le k, \text{ and } 4 \le j \le r, \\ (4r-1)i+4r-4j+7, & \text{if } 2 \le i \le k, \text{ and } r+2 \le j \le 2r-1, \\ (4r-1)i-4r+5, & \text{if } 2 \le i \le k, \text{ and } j=2r, \\ (4r-1)k, & \text{if } i=k, \text{ and } j=r+1. \end{cases}$$

We prove that the theorem is true when $r = 4$, $n = 8$.

When $r = 4$ the linear cyclic snake is a linear octagonal snake with $k, k \ge 2$ cycles of C_8.

Now, $p = 7k+1$ and $q = 8k$ and $p+q = 15k+1$. Define $f : E(kC_n) \to \{1, 2, 3, \cdots, 15k+1\}$ as follows:

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i=1, \text{ and } 1 \le j \le 3, \\ 13, & \text{if } i=1, \text{ and } j=4, \\ 36-4j, & \text{if } i=1, \text{ and } 5 \le j \le 6, \\ 7, & \text{if } i=1, \text{ and } j=7, \\ 1, & \text{if } i=1, \text{ and } j=8, \\ 15i-8, & \text{if } 2 \le i \le k, \text{ and } j=1, \\ 15i-12, & \text{if } 2 \le i \le k, \text{ and } j=2, \\ 15i-6, & \text{if } 2 \le i \le k, \text{ and } j=3, \\ 15i-2, & \text{if } 2 \le i \le k, \text{ and } j=4, \\ 15i-4j+21, & \text{if } 2 \le i \le k, \text{ and } 5 \le j \le 7, \\ 15i-15, & \text{if } i=k, \text{ and } j=8. \end{cases}$$

It can be easily verified that f is injective. The induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j - 1, & \text{if } i = 1, \text{ and } 2 \le j \le 4, \\ 38 - 4j, & \text{if } i = 1, \text{ and } 6 \le j \le 7, \\ 4, & \text{if } i = 1, \text{ and } j = 8, \\ 15i - 13, & \text{if } 2 \le i \le k, \text{ and } j = 1, \\ 15i - 10, & \text{if } 2 \le i \le k, \text{ and } j = 2, \\ 15i - 9, & \text{if } 2 \le i \le k, \text{ and } j = 3, \\ 15i - 4, & \text{if } 2 \le i \le k, \text{ and } j = 4, \\ 15i - 4j + 23, & \text{if } 2 \le i \le k, \text{ and } 6 \le j \le 7, \\ 15i - 11, & \text{if } 2 \le i \le k, \text{ and } j = 8, \\ 15k, & \text{if } i = k, \text{ and } j = 5. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 15k + 1\}$ as follows:

$$\begin{cases} f(E) & = \{3, 6, 9, 13, 16, 12, 7, 1\} \bigcup \\ & \{22, 18, 24, 28, 31, 27, 23, 15\} \bigcup \cdots, \\ & \{15k - 8, 15k - 12, 15k - 6, 15k - 2, 15k + 1, \\ & 15k - 3, 15k - 7, 15k - 15\}. \\ f^v(V) & = \{2, 5, 8, 11, 14, 10, 4\} \bigcup \\ & \{17, 20, 21, 26, 29, 25, 19\} \bigcup \cdots, \\ & \{15k - 13, 15k - 10, 15k - 9, 15k - 4, 15k - 1, \\ & 15k - 5, 15k - 11, 15k\}. \\ f(E) \bigcup f^v(V) & = \{1, 3, 6, 7, 9, 12, 13, 16\} \bigcup \\ & \{2, 4, 5, 8, 10, 11, 14\} \cup \\ & \{15, 18, 22, 23, 24, 27, 28, 31\} \bigcup \\ & \{17, 19, 20, 21, 25, 26, 29\} \bigcup \cdots, \\ & \{15k - 15, 15k - 12, 15k - 8, 15k - 7, 15k - 6, \\ & 15k - 3, 15k - 2, 15k + 1\} \bigcup \\ & \{15k - 13, 15k - 11, 15k - 10, 15k - 9, 15k - 5, \\ & 15k - 4, 15k - 1, 15k\}. \\ & = \{1, 2, 3, 4, \cdots, 29, 30, 31, \cdots, 15k - 2, 15k - 1, 15k, 15k + 1\}. \end{cases}$$

Thus the theorem is true when $r = 4$.

Now we assume that the theorem is true for $r-1, r \ge 5$ (i.e., for $n-2, n \ge 10$). In this case, $p = (n-3)k+1 = (2r-3)k+1$ and $q = (n-2)k = (2r-2)k$ and $p+q = (2n-5)k+1 = (4r-5)k+1$.

The induction hypothesis is that the edge labeling,

$$f : E(kC_{2r-2}) \to \{1, 2, 3, \cdots, (4r-5)k+1\}$$

defined as follows, is a super vertex mean labeling, where $r \geq 5, n \geq 10$, $n \equiv 0(mod\ 2)$ and $k \geq 2$.

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 4j-3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r-1, \\ 8r-4j-4, & \text{if } i = 1, \text{ and } r \leq j \leq 2r-4, \\ 7, & \text{if } i = 1, \text{ and } j = 2r-3, \\ 1, & \text{if } i = 1, \text{ and } j = 2r-2, \\ (4r-5)i-4r+12, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (4r-5)i-4r+8, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (4r-5)i-4r+14, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (4r-5)i-4r+4j+2, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r-1, \\ (4r-5)i+4r-4j+1, & \text{if } 2 \leq i \leq k, \text{ and } r \leq j \leq 2r-3, \\ (4r-5)(i-1), & \text{if } i = k, \text{ and } j = 2r-2. \end{cases}$$

and the induced vertex labeling is,

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j-1, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4, \\ 4j-5, & \text{if } i = 1, \text{ and } 5 \leq j \leq r-1, \\ 8r-4j-2, & \text{if } i = 1, \text{ and } r+1 \leq j \leq 2r-3, \\ 4, & \text{if } i = 1, \text{ and } j = 2r-2, \\ (4r-5)i-4r+7, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (4r-5)i-4r+10, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (4r-5)i-4r+11, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (4r-5)i-4r+4j, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r-1, \\ (4r-5)i+4r-4j+3, & \text{if } 2 \leq i \leq k, \text{ and } r+1 \leq j \leq 2r-3, \\ (4r-5)i-4r+9, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r-2, \\ (4r-5)k, & \text{if } i = k, \text{ and } j = r. \end{cases}$$

Now we prove that the result is true for any r. If we replace r with $r+1$ in the above mapping we get,

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 4j-3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r, \\ 8r-4j+4, & \text{if } i = 1, \text{ and } r+1 \leq j \leq 2r-2 \end{cases}$$

and

$$f(e_{i,j}) = \begin{cases} 7, & \text{if } i=1, \text{ and } j=2r-1,\\ 1, & \text{if } i=1, \text{ and } j=2r,\\ (4r-1)i-4r+8, & \text{if } 2\le i\le k, \text{ and } j=1,\\ (4r-1)i-4r+4, & \text{if } 2\le i\le k, \text{ and } j=2,\\ (4r-1)i-4r+10, & \text{if } 2\le i\le k, \text{ and } j=3,\\ (4r-1)i-4r+4j-2, & \text{if } 2\le i\le k, \text{ and } 4\le j\le r,\\ (4r-1)i+4r-4j+5, & \text{if } 2\le i\le k, \text{ and } r+1\le j\le 2r-1,\\ (4r-1)(i-1), & \text{if } i=k, \text{ and } j=2r. \end{cases}$$

and,

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i=1, \text{ and } j=1,\\ 3j-1, & \text{if } i=1, \text{ and } 2\le j\le 4,\\ 4j-5, & \text{if } i=1, \text{ and } 5\le j\le r,\\ 8r-4j+6, & \text{if } i=1, \text{ and } r+2\le j\le 2r-1,\\ 4, & \text{if } i=1, \text{ and } j=2r,\\ (4r-1)i-4r+3, & \text{if } 2\le i\le k, \text{ and } j=1,\\ (4r-1)i-4r+6, & \text{if } 2\le i\le k, \text{ and } j=2,\\ (4r-1)i-4r+7, & \text{if } 2\le i\le k, \text{ and } j=3,\\ (4r-1)i-4r+4j-4, & \text{if } 2\le i\le k, \text{ and } 4\le j\le r,\\ (4r-1)i+4r-4j+7, & \text{if } 2\le i\le k, \text{ and } r+2\le j\le 2r-1,\\ (4r-1)i-4r+5, & \text{if } 2\le i\le k, \text{ and } j=2r,\\ (4r-1)k, & \text{if } i=k, \text{ and } j=r+1. \end{cases}$$

This is equivalent to the original labeling in terms of n, which is given in the beginning of the proof, and it is clear that $f(E)\cup f^v(V)=\{1,2,3,\cdots,(4r-1)k-1,(4r-1)k,(4r-1)k+1\}$. Thus the theorem is proved by mathematical induction. □

Example 4.10 A graph given in Figure 6 is an SVM labeling of a linear cyclic snake $2C_{12}$.

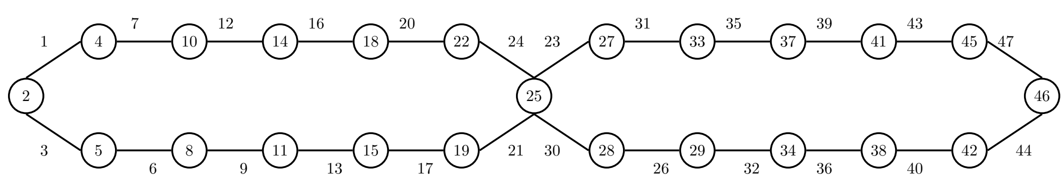

Figure 5 A super vertex mean labeling of a linear cyclic snake $2C_{12}$.

Theorem 4.11 *Let kC_n be a linear cyclic snake with $k, k\ge 2$ blocks of $C_n, n\ge 9$ and $n\equiv 1(mod\ 4)$. Then kC_n is a super vertex mean graph.*

Proof Let kC_n be a cyclic snake with $k, k\ge 2$ blocks of $C_n, n\ge 9$ and $n\equiv 1(mod\ 4)$. Let

$n = 2r+1, r \geq 4$, and $r = 2s, s \geq 2$ so that $n = 4s+1$. Now $p = (n-1)k+1$ and $q = nk$ and $p+q = (2n-1)k+1$. Define $f : E(kC_n) \to \{1, 2, 3, \cdots, (2n-1)k+1\}$ as follows:

$$f(e_{i,j}) = \begin{cases} n, & \text{if } i=1, \text{ and } j=1, \\ 2j+n-1, & \text{if } i=1, \text{ and } 2 \leq j \leq r+1, \\ 2j-n-2, & \text{if } i=1, \text{ and } r+2 \leq j \leq n, \\ (2n-1)i-n-1, & \text{if } 2 \leq i \leq k, \text{ and } j=1, \\ (2n-1)i-n+2j-2, & \text{if } 2 \leq i \leq k, \text{ and } 2 \leq j \leq r-3, \\ (2n-1)i-n+2j-1, & \text{if } 2 \leq i \leq k, \text{ and } r-2 \leq j \leq r-1, \\ (2n-1)i-n+2j, & \text{if } 2 \leq i \leq k, \text{ and } r \leq j \leq r+1, \\ (2n-1)i-8, & \text{if } 2 \leq i \leq k, n \neq 9 \text{ and } j=r+2, \\ (2n-1)i-7, & \text{if } 2 \leq i \leq k, n=9 \text{ and } j=r+2, \\ (2n-1)i-2n+1, & \text{if } 2 \leq i \leq k, \text{ and } j=r+3, \\ (2n-1)i-3n+2j-3, & \text{if } 2 \leq i \leq k, \text{ and } r+4 \leq j \leq n+1-s, \\ (2n-1)i-3n+2j-2, & \text{if } 2 \leq i \leq k, \text{ and } n+2-s \leq j \leq n-1, \\ (2n-1)i-n, & \text{if } 2 \leq i \leq k, \text{ and } j=n. \end{cases}$$

and, the induced vertex labels are as follows:

When $i = 1$, and $n \geq 9$,

$$f^v(v_{i,j}) = \begin{cases} n-1, & \text{if } j=1, \\ 2j+n-2, & \text{if } 2 \leq j \leq r, \\ n+1, & \text{if } j=r+2, \\ 2j-16, & \text{if } r+3 \leq j \leq n. \end{cases}$$

and when $2 \leq i \leq k$, and $n = 9$,

$$f^v(v_{i,j}) = \begin{cases} 17i-13, & \text{if } j=1, \\ 17i+3j-14, & \text{if } 2 \leq j \leq 4, \\ 17i, & \text{if } j=5 \text{ and } i=k, \\ 17i-3, & \text{if } j=6, \\ 22, & \text{if } j=7, \\ 19, & \text{if } j=8, \\ 23, & \text{if } j=9. \end{cases}$$

and when $2 \leq i \leq k$, and $n \geq 13$,

$$f^v(v_{i,j}) = \begin{cases} (2n-1)i-3r-1, & \text{if } j=1, \\ (2n-1)i-n+2j-3, & \text{if } 2 \leq j \leq r-3 \end{cases}$$

and

$$f^v(v_{i,j}) = \begin{cases} (2n-1)i-n+2j-2, & \text{if } r-2 \le j \le r-1, \\ (2n-1)i-2, & \text{if } j = r, \\ (2n-1)i, & \text{if } i = k, \text{ and } j = r+1, \\ (2n-1)i-3, & \text{if } j = r+2, \\ (2n-1)i-n-3, & \text{if } j = r+3, \\ (2n-1)i-3n+2j-4, & \text{if } r+4 \le j \le n+1-s, \\ (2n-1)i-3n+2j-3, & \text{if } n+2-s \le j \le n-1, \\ (2n-1)i-n-2, & \text{if } j = n. \end{cases}$$

We can easily prove the theorem by the technique of mathematical induction on s. The remaining of the proof is left as an exercise. □

Theorem 4.13 *Let kC_n be a linear cyclic snake with $k, k \ge 2$ blocks of $C_n, n \ge 11$ and $n \equiv 3(mod\ 4)$. Then kC_n is a super vertex mean graph.*

Proof Let kC_n be a linear cyclic snake with $k, k \ge 2$ blocks of $C_n, n \ge 11$ and $n \equiv 3(mod\ 4)$. Let $n = 2r+1, r \ge 5$, and $r = 2s+1, s \ge 2$ so that $n = 4s+3$. Now, $p = (n-1)k+1$ and $q = nk$ and $p+q = (2n-1)k+1$. Define $f : E(kC_n) \to \{1, 2, 3, \cdots, (2n-1)k+1\}$ as follows:

$$f(e_{i,j}) = \begin{cases} 4j-1, & \text{if } i = 1, \text{ and } 1 \le j \le r, \\ 4n-4j+2, & \text{if } i = 1, \text{ and } r+1 \le j \le 2r, \\ 1, & \text{if } i = 1, \text{ and } j = n, \\ (2n-1)i-2n+2j+7, & \text{if } 2 \le i \le k, \text{ and } 1 \le j \le 3, \\ (2n-1)i-2n+4j+1, & \text{if } 2 \le i \le k, \text{ and } 4 \le j \le r-1, \\ (2n-1)i-3r+3j-2, & \text{if } 2 \le i \le k, \text{ and } r \le j \le r+1, \\ (2n-1)i+2n-4j+2, & \text{if } 2 \le i \le k, \text{ and } r+2 \le j \le 2r-2, \\ (2n-1)i-2n+1, & \text{if } 2 \le i \le k, \text{ and } j = 2r-1, \\ (2n-1)i-4n+2j+7, & \text{if } 2 \le i \le k, \text{ and } 2r \le j \le n. \end{cases}$$

and, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 4j-3, & \text{if } i = 1, \text{ and } 2 \le j \le r, \\ 4n-4j+4, & \text{if } i = 1, \text{ and } r+2 \le j \le n, \\ (2n-1)i-2n+4, & \text{if } 2 \le i \le k, \text{ and } j = 1, \\ (2n-1)i-2n+2j+6, & \text{if } 2 \le i \le k, \text{ and } 2 \le j \le 3 \\ (2n-1)i-2n+4j-1, & \text{if } 2 \le i \le k, \text{ and } 4 \le j \le r, \\ (2n-1)i-1, & \text{if } 2 \le i \le k, \text{ and } j = r+2 \end{cases}$$

and

$$f^v(v_{i,j}) = \begin{cases} (2n-1)i + 2n - 4j + 4, & \text{if } 2 \le i \le k, \text{ and } r+3 \le j \le 2r-2, \\ (2n-1)i - 2n + 8, & \text{if } 2 \le i \le k, \text{ and } j = 2r-1, \\ (2n-1)i - 2n + 3, & \text{if } 2 \le i \le k, \text{ and } j = 2r, \\ (2n-1)i - 2n + 6, & \text{if } 2 \le i \le k, \text{ and } j = 2r+1, \\ (2n-1)k, & \text{if } i = k, \text{ and } j = r+1. \end{cases}$$

We can easily prove that the above labeling is an SVM labeling of kC_n, where $k \ge 2$ blocks of $C_n, n \ge 11$ and $n \equiv 3(mod\ 4)$, by using the technique of mathematical induction on s, where $n = 4s + 3$. Thus the theorem can easily be proved. □

§5. Super Vertex Mean Number

The concept of super vertex mean number arises from the earlier concepts such as, mean number, super mean number etc. M.Somasundaram and R.Ponraj have introduced the term mean number of a graph [16] and they have found the mean number of many standard graphs. Later on, A.Nagarajan et.al. introduced the concept of super mean number of a graph [9] and proved the existence of it for any graph by finding out the limit values of it. Encouraged by their works we introduce this new concept which we like to name as super vertex mean number or SVM number.

Definition 5.1 *Let f be a an injective function of a (p, q) - graph $G(V, E)$ defined from E to the set $\{1, 2, 3, \cdots, n\}$ that induces for each vertex v the label defined by the rule $f^v(v) =$ Round $\left(\frac{\sum_{e \in E_v} f(e)}{d(v)}\right)$, where E_v denotes the set of edges in G that are incident at the vertex v. Let $f(E) \cup f^v(V) \subseteq \{1, 2, 3, \cdots, n\}$. If n is the smallest positive integer satisfying these conditions together with the condition that all the vertex labels as well as the edge labels are distinct, then n is called the super vertex mean number (or SVM number) of the graph $G(V, E)$, and is denoted by $SV_m(G)$.*

Observation 5.2 It is observed that $SV_m(G) = p + q$, for all SVM graphs G whose order is p and size is q. And for other graphs G, $SV_m(G) \ge p + q + 1$. For graphs containing an isolated vertex or a leaf, the super vertex mean number is not defined.

Therefore, for any (p, q) − graph G, $p + q \le SV_m(G) \le \infty$. For example, the SVM number of C_4, $SV_m(C_4) = 9$.

§6. Conclusion

In this paper, we have proved that all the linear cyclic snakes are super vertex mean graphs. In the case of super mean labeling, the vertex analogue of SVM, it was easy to obtain a general formula for linear cyclic snakes as well as other cyclic snakes represented the string $s_1, s_2, s_3, \cdots, s_{k-2}$, where each s_i need not be equal. This is because when we calculate the induced edge label for an edge, by finding the average of the labels of the two vertices which are the end points of that particular edge, we need to consider only those two vertices. Therefore the average remains the same as in the case of cycles.

But for super vertex mean labeling, when we find the induced vertex labeling of the connecting vertices of a cyclic snake we have to consider four edges that are incident on those vertices to get the average. Thus it becomes pretty difficult to obtain a general formula for cyclic snakes represented the string $s_1, s_2, s_3, \cdots, s_{k-2}$, where each s_i need not be equal. Another possibility emerges is that we try to explore the SVM labeling of KC − snakes, which is defined as a connecting graph in which each of the k many blocks is isomorphic to a cycle C_n for some n and the block - cut point graph is a path. As in the case of kC_n − snakes, a kC –snake too can be represented by a string of integers, $s_1, s_2, \cdots, s-k-2$. It remains still an open problem to label a kC –snake which has either equal s_i or different s_i.

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