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A Generalization on
Product Degree Distance of Strong Product of Graphs

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Abstract: In this paper, the exact formulae for the generalized product degree distance, reciprocal product degree distance and product degree distance of strong product of a connected graph and the complete multipartite graph with partite sets of sizes $m_0, m_1, \cdots, m_{r-1}$ are obtained.

Key Words: Reciprocal product degree distance, product degree distance, strong product.

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§1. Introduction

All the graphs considered in this paper are simple and connected. For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u,v)$, is the length of a shortest (u,v)-path in G and let $d_G(v)$ be the degree of a vertex $v \in V(G)$. The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(u,v) : u \in V(G), v \in V(H)\}$ and $(u,x)(v,y)$ is an edge whenever (i) $u = v$ and $xy \in E(H)$, or (ii) $uv \in E(G)$ and $x = y$, or (iii) $uv \in E(G)$ and $xy \in E(H)$.

A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [12]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index.

Let G be a connected graph. Then *Wiener index* of G is defined as

$$W(G) = \frac{1}{2} \sum_{u,\, v \,\in\, V(G)} d_G(u,v)$$

with the summation going over all pairs of distinct vertices of G. This definition can be further

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generalized in the following way:

$$W_\lambda(G) = \frac{1}{2} \sum_{u,\, v \in V(G)} d_G^\lambda(u,v),$$

where $d_G^\lambda(u,v) = (d_G(u,v))^\lambda$ and λ is a real number [13, 14]. If $\lambda = -1$, then $W_{-1}(G) = H(G)$, where $H(G)$ is Harary index of G. In the chemical literature also $W_{\frac{1}{2}}$ [29] as well as the general case W_λ were examined [10, 15].

Dobrynin and Kochetova [6] and Gutman [11] independently proposed a vertex-degree-weighted version of Wiener index called *degree distance*, which is defined for a connected graph G as

$$DD(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G(u,v),$$

where $d_G(u)$ is the degree of the vertex u in G. Similarly, the *product degree distance* or *Gutman index* of a connected graph G is defined as

$$DD_*(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u) d_G(v) d_G(u,v).$$

The *additively weighted Harary index*(H_A) or *reciprocal degree distance*(RDD) is defined in [3] as

$$H_A(G) = RDD(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{(d_G(u) + d_G(v))}{d_G(u,v)}.$$

Similarly, Su et al. [28] introduce the *reciprocal product degree distance* of graphs, which can be seen as a product-degree-weight version of Harary index

$$RDD_*(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{d_G(u) d_G(v)}{d_G(u,v)}.$$

In [16], Hamzeh et al. recently introduced generalized degree distance of graphs. Hua and Zhang [18] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants. Pattabiraman et al. [22, 23] have obtained the reciprocal degree distance of join, tensor product, strong product and wreath product of two connected graphs in terms of other graph invariants. The chemical applications and mathematical properties of the reciprocal degree distance are well studied in [3, 20, 27].

The *generalized degree distance*, denoted by $H_\lambda(G)$, is defined as

$$H_\lambda(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G^\lambda(u,v),$$

where λ is a real number. If $\lambda = 1$, then $H_\lambda(G) = DD(G)$ and if $\lambda = -1$, then $H_\lambda(G) =$

$RDD(G)$. Similarly, *generalized product degree distance*, denoted by $H_\lambda^*(G)$, is defined as

$$H_\lambda^*(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u) d_G(v) d_G^\lambda(u,v).$$

If $\lambda = 1$, then $H_\lambda^*(G) = DD_*(G)$ and if $\lambda = -1$, then $H_\lambda^*(G) = RDD_*(G)$. Therefore the study of the above topological indices are important and we try to obtain the results related to these indices. The generalized degree distance of unicyclic and bicyclic graphs are studied by Hamzeh et al. [16, 17]. Also they are given the generalized degree distance of Cartesian product, join, symmetric difference, composition and disjunction of two graphs. The generalized degree distance and generalized product degree distance of some classes of graphs are obtained in [24, 25, 26]. In this paper, the exact formulae for the generalized product degree distance, reciprocal product degree distance and product degree distance of strong product $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$, where $K_{m_0, m_1, \cdots, m_{r-1}}$ is the complete multipartite graph with partite sets of sizes m_0, m_1, $\cdots$, m_{r-1} are obtained.

The *first Zagreb index* is defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2$$

and the *second Zagreb index* is defined as

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

In fact, one can rewrite the first Zagreb index as

$$M_1(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)).$$

The Zagreb indices were found to be successful in chemical and physico-chemical applications, especially in QSPR/QSAR studies, see [8, 9].

For $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph of G induced by S. For two subsets $S, T \subset V(G)$, not necessarily disjoint, by $d_G(S,T)$, we mean the sum of the distances in G from each vertex of S to every vertex of T, that is, $d_G(S,T) = \sum_{s \in S, t \in T} d_G(s,t)$.

§2. Generalized Product Degree Distance of Strong Product of Graphs

In this section, we obtain the Generalized product degree distance of $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$. Let G be a simple connected graph with $V(G) = \{v_0, v_1, \cdots, v_{n-1}\}$ and let $K_{m_0, m_1, \cdots, m_{r-1}}$, $r \geq 2$, be the complete multiparite graph with partite sets V_0, V_1, $\cdots$, V_{r-1} and let $|V_i| = m_i$, $0 \leq i \leq r-1$. In the graph $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$, let $B_{ij} = v_i \times V_j, v_i \in V(G)$ and $0 \leq j \leq r-1$.

For our convenience, the vertex set of $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$ is written as

$$V(G) \times V(K_{m_0, m_1, \ldots, m_{r-1}}) = \bigcup_{\substack{i=0 \\ j=0}}^{\substack{r-1 \\ n-1}} B_{ij}.$$

Let $\mathscr{B} = \{B_{ij}\}_{\substack{i=0,1,\cdots,n-1 \\ j=0,1,\cdots,r-1}}$. Let $X_i = \bigcup\limits_{j=0}^{r-1} B_{ij}$ and $Y_j = \bigcup\limits_{i=0}^{n-1} B_{ij}$; we call X_i and Y_j as *layer* and *column* of $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, respectively. If we denote $V(B_{ij}) = \{x_{i1}, x_{i2}, \cdots, x_{im_j}\}$ and $V(B_{kp}) = \{x_{k1}, x_{k2}, \cdots, x_{k\,m_p}\}$, then $x_{i\ell}$ and $x_{k\ell}, 1 \leq \ell \leq j$, are called the *corresponding vertices* of B_{ij} and B_{kp}. Further, if $v_iv_k \in E(G)$, then the induced subgraph $\langle B_{ij} \bigcup B_{kp}\rangle$ of $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$ is isomorphic to $K_{|V_j||V_p|}$ or, m_p independent edges joining the corresponding vertices of B_{ij} and B_{kj} according as $j \neq p$ or $j = p$, respectively.

The following remark is follows from the structure of the graph $K_{m_0, m_1, \cdots, m_{r-1}}$.

Remark 2.1 *Let n_0 and q be the number of vertices and edges of $K_{m_0, m_1, \cdots, m_{r-1}}$. Then the sums*

$$\begin{aligned} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} m_j m_p &= 2q, \\ \sum_{j=0}^{r-1} m_j^2 &= n_0^2 - 2q, \\ \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} m_j^2 m_p = n_0 q - 3t &= \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} m_j m_p^2, \\ \sum_{j=0}^{r-1} m_j^3 &= n_0^3 - 3n_0 q + 3t \end{aligned}$$

and

$$\sum_{j=0}^{r-1} m_j^4 = n_0^4 - 4n_0^2 q + 2q^2 + 4n_0 t - 4\tau,$$

where t and τ are the number of triangles and $K_4^{'s}$ in $K_{m_0, m_1, \cdots, m_{r-1}}$.

The proof of the following lemma follows easily from the properties and structure of $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$.

Lemma 2.2 *Let G be a connected graph and let $B_{ij},\ B_{kp} \in \mathscr{B}$ of the graph $G' = G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$, where $r \geq 2$. Then*

(i) If $v_iv_k \in E(G)$ and $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kj}$, then

$$d_{G'}(x_{it}, x_{k\ell}) = \begin{cases} 1, & \text{if } t = \ell, \\ 2, & \text{if } t \neq \ell, \end{cases}$$

and if $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kp},\ j \neq p$, *then* $d_{G'}(x_{it}, x_{k\ell}) = 1$.

(*ii*) *If* $v_iv_k \notin E(G)$, *then for any two vertices* $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kp}$, $d_{G'}(x_{it}, x_{k\ell}) = d_G(v_i, v_k)$.

(*iii*) *For any two distinct vertices in* B_{ij}, *their distance is 2.*

The proof of the following lemma follows easily from Lemma 2.2, which is used in the proof of the main theorems of this section.

Lemma 2.3 *Let* G *be a connected graph and let* $B_{ij},\ B_{kp} \in \mathscr{B}$ *of the graph* $G' = G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, *where* $r \geq 2$.

(*i*) *If* $v_iv_k \in E(G)$, *then*

$$d^H_{G'}(B_{ij}, B_{kp}) = \begin{cases} m_jm_p, \ \ if \ \ j \neq p, \\ \frac{m_j(m_j+1)}{2}, \ \ if \ \ j = p, \end{cases}$$

(*ii*) *If* $v_iv_k \notin E(G)$, *then*

$$d^H_{G'}(B_{ij}, B_{kp}) = \begin{cases} \frac{m_jm_p}{d_G(v_i,v_k)}, \quad if \ j \neq p, \\ \frac{m_j^2}{d_G(v_i,v_k)}, \quad if \ j = p. \end{cases}$$

(*iii*) $d^H_{G'}(B_{ij}, B_{ip}) = \begin{cases} m_jm_p, \quad if \ j \neq p, \\ \frac{m_j(m_j-1)}{2}, \quad if \ j = p. \end{cases}$

Lemma 2.4 *Let* G *be a connected graph and let* B_{ij} *in* $G' = G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$. *Then the degree of a vertex* $(v_i, u_j) \in B_{ij}$ *in* G' *is*

$$d_{G'}((v_i, u_j)) = d_G(v_i) + (n_0 - m_j) + d_G(v_i)(n_0 - m_j),$$

where $n_0 = \sum\limits_{j=0}^{r-1} m_j$.

Now we obtain the generalized product degree distance of $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$.

Theorem 2.5 *Let* G *be a connected graph with* n *vertices and* m *edges. Then*

$$\begin{aligned} &H^*_\lambda(G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}) \\ &= (4q^2 + n_0^2 + 4n_0q)H^*_\lambda(G) + 4q^2W_\lambda(G) + (4q^2 + 2n_0q)H_\lambda(G) + \frac{n}{2}(4q^2 - n_0q - 3t) \\ &+ \frac{M_1(G)}{2}\Big[4n_0^2q - 2q^2 + 4n_0t + 9t + 7n_0q - n_0 - 3n_0^2 - 2n_0^3 + 8\tau\Big] \\ &+ m\Big[3n_0q + 2n_0t - 2q^2 - 3t - 4q + 4\tau\Big] \\ &+ 2^\lambda\Big[M_1(G)(2q^2 - 2n_0t - 6t - 2q - 4\tau) + m(2q^2 - 2n_0t - n_0q - 3t - 4\tau)\Big] \\ &+ (2^\lambda - 1)M_2(G)\big[2q^2 - 2n_0t - 3n_0^3 + 10n_0q + n_0^2 - 18t - 6q - n_0 - 4\tau\Big]. \end{aligned}$$

Proof Let $G' = G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$. Clearly,

$$
\begin{aligned}
H^*_\lambda(G') &= \frac{1}{2} \sum_{B_{ij}, B_{kp} \in \mathscr{B}} d_{G'}(B_{ij}) d_{G'}(B_{kp}) d^\lambda_{G'}(B_{ij}, B_{kp}) \\
&= \frac{1}{2} \Bigg(\sum_{i=0}^{n-1} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} d_{G'}(B_{ij}) d_{G'}(B_{ip}) d^\lambda_{G'}(B_{ij}, B_{ip}) \\
&\quad + \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum_{j=0}^{r-1} d_{G'}(B_{ij}) d_{G'}(B_{kj}) d^\lambda_{G'}(B_{ij}, B_{kj}) \\
&\quad + \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} d_{G'}(B_{ij}) d_{G'}(B_{kp}) d^\lambda_{G'}(B_{ij}, B_{kp}) \\
&\quad + \sum_{i=0}^{n-1} \sum_{j=0}^{r-1} d_{G'}(B_{ij}) d_{G'}(B_{ij}) d^\lambda_{G'}(B_{ij}, B_{ij}) \Bigg). \qquad (2.1)
\end{aligned}
$$

We shall obtain the sums of (2.1) are separately.

First we calculate $A_1 = \sum_{i=0}^{n-1} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} d_{G'}(B_{ij}) d_{G'}(B_{ip}) d^\lambda_{G'}(B_{ij}, B_{ip})$. For that first we find T'_1. By Lemma 2.4, we have

$$
\begin{aligned}
T'_1 &= d_{G'}(B_{ij}) d_{G'}(B_{ip}) \\
&= \Big(d_G(v_i)(n_0 - m_j + 1) + (n_0 - m_j) \Big) \Big(d_G(v_i)(n_0 - m_p + 1) + (n_0 - m_p) \Big) \\
&= \Big((n_0 + 1)^2 - (n_0 + 1) m_j - (n_0 + 1) m_p + m_j m_p \Big) d^2_G(v_i) \\
&\quad + \Big(2n_0(n_0 + 1) - (2n_0 + 1) m_j - (2n_0 + 1) m_p + 2 m_j m_p \Big) d_G(v_i) \\
&\quad + \Big(n_0^2 - n_0 m_p - n_0 m_j + m_j m_p \Big).
\end{aligned}
$$

From Lemma 2.3, we have $d^\lambda_{G'}(B_{ij}, B_{ip}) = m_j m_p$. Thus

$$
\begin{aligned}
& T'_1 d^\lambda_{G'}(B_{ij}, B_{ip}) = T'_1 m_j m_p \\
&= \Big((n_0 + 1)^2 m_j m_p - (n_0 + 1) m_j^2 m_p - (n_0 + 1) m_j m_p^2 + m_j^2 m_p^2 \Big) d^2_G(v_i) \\
&\quad + \Big(2n_0(n_0 + 1) m_j m_p - (2n_0 + 1) m_j^2 m_p - (2n_0 + 1) m_j m_p^2 + 2 m_j^2 m_p^2 \Big) d_G(v_i) \\
&\quad + \Big(n_0^2 m_j m_p - n_0 m_j^2 m_p - n_0 m_j m_p^2 + m_j^2 m_p^2 \Big).
\end{aligned}
$$

By Remark 2.1, we have

$$\begin{aligned} T_1 &= \sum_{\substack{j,p=0 \\ j\neq p}}^{r-1} T_1' d^{\lambda}_{G'}(B_{ij}, B_{ip}) \\ &= \Big(2q^2+2qn_0+2n_0t+2q+4\tau+6t\Big) d^2_G(v_i) \\ &\quad +\Big(2qn_0+4n_0t-4q^2+6t+8\tau\Big) d_G(v_i) \\ &\quad +\Big(2n_0t+2q^2+4\tau\Big). \end{aligned}$$

From the definition of the first Zagreb index, we have

$$\begin{aligned} A_1 &= \sum_{i=0}^{n-1} T_1 \\ &= \Big(2q^2+2qn_0+2n_0t+2q+4\tau+6t\Big) M_1(G) \\ &\quad +2m\Big(2qn_0+4n_0t-4q^2+6t+8\tau\Big) \\ &\quad +n\Big(2n_0t+2q^2+4\tau\Big). \end{aligned}$$

Next we obtain $A_2 = \sum_{\substack{i,k=0 \\ i\neq k}}^{n-1} \sum_{j=0}^{r-1} d_{G'}(B_{ij}) d_{G'}(B_{kj}) d^{\lambda}_{G'}(B_{ij}, B_{kj})$. For that first we find T_2'. By Lemma 2.4, we have

$$\begin{aligned} T_2' &= d_{G'}(B_{ij}) d_{G'}(B_{kj}) \\ &= \Big(d_G(v_i)(n_0-m_j+1)+(n_0-m_j)\Big)\Big(d_G(v_k)(n_0-m_j+1)+(n_0-m_j)\Big) \\ &= (n_0-m_j+1)^2 d_G(v_i) d_G(v_k) + (n_0-m_j)(n_0-m_j+1)(d_G(v_i)+d_G(v_k)) \\ &\quad +(n_0-m_j)^2. \end{aligned}$$

Thus

$$\begin{aligned} A_2 &= \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i\neq k}}^{n-1} T_2' d^{\lambda}_{G'}(B_{ij}, B_{kj}) \\ &= \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\in E(G)}}^{n-1} T_2' d^{\lambda}_{G'}(B_{ij}, B_{kj}) + \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\notin E(G)}}^{n-1} T_2' d^{\lambda}_{G'}(B_{ij}, B_{kj}) \end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned} A_2 &= \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\in E(G)}}^{n-1} T_2'\Big(1-2^{\lambda}+2^{\lambda}m_j\Big)m_j + \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\notin E(G)}}^{n-1} T_2'm_j^2\ d_G^{\lambda}(v_i,v_k), \\ &= \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\in E(G)}}^{n-1} T_2'\Big(\Big(1-2^{\lambda}+2^{\lambda}m_j\Big)m_j + m_j^2 - m_j^2\Big) + \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\notin E(G)}}^{n-1} T_2'm_j^2\ d_G^{\lambda}(v_i,v_k) \\ &= \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\in E(G)}}^{n-1} T_2'(2^{\lambda}-1)\Big(m_j^2 - m_j\Big) + \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i\neq k}}^{n-1} T_2'm_j^2\ d_G^{\lambda}(v_i,v_k) \\ &= S_1 + S_2, \end{aligned} \tag{2.2}$$

where S_1 and S_2 are the sums of the terms of the above expression, in order.

Now we calculate S_1. For that first we find the following.

$$\begin{aligned} (2^{\lambda}-1)T_2'\Big(m_j^2-m_j\Big) &= (2^{\lambda}-1)\Big[\Big(m_j^4-(2n_0+3)m_j^3+(n_0^2+4n_0+3)m_j^2 \\ &\quad -(n_0+1)^2m_j\Big)d_G(v_i)d_G(v_k) \\ &\quad +\Big(m_j^4-(2n_0+2)m_j^3+(n_0^2+3n_0+1)m_j^2-(n_0^2+n_0)m_j\Big)(d_G(v_i)+d_G(v_k)) \\ &\quad +\Big(m_j^4-(2n_0+1)m_j^3+(n_0^2+2n_0)m_j^2-n_o^2m_j\Big)\Big]. \end{aligned}$$

By Remark 2.1, we have

$$\begin{aligned} T_2'' &= \sum_{j=0}^{r-1}(2^{\lambda}-1)T_2'\Big(m_j^2-m_j\Big) \\ &= (2^{\lambda}-1)\Big[\Big(2q^2-2n_0t-4\tau-3n_0^3+10n_0q-18t+n_0^2-6q-n_0\Big)d_G(v_i)d_G(v_k) \\ &\quad +\Big(2q^2-4\tau-2n_0t-6t-2q\Big)(d_G(v_i)+d_G(v_k)) \\ &\quad +\Big(2q^2-4\tau-2n_0t-n_0q-3t\Big)\Big]. \end{aligned}$$

Hence

$$\begin{aligned} S_1 &= \sum_{\substack{i,k=0 \\ i\neq k \\ v_iv_k\in E(G)}}^{n-1} T_2'' \\ &= (2^{\lambda}-1)\Big[\Big(2q^2-2n_0t-4\tau-3n_0^3+10n_0q-18t+n_0^2-6q-n_0\Big)2M_2(G) \\ &\quad +\Big(2q^2-4\tau-2n_0t-6t-2q\Big)2M_1(G) \\ &\quad +2m\Big(2q^2-4\tau-2n_0t-n_0q-3t\Big)\Big]. \end{aligned}$$

Next we calculate S_2. For that we need the following.

$$\begin{aligned} T_2'm_j^2 &= \Big(m_j^4 - (2n_0+2)m_j^3 + (n_0+1)^2 m_j^2\Big) d_G(v_i)d_G(v_k) \\ &\quad + \Big(m_j^4 - (2n_0+1)m_j^3 + (n_0^2+n_0)m_j^2\Big)(d_G(v_i)+d_G(v_k)) \\ &\quad + \Big(m_j^4 - 2n_0 m_j^3 + n_0^2 m_j^2\Big). \end{aligned}$$

By Remark 2.1, we have

$$\begin{aligned} T_2 &= \sum_{j=0}^{r-1} T_2'm_j^2 \\ &= \Big(2q^2 - 4\tau - 2n_0 t - 6t + 2n_0 q - 2q + n_0^2\Big) d_G(v_i)d_G(v_k) \\ &\quad + \Big(2q^2 - 4\tau - 2n_0 t - 3t + n_0 q\Big)(d_G(v_i)+d_G(v_k)) \\ &\quad + \Big(2q^2 - 4\tau - 2n_0 t\Big). \end{aligned}$$

From the definitions of H_λ^*,H_λ and W_λ, we obtain

$$\begin{aligned} S_2 &= \sum_{\substack{i,k=0 \\ i\neq k}}^{n-1} T_2 d_G^\lambda(v_i,v_k) \\ &= 2\Big(2q^2 - 4\tau - 2n_0 t - 6t + 2n_0 q - 2q + n_0^2\Big) H_\lambda^*(G) \\ &\quad + 2\Big(2q^2 - 4\tau - 2n_0 t - 3t + n_0 q\Big) H_\lambda(G) \\ &\quad + 2\Big(2q^2 - 4\tau - 2n_0 t\Big) W_\lambda(G). \end{aligned}$$

Now we calculate $A_3 = \sum\limits_{\substack{i,k=0 \\ i\neq k}}^{n-1} \sum\limits_{\substack{j,p=0 \\ j\neq p}}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{kp})d_{G'}^\lambda(B_{ij},B_{kp})$. For that first we compute T_3'. By Lemma 2.4, we have

$$\begin{aligned} T_3' &= d_{G'}(B_{ij})d_{G'}(B_{kp}) \\ &= \Big(d_G(v_i)(n_0 - m_j + 1) + (n_0 - m_j)\Big)\Big(d_G(v_k)(n_0 - m_p + 1) + (n_0 - m_p)\Big) \\ &= d_G(v_i)d_G(v_k)(n_0 - m_j + 1)(n_0 - m_p + 1) + d_G(v_i)(n_0 - m_j + 1)(n_0 - m_p) \\ &\quad + d_G(v_k)(n_0 - m_p + 1)(n_0 - m_j) + (n_0 - m_j)(n_0 - m_p). \end{aligned}$$

Since the distance between B_{ij} and B_{kp} is $m_jm_pd^{\lambda}_G(v_i,v_k)$. Thus

$$\begin{aligned} T'_3m_jm_p &= d_G(v_i)d_G(v_k)\Big((n_0^2+2n_0+1)m_jm_p-(n_0+1)m_j^2m_p-(n_0+1)m_jm_p^2+m_j^2m_p^2\Big)\\ &+d_G(v_i)\Big((n_0^2+n_0)m_jm_p-(n_0+1)m_jm_p^2-n_0m_j^2m_p+m_j^2m_p^2\Big)\\ &+d_G(v_k)\Big((n_0^2+n_0)m_jm_p-n_0m_jm_p^2-(n_0+1)m_j^2m_p+m_j^2m_p^2\Big)\\ &+\Big(n_0^2m_jm_p-n_0m_jm_p^2-n_0m_j^2m_p+m_j^2m_p^2\Big). \end{aligned}$$

By Remark 2.1, we obtain

$$\begin{aligned} T_3 &= \sum_{\substack{j,\,p\,=\,0,\\ j\neq p}}^{r-1} T'_3m_jm_p = d_G(v_i)d_G(v_k)\Big(2n_0q+2n_0t+2q+2q^2+6t+4\tau\Big)\\ &+(d_G(v_i)+d_G(v_k))\Big(qn_0+2n_0t+3t+2q^2+4\tau\Big)\\ &+\Big(2n_0t+2q^2+4\tau\Big). \end{aligned}$$

Hence

$$\begin{aligned} A_3 &= \sum_{\substack{i,\,k\,=\,0\\ i\neq k}}^{n-1} T_3d^{\lambda}_G(v_i,v_k) = 2H^*_{\lambda}(G)\Big(2n_0q+2n_0t+2q+2q^2+6t+4\tau\Big)\\ &+2H_{\lambda}(G)\Big(qn_0+2n_0t+3t+2q^2+4\tau\Big)\\ &+2W_{\lambda}(G)\Big(2n_0t+2q^2+4\tau\Big). \end{aligned}$$

Finally, we obtain $A_4=\sum\limits_{i\,=\,0}^{n-1}\sum\limits_{j\,=\,0}^{r-1} d_{G'}(B_{ij})d_{G'}(B_{ij})d^{\lambda}_{G'}(B_{ij},B_{ij})$. For that first we calculate T'_4. By Lemma 2.4, we have

$$\begin{aligned} T'_4 &= d_{G'}(B_{ij})d_{G'}(B_{ij})\\ &= \Big(d_G(v_i)(n_0-m_j+1)+(n_0-m_j)\Big)^2\\ &= d^2_G(v_i)(n_0-m_j+1)^2+2d_G(v_i)(n_0-m_j)(n_0-m_j+1)+(n_0-m_j)^2. \end{aligned}$$

From Lemma 2.3, the distance between $(B_{ij}$ and $(B_{ij}$ is $m_j(m_j-1)$. Thus

$$\begin{aligned} T'_4m_j(m_j-1) &= d^2_G(v_i)\Big(m_j^4-(2n_0+3)m_j^3+((n_0+1)^2+2)m_j^2-(n_0+1)^2m_j\Big)\\ &+2d_G(v_i)\Big(m_j^4-(2n_0+2)m_j^3+(n_0^2+3n_0+1)m_j^2-(n_0^2+n_0)m_j\Big)\\ &+\Big(m_j^4-(2n_0+1)m_j^3+(n_0^2+2n_0)m_j^2-n_0^2m_j\Big). \end{aligned}$$

By Remark 2.1, we obtain

$$\begin{aligned} T_4 &= \sum_{j=0}^{r-1} T_4' m_j(m_j-1) \\ &= d_G^2(v_i)\Big(4n_0^2q - 2n_0^3 - 3n_0^2 - 2n_0t + 5n_0q - 9t - 6q - n_0 - 4\tau\Big) \\ &\quad +2d_G(v_i)\Big(2q^2 - 2n_0t - 2q - 6t - 4\tau\Big) \\ &\quad +\Big(2q^2 - 2n_0t - n_0q - 3t - 4\tau\Big). \end{aligned}$$

Hence

$$\begin{aligned} A_4 &= \sum_{i=0}^{n-1} T_4 d_{G'}^{\lambda}(B_{ij}, B_{ij}) \\ &= M_1(G)\Big(4n_0^2q - 2n_0^3 - 3n_0^2 - 2n_0t + 5n_0q - 9t - 6q - n_0 - 4\tau\Big) \\ &\quad +4m\Big(2q^2 - 2n_0t - 2q - 6t - 4\tau\Big) \\ &\quad +n\Big(2q^2 - 2n_0t - n_0q - 3t - 4\tau\Big). \end{aligned}$$

Adding A_1,S_1,S_2,A_3 and A_4 we get the required result. □

If we set $\lambda = 1$ in Theorem 2.5, we obtain the product degree distance of $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$.

Theorem 2.6 *Let G be a connected graph with n vertices and m edges. Then*

$$\begin{aligned} &DD_*(G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}) \\ &= (4q^2 + n_0^2 + 4n_0q)DD_*(G) + 4q^2W(G) \\ &\quad +(4q^2 + 2n_0q)DD(G) + \frac{n}{2}(4q^2 - n_0q - 3t) \\ &\quad +\frac{M_1(G)}{2}\Big[4n_0^2q + 6q^2 - 4n_0t - 15t + 7n_0q - n_0 - 3n_0^2 - 2n_0^3 - 8\tau\Big] \\ &\quad +m\Big[n_0q - 2n_0t + 2q^2 - 9t - 4q - 4\tau\Big] \\ &\quad +M_2(G)\Big[2q^2 - 2n_0t - 3n_0^3 + 10n_0q + n_0^2 - 18t - 6q - n_0 - 4\tau\Big] \end{aligned}$$

for $r \geq 2$.

Setting $\lambda = -1$ in Theorem 2.5, we obtain the reciprocal product degree distance of $G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}$.

Theorem 2.7 *Let G be a connected graph with n vertices and m edges. Then*

$$\begin{aligned} &RDD_*(G \boxtimes K_{m_0, m_1, \cdots, m_{r-1}}) \\ &= (4q^2 + n_0^2 + 4n_0q)RDD_*(G) + 4q^2H(G) \\ &\quad +(4q^2 + 2n_0q)RDD(G) + \frac{n}{2}(4q^2 - n_0q - 3t) \end{aligned}$$

$$+\frac{M_1(G)}{2}\Big[4n_0^2q+2n_0t+3t+7n_0q-n_0-3n_0^2-2n_0^3-2q+4\tau\Big]$$
$$+m\Big[\frac{5n_0q}{2}+n_0t-q^2-\frac{9t}{2}-4q+2\tau\Big]$$
$$-\frac{M_2(G)}{2}\Big[2q^2-2n_0t-3n_0^3+10n_0q+n_0^2-18t-6q-n_0-4\tau\Big]$$

for $r \geq 2$.

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