Question 476 : A Trigonometric Integral

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02-09-2018 13:07:23

ABSTRACT. Some remarks on a trigonometric integral

The Number Pi

$$\pi = 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...\right)=3.1415926535... \quad (1)$$

A Formula

$$2\pi = \int_0^{25} \sin^{-1}\sqrt{\frac{5-\sqrt{5+4\sqrt{x}}}{8}}\,dx - \int_0^{25/16} \sin^{-1}\sqrt{\frac{5-\sqrt{5-4\sqrt{x}}}{8}}\,dx + \\ + \int_0^{25/16} \sin^{-1}\sqrt{\frac{5+\sqrt{5-4\sqrt{x}}}{8}}\,dx - \int_0^{1} \sin^{-1}\sqrt{\frac{5+\sqrt{5+4\sqrt{x}}}{8}}\,dx \quad (2)$$

Change of Variables

$$2\pi = 16\int_0^{\sqrt{(5-\sqrt{5})/8}} x\left(5-8x^2\right)\left(5-20x^2+16x^4\right)\sin^{-1}x\,dx + \\ +16\int_{\sqrt{(5-\sqrt{5})/8}}^{\sqrt{10}/4} x\left(5-8x^2\right)\left(5-20x^2+16x^4\right)\sin^{-1}x\,dx + \\ +16\int_{\sqrt{10}/4}^{\sqrt{(5+\sqrt{5})/8}} x\left(5-8x^2\right)\left(5-20x^2+16x^4\right)\sin^{-1}x\,dx + \\ +16\int_{\sqrt{(5+\sqrt{5})/8}}^{1} x\left(5-8x^2\right)\left(5-20x^2+16x^4\right)\sin^{-1}x\,dx \quad (3)$$

Simplify (3)

$$2\pi = 16\int_0^1 x\left(5-8x^2\right)\left(5-20x^2+16x^4\right)\sin^{-1}x\,dx \tag{4}$$

Change of Variables

$$2\pi = 16\int_0^{\pi/2} x\left(5\sin x-20\sin^3 x+16\sin^5 x\right)\left(5-8\sin^2 x\right)\cos x\,dx \tag{5}$$

Integration by Parts

- Integration by parts:

$$\int u\,dv = uv - \int v\,du \tag{6}$$

$$u = x \quad , \quad dv = \left(5\sin x-20\sin^3 x+16\sin^5 x\right)\left(5-8\sin^2 x\right)\cos x\,dx \tag{7}$$

$$2\pi = 12\pi + \int_0^{\pi/2}\left(256\sin^8 x-640\sin^6 x+560\sin^4 x-200\sin^2 x\right)dx \tag{8}$$

$$2\pi = 12\pi + \int_0^{\pi/2}\left(\left(5-20\sin^2 x+16\sin^4 x\right)^2-25\right)dx \tag{9}$$

$$\frac{5\pi}{2} = \int_0^{\pi/2}\left(5-20\sin^2 x+16\sin^4 x\right)^2 dx = \int_0^{\pi/2}\left(\frac{\sin 5x}{\sin x}\right)^2 dx \tag{10}$$

An Integral in Gradshteyn and Ryzhik

$$\int_0^{\pi/2}\left(\frac{\sin ax}{\sin x}\right)^2 dx = \frac{a\pi}{2} - \frac{1}{2}\sin a\pi\left[2a\beta(a)-1\right] \quad ,a>0 \quad (*) \tag{11}$$

$(*)$ Gradshteyn and Ryzhik, seventh edition,2007,page 396,formula 3.624.6 .

Formula (4) $\to$ Series

$$\frac{\pi}{8}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{25}{2n+3}-\frac{140}{2n+5}+\frac{240}{2n+7}-\frac{128}{2n+9}\right) \tag{12}$$

Other Formulas

$$\int_0^{\pi/4}\left(\frac{\sin 5x}{\sin x}\right)^2 dx=\frac{10}{3}+\frac{5}{4}\pi \tag{13}$$

$$\int_0^{25}\sin^{-1}\sqrt{\frac{5-\sqrt{5+4\sqrt{x}}}{8}}\,dx-\int_0^{1}\sin^{-1}\sqrt{\frac{5-\sqrt{5-4\sqrt{x}}}{8}}\,dx=\frac{10}{3}+\pi \tag{14}$$

$$\sqrt{2}\left(\pi+\frac{10}{3}\right)=8\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-3n}}{2n+1}\left(\frac{25}{2n+3}-\frac{70}{2n+5}+\frac{60}{2n+7}-\frac{16}{2n+9}\right) \tag{15}$$

$$\int_0^{\pi/6}\left(\frac{\sin 5x}{\sin x}\right)^2 dx=\frac{21}{8}\sqrt{3}+\frac{5}{6}\pi \tag{16}$$

$$\int_1^{25}\sin^{-1}\sqrt{\frac{5-\sqrt{5+4\sqrt{x}}}{8}}\,dx=\frac{21}{8}\sqrt{3}+\frac{2}{3}\pi \tag{17}$$

$$\frac{21}{16}\sqrt{3}+\frac{\pi}{3}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-4n}}{2n+1}\left(\frac{25}{2n+3}-\frac{35}{2n+5}+\frac{15}{2n+7}-\frac{2}{2n+9}\right) \tag{18}$$

References

1. Gradshteyn, I.S. and Ryzhik, I.M.: Table of Integrals,Series, and Products. Seventh Edition. Edited by Alan Jeffrey and Daniel Zwillinger. Academic Press, 2007.