Complex Programming

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Abstract

We introduce and suggest to research a special class of optimization problems, wherein an objective function is a real-valued complex variables function and constraints comprising complex-valued complex variables functions.

Keywords: optimization, target function, constraints, minimizers, complex plane, real-valued complex variables target.

1. Introduction

Its well-known that an optimization problem can be represented in the following way:

Given: a function f: $\mathbf{A} \rightarrow \mathbf{R}$ from some set **A** to the real numbers
Sought*:* an element $x_0 \in \mathbf{A}$ such that $f(x_0) \leq f(x)$ for all $x \in \mathbf{A}$ ("minimization") or such that $f(x_0) \geq f(x)$ for all $x \in \mathbf{A}$ ("maximization").

The following are the major subfields of the Optimization Theory: convex programming, non-linear programming, integer programming, quadratic programming, combinatorial optimization, stochastic optimization, etc (see, e.g., Boyd and Vandenberghe [1], Hemmecke et al. [2]).

Typically, **A** is some subset of the Euclidean space $\mathbf{R}^n$, specified by a set of constraints and the function f is called an objective function, target function.

The case, when **A** is some subset of two-dimensional complex plane and target function f: $\mathbf{C} \to \mathbf{R}$ is real-valued complex variable function is not investigated yet. Accordingly, **A** is supposed to be defined by complex-valued, complex variable constraints. In more general models, function f: $\mathbf{C}^n \to \mathbf{R}$ is supposed to be defined on the multi-dimensional complex space.

The purpose of this paper is to introduce and describe such optimization problems of f: $\mathbf{C} \to \mathbf{R}$ and f: $\mathbf{C}^n \to \mathbf{R}$ target functions over subsets of C and $\mathbf{C}^n$.

2. Complex Programming

Let | z | be the absolute value of a complex number z = Re(z) + Im(z)i = x + iy and arg(z) the argument of z(the principal value).
(See, e.g., Scheidemann [3], Shaw [4]).

Let us introduce and demonstrate optimization problems, defined in terms of complex numbers and functions.

Example 1.

cp1 = { minimize | z | subject to | z | ≥ 1 }, argmin(cp1) = {z: | z | = 1}.

Example 2.

cp2 = { minimize - Im(z) subject to | z | ≤ 1 }, argmin(cp2) = i.

Example 3.

cp3 = { minimize Re(z) subject to | z | ≤ 1 }, argmin(cp3) = -1.

Example 4.

cp4 = { maximize | z | subject to 0 ≤ Re(z) ≤ 1, to 0 ≤ Im(z) ≤ 1 },
argmax(cp4) = 1 + i.

Example 5.

cp5 = { maximize Re(z) + Im(z) subject to $0 \le \text{Re}(z) \le 1$, $0 \le \text{Im}(z) \le 1$ },
argmax(cp5) = 1 + i.

Example 6.

cp6 = { maximize Re(z) + Im(z) subject to $0 \le \text{Re}(z) \le 1$, $0 \le \text{Im}(z) \le 1$,
arg(z) = 0 }, argmax(cp6) = 1.

Example 7.

cp7 = { maximize Im(z) subject to $0 \le \text{Re}(z) \le 1$, $0 \le \text{Im}(z) \le 1$,
$\text{Im}(z) \le \text{Re}(z)$ }, argmax(cp7) = 1 + i.

Example 8.

cp8 = { maximize | z | subject to $\text{Im}(z) \ge \text{Re}^2(z)$, $\text{Re}(z) \ge \text{Im}^2(z)$ },
argmax(cp8) = 1 + i.

Example 9. Several complex variables.

cp9 = { maximize $| z_1 + z_2 |$ subject to $| z_1 | \le 1$, $| z_2 | \le 1$ }.

Example 10.

cp10 = { maximize $| z_1 + z_2 |$ subject to $| z_1 | \le 1$, $| z_2 | \le 1$, $\arg(z_1 z_2) \le \pi/4$ }.

Example 11. "Linear" Complex Programming.

cp11 = { maximize $| z_1 + \dots + z_n |$ subject to

$$| a_{11}z_1 + \dots + a_{1n}z_n | \le b_1,$$
$$\dots \quad \dots \quad \dots$$
$$| a_{m1}z_1 + \dots + a_{mn}z_n | \le b_m,\ z_i \in \mathbf{C},\ a_{ij} \in \mathbf{C},\ b_i \in \mathbf{R},\ n \in \mathbf{N},\ m \in \mathbf{N} \}.$$

Example 12.

cp12 = { maximize $| z_1 + \dots + z_n |$ subject to

$$\mathrm{Re}(a_{11}z_1 + \ldots + a_{1n}z_n) \leq b_1,$$
$$\ldots \quad \ldots \quad \ldots$$
$$\mathrm{Re}(a_{m1}z_1 + \ldots + a_{mn}z_n) \leq b_m,$$

$$\mathrm{Im}(a_{11}z_1 + \ldots + a_{1n}z_n) \leq c_1,$$
$$\ldots \quad \ldots \quad \ldots$$
$$\mathrm{Im}(a_{m1}z_1 + \ldots + a_{mn}z_n) \leq c_m,$$

$$z_i \in \mathbf{C},\ a_{ij} \in \mathbf{C},\ b_i \in \mathbf{R},\ c_i \in \mathbf{R},\ n \in \mathbf{N},\ m \in \mathbf{N}\ \}.$$

Example 13.

cp13 = **{** maximize $|z_1 + \ldots + z_n|$ subject to

$$\arg(a_{11}z_1 + \ldots + a_{1n}z_n) \leq b_1,$$
$$\ldots \quad \ldots \quad \ldots$$
$$\arg(a_{m1}z_1 + \ldots + a_{mn}z_n) \leq b_m,$$

$$\mathrm{Im}(a_{11}z_1 + \ldots + a_{1n}z_n) \leq c_1,$$
$$\ldots \quad \ldots \quad \ldots$$
$$\mathrm{Im}(a_{m1}z_1 + \ldots + a_{mn}z_n) \leq c_m,$$

$$z_i \in \mathbf{C},\ a_{ij} \in \mathbf{C},\ b_i \in \mathbf{R},\ c_i \in \mathbf{R},\ n \in \mathbf{N},\ m \in \mathbf{N}\ \}.$$

Example 14.

cp14 = **{** maximize $\arg(z_1 \ldots z_n)$ subject to

$$\mathrm{Re}(a_{11}z_1 + \ldots + a_{1n}z_n) \leq b_1,$$
$$\ldots \quad \ldots \quad \ldots$$
$$\mathrm{Re}(a_{m1}z_1 + \ldots + a_{mn}z_n) \leq b_m,$$

$$\mathrm{Im}(a_{11}z_1 + \ldots + a_{1n}z_n) \leq c_1,$$
$$\ldots \quad \ldots \quad \ldots$$
$$\mathrm{Im}(a_{m1}z_1 + \ldots + a_{mn}z_n) \leq c_m,$$

$\arg(z_i)\ ) \leq d_i$,

$z_i \in \mathbf{C}$, $a_{ij} \in \mathbf{C}$, $b_i \in \mathbf{R}$, $c_i \in \mathbf{R}$, $d_i \in \mathbf{R}$, $1 \leq i \leq n$, $n \in \mathbf{N}$, $m \in \mathbf{N}$ }.

Example 15. "Quadratic" Complex Programming.

cp15 = **{** maximize $| z_1^2 + ... + z_n^2 |$ subject to

$| a_{11}z_1 + ... + a_{1n}z_n | \leq b_1$,
... ...
$| a_{m1}z_1 + ... + a_{mn}z_n | \leq b_m$, $z_i \in \mathbf{C}$, $a_{ij} \in \mathbf{C}$, $b_i \in \mathbf{R}$, $n \in \mathbf{N}$, $m \in \mathbf{N}$ }.

Example 16. "Non-Linear" Complex Programming.

cp16 = **{** maximize $| e^z + \sin(\pi z) |$ subject to
$| \cos((\pi z) | \leq a$, $0 \leq \mathrm{Re}(z) \leq 1$, $0 \leq \mathrm{Im}(z) \leq 1$, $z \in \mathbf{C}$, $a \in \mathbf{R}$ }.

Example 17. "Integer" Complex Programming(Over Gaussian Integers).

cp17 = **{** maximize $| z_1^4 + ... + z_n^4 |$ subject to
$| a_{11}z_1 + ... + a_{1n}z_n | \leq b_1$,
... ...
$| a_{m1}z_1 + ... + a_{mn}z_n | \leq b_m$, $z_i \in \mathbf{C} \cap \mathbf{Z}$, $a_{ij} \in \mathbf{C}$, $b_i \in \mathbf{R}$, $n \in \mathbf{N}$, $m \in \mathbf{N}$ }.

Despite such optimization problems actually could be translated and considered in terms of optimization problems over the Euclidean space, it may be not so easy task(complexity problems, etc).

That is why, it would be preferable to develop specific, "direct" methods for complex programming problems using complex analysis.

The corresponding complexity evaluations for the Complex Programming problems would be developed: for example in binary encoded length of the coefficients.

References

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