Visualizing the distributions of the escape paths of quaternion fractals

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Abstract

The length, displacement, and magnitude distributions of the escape paths of the points in some quaternion fractal sets are visualized.

1 Length, displacement, and magnitude histograms

As discussed in [1], a 3D scalar field of quaternion magnitudes $|Z| = \sqrt{Z_x^2 + Z_y^2 + Z_z^2 + Z_w^2}$ results from calculating a quaternion fractal set when using a finite 3D lattice of regularly spaced points as input. Here we will visualize the distributions of the escape paths' lengths, displacements, and magnitudes for those points where $|Z|$ remains below the threshold of 4.0 during 8 iterations. A small C++ code is given in the next section, which shows how to perform the iteration process.

See Fig. 1 for a simplified 2D illustration of length, displacement, and magnitude per escape path. The escape paths used to calculate the histograms given in this paper are in 4D.

The histograms in Figs. 2 - 10 together show how the maximum length is generally greater than the maximum displacement. This is also generally the case for the length and displacement per individual escape path, which is generally indicative of curved escape paths – the escape paths generally meander because there are bends.

In a lot of the cases (but not all cases) a curved escape path forms a loop (see pages 7, 12, and 13 in [2]), which gives rise to the commonly-used name 'orbit' (see [3]). However, most of the time the loop is not quite exact, and so all 'maximum iteration count'+1 = 9 points per escape path end up being distinct. This means that when a curved escape path forms an orbit, the orbit is generally not quite perfect – the curved escape path is likely jittery, or precessing, or spiral-shaped, or all three.

Are there new fractals to be discovered by limiting the length or displacement, like we do with magnitude? It turns out that the answer is not really. In most cases, a similar shape is produced by all three criteria. This is interesting because all three criteria are represented by vastly different (but equally beautiful) histograms. Analysis of this behaviour will be the focus of future research.

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2 Core C++ code

The following code performs the iteration process:

```cpp
float quaternion_fractal_set::iterate(
    vector<vector_4> &escape_path_points,
    const quaternion &seed_Z,
    const short unsigned int maximum_iteration_count = 8,
    const float threshold = 4.0f)
{
    // seed_Z constains:
    // 1) a 3D lattice location for seed_Z.xyz, and
    // 2) a constant for seed_Z.w
    Z = seed_Z;

    escape_path_points.clear();

    // Add first point to escape path
    vector_4 p;
    p.x = Z.x;
    p.y = Z.y;
    p.z = Z.z;
    p.w = Z.w;
    escape_path_points.push_back(p);

    // Use squared values to avoid using sqrtf() during the iteration
    float magnitude_squared = Z.self_dot();
    const float threshold_squared = threshold*threshold;

    for(short unsigned int i = 0; i < maximum_iteration_count; i++)
    {
        // Iterative equation
        Z = Z*Z + C;

        // Add additional point(s) to escape path
        p.x = Z.x;
        p.y = Z.y;
        p.z = Z.z;
        p.w = Z.w;
        escape_path_points.push_back(p);

        magnitude_squared = Z.self_dot();

        // Abort early if magnitude tends toward infinity
        if(magnitude_squared >= threshold_squared)
            break;
    }

    return sqrtf(magnitude_squared);
}
```

The following code shows the quaternion implementation:

```cpp
class quaternion
{
public:

    // Constructors omitted for brevity ...

    inline float self_dot(void) const
    {
        return x*x + y*y + z*z + w*w;
    }

    quaternion operator*(const quaternion &right) const
    {
        quaternion ret;

        ret.x = x*right.x - y*right.y - z*right.z - w*right.w;
        ret.y = x*right.y + y*right.x + z*right.w - w*right.z;
        ret.z = x*right.z - y*right.w + z*right.x + w*right.y;
        ret.w = x*right.w + y*right.z - z*right.y + w*right.x;

        return ret;
    }

    quaternion operator+(const quaternion &right) const
    {
        quaternion ret;

        ret.x = x + right.x;
        ret.y = y + right.y;
        ret.z = z + right.z;
        ret.w = w + right.w;

        return ret;
    }

    float x, y, z, w;
};

// For iterative equations like Z = sin(Z) + C*sin(Z)
quaternion sin(const quaternion &in)
{
    quaternion ret;

    const float mag_vector = sqrtf(in.y*in.y + in.z*in.z + in.w*in.w);

    ret.x = sin(in.x) * cosh(mag_vector);
    ret.y = cos(in.x) * sinh(mag_vector) * in.y / mag_vector;
    ret.z = cos(in.x) * sinh(mag_vector) * in.z / mag_vector;
    ret.w = cos(in.x) * sinh(mag_vector) * in.w / mag_vector;

    return ret;
}
```
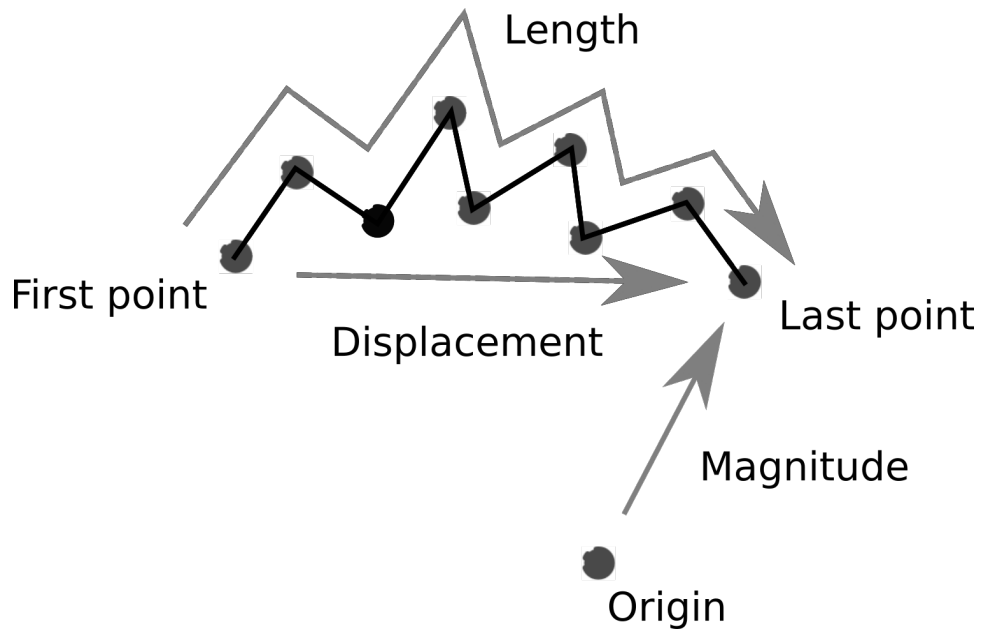
Figure 1: Length, displacement, and magnitude of a meandering escape path that consists of 'maximum iteration count'+1 = 9 points.

References

[1] Halayka S. *Some visually interesting non-standard quaternion fractal sets* Chaos, Solitons & Fractals Vol. 41, Issue 5

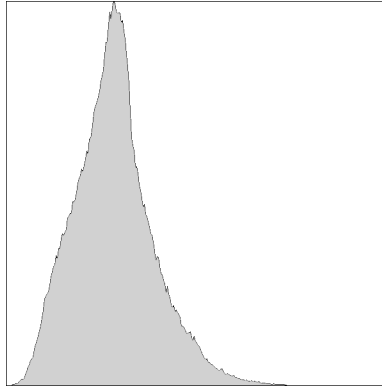[2] https://iquilezles.org/www/articles/arquimedes/arquimedes.pdf

[3] http://math.bu.edu/DYSYS/FRACGEOM/FRACGEOM.html

[4] http://www.theworld.com/%7Esweetser/quaternions/intro/tools/tools.html

Figure 2: Lengths of $Z' = Z^2 + C$, where $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$. For this histogram the maximum length is 21.2391; mean: 7.05497; mode: 5.95688.
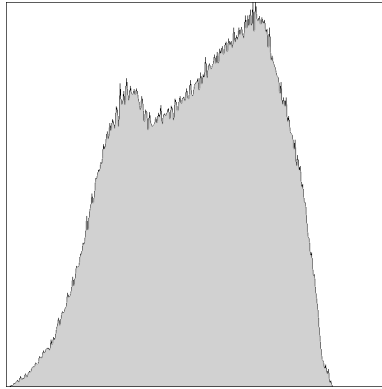
Figure 3: Displacements of $Z' = Z^2 + C$, where $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$. For this histogram the maximum displacement is 2.36506; mean: 1.38231; mode: 1.53986.
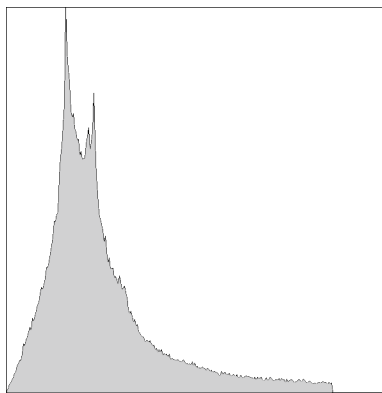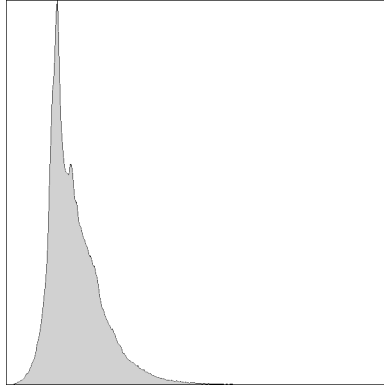

Figure 4: Magnitudes of $Z' = Z^2 + C$, where $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$. For this histogram the maximum magnitude is 3.99997; mean: 1.18208; mode 0.614353.

Figure 5: Lengths of $Z' = Z^5 + C$, where $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$. For this histogram the maximum length is 14.8154; mean: 3.15715; mode: 1.97868.
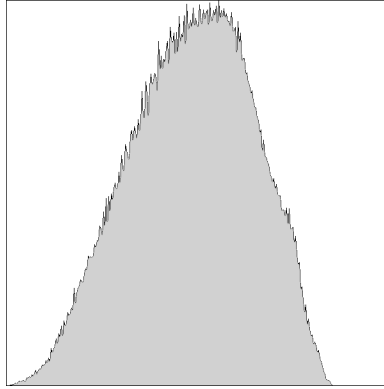

Figure 6: Displacements of $Z' = Z^5 + C$, where $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$. For this histogram the maximum displacement is 2.04231; mean: 1.16552; mode: 1.13196.
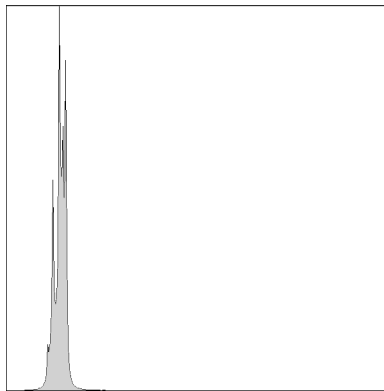

Figure 7: Magnitudes of $Z' = Z^5 + C$, where $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$. For this histogram the maximum magnitude is 3.99525; mean: 0.667052; mode 0.546928.
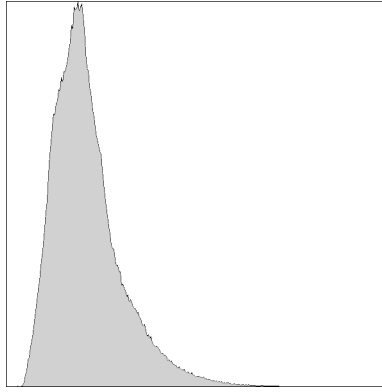
Figure 8: Lengths of $Z' = \sin(Z) + C\sin(Z)$, where $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$. For this histogram the maximum length is 28.7689; mean: 7.19823; mode: 5.37916.
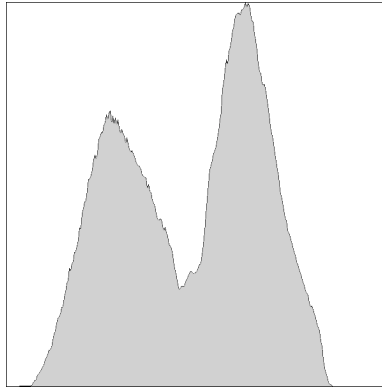

Figure 9: Displacements of $Z' = \sin(Z) + C\sin(Z)$, where $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$. For this histogram the maximum displacement is 3.93216; mean: 2.22363; mode: 2.45514.

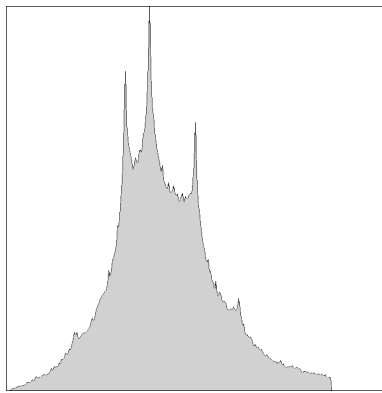
Figure 10: Magnitudes of $Z' = \sin(Z) + C\sin(Z)$, where $C_{xyzw} = 0.3, 0.5, 0.4, 0.2$. For this histogram the maximum magnitude is 3.99999; mean: 1.95538; mode 1.49582.