Dirac Theory's Breaches of Quantum Correspondence and Relativity; Nonrelativistic Pauli Theory's Unique Relativistic Extension

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Abstract

A single-particle Hamiltonian independent of the particle's coordinate ensures the particle conserves momentum, i.e., is free. This free-particle Hamiltonian is completely determined by Lorentz covariance of its energy-momentum and the particle's rest-energy value; such a free particle has velocity which vanishes when its momentum vanishes. Dirac required his free-particle Hamiltonian to be inhomogeneously linear in momentum, which contrariwise produces velocity that is independent of momentum; he also required his Hamiltonian's square to equal the above relativistic Hamiltonian's square, forcing many observables to anticommute and breach the quantum correspondence principle, as well as forcing the speed of any Dirac "free particle" to be c times the square root of three, which remains true when the particle interacts electromagnetically. The quantum correspondence principle breach causes a Dirac "free particle" to exhibit spontaneous acceleration that becomes unbounded in the classical limit; an artificial "spin" is also made available. Unlike the Dirac Hamiltonian, the nonrelativistic Pauli Hamiltonian is free of unphysical anomalies. Its relativistic extension is worked out via Lorentz-invariant upgrade of its associated action functional at zero particle velocity, and is obtained in closed form when there is no applied magnetic field; when there is, a successive approximation scheme must be used.

Introduction

The *relativistic Hamiltonian* $H(\mathbf{p})$ *for a free particle* ensures *conservation of the particle's momentum* $\mathbf{p}$ through *its independence of the particle's coordinate* $\mathbf{r}$. The *Lorentz covariance of the Hamiltonian's associated energy-momentum four-vector* $H^\mu = (H(\mathbf{p}), c\mathbf{p})$ allows it to be worked out for an arbitrary value of $\mathbf{p}$ *from its value at* $\mathbf{p} = \mathbf{0}$ *where* $H^\mu = (H_0, \mathbf{0})$*, and* $H_0 = H(\mathbf{p} = \mathbf{0})$ *is the particle's rest energy*. We now Lorentz transform $(H_0, \mathbf{0})$ *from* an inertial frame where the free particle has zero momentum *to* an inertial frame where it has some arbitrary velocity $\dot{\mathbf{r}}$ such that $|\dot{\mathbf{r}}| < c$. Zero momentum *corresponds to zero velocity because as* $|\mathbf{p}| \to 0$*, we have the familiar nonrelativistic relation of velocity to momentum*, i.e., $\dot{\mathbf{r}} = (\mathbf{p}/m)$. The Lorentz transformation of $(H_0, \mathbf{0})$ *from* the inertial frame where the particle has zero velocity, $\dot{\mathbf{r}} = \mathbf{0}$, *to* the inertial frame where the particle has an arbitrary velocity $\dot{\mathbf{r}}$ such that $|\dot{\mathbf{r}}| < c$ is,

$$(H_0, \mathbf{0}) \to H_0 \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{-\frac{1}{2}} (1, (\dot{\mathbf{r}}/c)) = H^\mu = (H(\mathbf{p}), c\mathbf{p}). \tag{1a}$$

We read off from Eq. (1a) that,

$$\mathbf{p} = (H_0/c)(\dot{\mathbf{r}}/c) \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{-\frac{1}{2}}, \tag{1b}$$

which is readily inverted to obtain $(\dot{\mathbf{r}}/c)$,

$$(\dot{\mathbf{r}}/c) = (c\mathbf{p}/H_0) \left(1 + |c\mathbf{p}/H_0|^2\right)^{-\frac{1}{2}}, \tag{1c}$$

which permits us to in addition obtain,

$$\left(1 - |\dot{\mathbf{r}}/c|^2\right)^{-\frac{1}{2}} = \left(1 + |c\mathbf{p}/H_0|^2\right)^{\frac{1}{2}}. \tag{1d}$$

We now insert Eqs. (1d) and (1c) into Eq. (1a) in order to obtain H^μ in terms of $\mathbf{p}$ instead of in terms of $\dot{\mathbf{r}}$,

$$H^\mu = \left(H_0 \left(1 + |c\mathbf{p}/H_0|^2\right)^{\frac{1}{2}},\, c\mathbf{p}\right) = (H(\mathbf{p}),\, c\mathbf{p}), \tag{1e}$$

which yields the relativistic free-particle Hamiltonian $H(\mathbf{p})$,

$$H(\mathbf{p}) = H_0 \left(1 + |c\mathbf{p}/H_0|^2\right)^{\frac{1}{2}}, \tag{1f}$$

where, as pointed out in the paragraph preceding Eq. (1a), $H_0 = H(\mathbf{p} = \mathbf{0})$ is the particle's *rest energy*. Our relativistic free-particle energy $H(\mathbf{p})$ must be asymptotically consistent with nonrelativistic free-particle

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kinetic energy $\left(|\mathbf{p}|^2/(2m)\right)$ as $|\mathbf{p}| \to 0$. Therefore we can determine the free-particle rest energy H_0 by setting the following $|\mathbf{p}| \to 0$ asymptotic result,

$$(H(\mathbf{p}) - H_0) \sim \left(|c\mathbf{p}|^2/(2H_0)\right), \tag{1g}$$

equal to the nonrelativistic free-particle kinetic energy $\left(|\mathbf{p}|^2/(2m)\right)$, which yields,

$$H_0 = mc^2, \tag{1h}$$

and this together with Eq. (1f) implies that,

$$H(\mathbf{p}) = mc^2 \left(1 + |\mathbf{p}/(mc)|^2\right)^{\frac{1}{2}} = \left(m^2c^4 + |c\mathbf{p}|^2\right)^{\frac{1}{2}}. \tag{1i}$$

We now use $H(\mathbf{p})$ together with the Heisenberg equation of motion to calculate the relativistic free-particle velocity $\dot{\mathbf{r}}$,

$$\dot{\mathbf{r}} = (-i/\hbar)\left[\mathbf{r},\, mc^2 \left(1 + |\mathbf{p}/(mc)|^2\right)^{\frac{1}{2}}\right] = (\mathbf{p}/m)\left(1 + |\mathbf{p}/(mc)|^2\right)^{-\frac{1}{2}}, \tag{2a}$$

whose $|\mathbf{p}| \to 0$ asymptotic form is,

$$\dot{\mathbf{r}} \sim (\mathbf{p}/m) \text{ as } |\mathbf{p}| \to 0, \tag{2b}$$

in agreement with the nonrelativistic result for $\dot{\mathbf{r}}$.

By using the fact that $H_0 = mc^2$, we can see that Eq. (1c) above *already expresses the result which has just been given by* Eq. (2a). In Eq. (2a) that result was obtained by using a Heisenberg equation of motion, whereas in Eq. (1c) it emerges in the course of a Lorentz transformation.

We also note that the $\dot{\mathbf{r}}$ of Eq. (2a) *is parallel to* $\mathbf{p}$, which implies that since the momentum $\mathbf{p}$ of a relativistic free particle is conserved, namely that $d\mathbf{p}/dt = \mathbf{0}$, *its orbital angular momentum* $\mathbf{L} = (\mathbf{r} \times \mathbf{p})$ *is conserved as well* because,

$$d\mathbf{L}/dt = d(\mathbf{r} \times \mathbf{p})/dt = (\dot{\mathbf{r}} \times \mathbf{p}) + (\mathbf{r} \times (d\mathbf{p}/dt)) = \mathbf{0}. \tag{2c}$$

We *further* note that *the three components* of the Eq. (2a) velocity $\dot{\mathbf{r}}$ of a relativistic free particle mutually *commute* with each other.

Finally, we can see from perusing Eq. (2a) that since $\mathbf{p}$ is conserved because we are dealing with a free particle, it is clear that because $\dot{\mathbf{r}}$ depends on only $\mathbf{p}$ and the constants m and c, $\dot{\mathbf{r}}$ will be conserved. Since $H(\mathbf{p}) = \left(m^2c^4 + |c\mathbf{p}|^2\right)^{\frac{1}{2}}$ likewise only depends on $\mathbf{p}$ and the constants m and c, that result also immediately follows from the Heisenberg equation of motion,

$$\ddot{\mathbf{r}} = (-i/\hbar)[\dot{\mathbf{r}},\, H(\mathbf{p})] = (-i/\hbar)\left[(\mathbf{p}/m)\left(1 + |\mathbf{p}/(mc)|^2\right)^{-\frac{1}{2}},\, \left(m^2c^4 + |c\mathbf{p}|^2\right)^{\frac{1}{2}}\right] = \mathbf{0}, \tag{2d}$$

which expresses Newton's First Law for our relativistic free particle, namely that it doesn't undergo spontaneous acceleration.

We now turn to *comparison* of the Dirac "relativistic" free-particle Hamiltonian $H_D(\mathbf{p})$ and its consequences with those of the *actual* relativistic free-particle Hamiltonian $H(\mathbf{p}) = \left(m^2c^4 + |c\mathbf{p}|^2\right)^{\frac{1}{2}}$ of Eq. (1i), which we have just developed and discussed. The *central idea* which guided Dirac's 1928 development of his purportedly "relativistic" free-particle Hamiltonian operator $H_D(\mathbf{p})$ was his *intuitive impression* that the resulting *free-particle* Schrödinger equation,

$$i\hbar\partial\psi/\partial t = H_D(\mathbf{p})\psi, \tag{3a}$$

(whose Hamiltonian $H_D(\mathbf{p})$ is of course *independent* of $\mathbf{r}$ to render the particle's momentum *constant* in accord with the particle's being *free*), *must be space-time symmetric in configuration representation in order to accord with special relativity* [1]. Since in configuration representation, $\mathbf{p}\psi$ is given by,

$$\mathbf{p}\psi = -i\hbar\nabla_{\mathbf{r}}\psi, \tag{3b}$$

Dirac *specifically implemented* his somewhat vague *intuitive impression* that the Eq. (3a) free-particle Schrödinger equation is space-time symmetric by *postulating* that $H_D(\mathbf{p})$ *is inhomogeneously linear in* $\mathbf{p}$, namely that [1, 2, 3, 4],

$$H_D(\mathbf{p}) = c\vec{\alpha} \cdot \mathbf{p} + \beta mc^2, \tag{3c}$$

where $\vec{\alpha}$ and β are, of course, Hermitian, dimensionless and *independent of* $\mathbf{p}$ *and* $\mathbf{r}$.

The Heisenberg equations of motion with this $H_D(\mathbf{p})$ then yield,

$$\dot{\mathbf{p}} = (-i/\hbar)[\mathbf{p},\, H_D(\mathbf{p})] = (-i/\hbar)\left[\mathbf{p},\, c\vec{\alpha}\cdot\mathbf{p} + \beta mc^2\right] = \mathbf{0}, \tag{3d}$$

which of course is the basic property of a free particle, namely that its momentum is conserved, and they *also* yield [5, 6],

$$\dot{\mathbf{r}} = (-i/\hbar)[\mathbf{r},\, H_D(\mathbf{p})] = (-i/\hbar)\left[\mathbf{r},\, c\vec{\alpha}\cdot\mathbf{p} + \beta mc^2\right] = c\vec{\alpha}, \tag{3e}$$

which, since $\vec{\alpha}$ is *independent of* $\mathbf{p}$, unfortunately outright *contradicts* the particular fundamental attribute of free-particle special relativity that the $|\mathbf{p}| \to 0$ asymptotic form of $\dot{\mathbf{r}}$ *must be the nonrelativistic result* $(\mathbf{p}/m)$ *for* $\dot{\mathbf{r}}$, i.e.,

$$\dot{\mathbf{r}} \sim (\mathbf{p}/m) \text{ as } |\mathbf{p}| \to 0. \tag{3f}$$

exactly as is the case which is illustrated by Eqs. (2a) and (2b). The *incompatibility with* Eq. (3f) of the Eq. (3e) consequence $\dot{\mathbf{r}} = c\vec{\alpha}$ of Dirac's free-particle Hamiltonian $H_D(\mathbf{p})$ of Eq. (3c), where $\vec{\alpha}$ *is independent of* $\mathbf{p}$, shows that Dirac's free-particle Hamiltonian *breaches special relativity*.

Gross breaches of the quantum correspondence principle by Dirac's free-particle Hamiltonian $H_D(\mathbf{p})$, as well as further breaches of special relativity, flow from the well-known algebraic properties of $\vec{\alpha}$ and β that follow from Dirac's *second* postulate [1, 7, 8],

$$(H_D(\mathbf{p}))^2 = (c\vec{\alpha}\cdot\mathbf{p} + \beta mc^2)^2 = (H(\mathbf{p}))^2 = m^2c^4 + |c\mathbf{p}|^2, \tag{4a}$$

which ensures that any solution of the Dirac equation satisfies the Klein-Gordon equation. (It as well ensures that the Dirac equation *shares* the Klein-Gordon equation's property of having *negative-energy free-particle solutions*.) The well-known consequences of Eq. (4a) for the algebraic properties of $\vec{\alpha}$ and β are [1, 7, 8],

$$(\alpha_x)^2 = (\alpha_y)^2 = (\alpha_z)^2 = (\beta)^2 = 1 \text{ and } \alpha_x,\, \alpha_y,\, \alpha_z \text{ and } \beta \text{ all mutually anticommute.} \tag{4b}$$

Eq. (4b) implies that the three observable components of the Dirac free-particle velocity $\dot{\mathbf{r}} = c\vec{\alpha}$ and the observable term βmc^2 of Dirac's free-particle Hamiltonian $H_D(\mathbf{p})$ all mutually anticommute, so a commutator of a pair of these observables equals twice its product, which fails to vanish in the limit that $\hbar \to 0$ *in gross breach of the quantum correspondence-principle requirement that commutators of observables must vanish when* $\hbar \to 0$. Thus Dirac's free-particle Hamiltonian postulates of Eqs. (3c) and (4a) *are in utterly hopeless conflict with the quantum correspondence principle*.

Eq. (4b) also yields the following result for the Dirac free-particle speed $|\dot{\mathbf{r}}| = c|\vec{\alpha}|$,

$$|\dot{\mathbf{r}}| = c|\vec{\alpha}| = c\left((\alpha_x)^2 + (\alpha_y)^2 + (\alpha_z)^2\right)^{\frac{1}{2}} = c(1+1+1)^{\frac{1}{2}} = c\sqrt{3}, \tag{4c}$$

a fixed c-number whose value $c\sqrt{3}$ not only breaches the nonrelativistic asymptotic free-particle requirement that $|\dot{\mathbf{r}}| \sim (|\mathbf{p}|/m)$ as $|\mathbf{p}| \to 0$, but as well breaches the special-relativistic free-particle *speed limit* $|\dot{\mathbf{r}}| < c$. Thus Dirac's free-particle Hamiltonian postulates of Eqs. (3c) and (4a) *are in utterly hopeless conflict with special relativity*.

Since its implications for the physical legitimacy of Dirac theory are devastating, the result that $|\dot{\mathbf{r}}| = c\sqrt{3}$ isn't written down in any textbook, but *the fact that the eigenvalues of the three components of* $\dot{\mathbf{r}} = c\vec{\alpha}$ *are* $\pm c$ *is indeed pointed out in some textbooks* [5], and $|\dot{\mathbf{r}}| = c\sqrt{3}$ immediately mathematically follows.

We noted in Eq. (2d) that the *actual* relativistic free-particle Hamiltonian $H(\mathbf{p}) = \left(m^2c^4 + |c\mathbf{p}|^2\right)^{\frac{1}{2}}$ implies that $\ddot{\mathbf{r}} = \mathbf{0}$, and therefore produces *no* spontaneous acceleration of free particles, in accord with Newton's First Law. However, because Dirac's free-particle Hamiltonian $H_D(\mathbf{p})$ *grossly breaches the quantum correspondence principle*, it produces the spontaneous free-particle acceleration,

$$\ddot{\mathbf{r}} = (-i/\hbar)[\dot{\mathbf{r}},\, H_D(\mathbf{p})] = (-i/\hbar)\left[c\vec{\alpha},\, c\vec{\alpha}\cdot\mathbf{p} + \beta mc^2\right] = \left(-ic^2/\hbar\right)((\mathbf{p}\times(\vec{\alpha}\times\vec{\alpha})) + (2\vec{\alpha}\beta mc)), \tag{4d}$$

which implies that *when* $\mathbf{p} = \mathbf{0}$, namely in the case of a Dirac free particle of *zero momentum*,

$$\ddot{\mathbf{r}} = -2i\vec{\alpha}\beta\left(mc^3/\hbar\right), \tag{4e}$$

and therefore,

$$|\ddot{\mathbf{r}}| = 2\sqrt{3}\left(mc^3/\hbar\right). \tag{4f}$$

Eq. (4f) tells us that *due to varying direction of travel*, a $\mathbf{p} = \mathbf{0}$ Dirac "free particle", *which has special-relativity breaching fixed speed* $c\sqrt{3}$, undergoes spontaneous acceleration *whose magnitude has no upper bound in the classical limit* $\hbar \to 0$. *Already for a* $\mathbf{p} = \mathbf{0}$ *electron*, Eq. (4f) implies a *zitterbewegung* spontaneous-acceleration magnitude $|\ddot{\mathbf{r}}|$ of *the mind-boggling order of* 10^{28} *times* g, where $g = 9.8\,m/s^2$, the acceleration of gravity at the Earth's surface. However, if the observables $\dot{\mathbf{r}} = c\vec{\alpha}$ and βmc^2 sensibly *commuted* instead of *grossly breaching the quantum correspondence principle because of the unphysical anticommutation that is imposed on them by Dirac's misguided postulates*, we see from Eq. (4d) that the $\mathbf{p} = \mathbf{0}$ particle *zitterbewegung* spontaneous-acceleration $\ddot{\mathbf{r}}$ of course *would vanish altogether*.

Furthermore, if the observable components of the Dirac "free particle" Hamiltonian's velocity operator $\dot{\mathbf{r}} = c\vec{\alpha}$ sensibly *commuted with each other*, as do the observable components of the *actual* relativistic free-particle Hamiltonian's velocity operator $\dot{\mathbf{r}} = (\mathbf{p}/m)\left(1 + |\mathbf{p}/(mc)|^2\right)^{-\frac{1}{2}}$ of Eq. (2a), instead of *grossly breaching the quantum correspondence principle because of the unphysical anticommutation that is imposed on them by Dirac's misguided postulates*, the "famous" Dirac spin-1/2 operator $\mathbf{S}$, namely,

$$\mathbf{S} = -i(\hbar/4)(\vec{\alpha} \times \vec{\alpha}) = -i\left(\hbar/\left(4c^2\right)\right)(\dot{\mathbf{r}} \times \dot{\mathbf{r}}),$$

would vanish altogether. Thus *the very existence* of the "famous" Dirac spin-1/2 operator $\mathbf{S}$ *is the direct consequence of the Dirac theory's completely unphysical anticommutation rules which grossly breach the quantum correspondence principle*.

Moreover, scrutiny of Eq. (4d) above, reveals that the Dirac spin-1/2 operator-related entity $(\vec{\alpha} \times \vec{\alpha})$ *contributes to the spontaneous acceleration* $\ddot{\mathbf{r}}$ *of a Dirac "free particle"*, which of course breaches the Newton's First Law property of free-particle special relativity.

The "automatic emergence" of the spin-1/2 operator $\mathbf{S} = -i(\hbar/4)(\vec{\alpha} \times \vec{\alpha}) = -i\left(\hbar/\left(4c^2\right)\right)(\dot{\mathbf{r}} \times \dot{\mathbf{r}})$ in Dirac theory is traditionally *touted* as "a great accomplishment" of that theory, but (1) *its very existence* depends on *the the Dirac theory's completely unphysical gross breach of the quantum correspondence principle*, and (2) the spin-1/2 operator-related entity $(\vec{\alpha} \times \vec{\alpha})$ *is a contributor* to the *special-relativity breaching* spontaneous acceleration $\ddot{\mathbf{r}}$ of a Dirac "free particle", as is seen from Eq. (4d).

Turning now to the electromagnetically minimally coupled Dirac Hamiltonian [9, 10],

$$H_D(\mathbf{r}, \mathbf{P}) = \vec{\alpha} \cdot (c\mathbf{P} - e\mathbf{A}) + e\phi + \beta mc^2, \tag{5a}$$

we immediately see that *it has exactly the same velocity operator* $\dot{\mathbf{r}} = c\vec{\alpha}$ [6],

$$\dot{\mathbf{r}} = (-i/\hbar)[\mathbf{r},\, H_D(\mathbf{r}, \mathbf{P})] = (-i/\hbar)[\mathbf{r},\, c\vec{\alpha} \cdot \mathbf{P}] = c\vec{\alpha}, \tag{5b}$$

as the "free-particle" Dirac Hamiltonian (see Eq. (3e)), so *any* electromagnetically coupled Dirac particle *always* has the speed $|\dot{\mathbf{r}}| = c\sqrt{3}$ that breaches the special-relativistic particle speed limit $|\dot{\mathbf{r}}| < c$. The electromagnetically minimally coupled Dirac Hamiltonian of Eq. (5a) also grossly breaches the quantum correspondence principle in exactly the same way as the free-particle Dirac Hamiltonian, namely the three observable components of the velocity operator $\dot{\mathbf{r}} = c\vec{\alpha}$ and the observable Hamiltonian term βmc^2 all mutually anticommute, so a commutator of a pair of these observables equals twice its product, which fails to vanish in the limit that $\hbar \to 0$.

The *speed result*, $|\dot{\mathbf{r}}| = c\sqrt{3}$, for the electromagnetically minimally coupled Dirac Hamiltonian of Eq. (5a) *immediately contradicts* the well-known textbook "theorem" that that Hamiltonian *effectively reduces* to the electromagnetically coupled nonrelativistic Pauli Hamiltonian [11, 12],

$$H = \left(|\mathbf{P} - (e/c)\mathbf{A}|^2/(2m)\right) + e\phi - (e\hbar/(2mc))(\vec{\sigma} \cdot \mathbf{B}), \tag{6a}$$

in the latter's *region of special-relativistic validity*, which is, of course, when,

$$|\dot{\mathbf{r}}| = (|\mathbf{P} - (e/c)\mathbf{A}|/m) \ll c, \tag{6b}$$

as a consequence of the fact that,

$$\dot{\mathbf{r}} = (-i/\hbar)[\mathbf{r},\, H] = (-i/\hbar)\left[\mathbf{r},\, \left(|\mathbf{P} - (e/c)\mathbf{A}|^2/(2m)\right)\right] = ((\mathbf{P} - (e/c)\mathbf{A})/m). \tag{6c}$$

However, *since there is no overlap whatsoever between* $|\dot{\mathbf{r}}| = c\sqrt{3}$ *and* $|\dot{\mathbf{r}}| \ll c$, this well-known textbook "theorem" *comically falls flat on its face*.

The purported "proof" which textbooks proffer for this well-known "theorem" relies on the ostensibly "plausible" supposition for the Dirac Hamiltonian that if [13, 14],

$$|\mathbf{P} - (e/c)\mathbf{A}| \ll mc, \tag{7a}$$

then,

$$|E - mc^2| \ll mc^2. \tag{7b}$$

The *difficulty* with this "plausible" supposition becomes apparent when the Dirac equation's *unavoidable negative-energy solutions* are taken into consideration. For example, it is *entirely feasible* to have the condition given by Eq. (7a) *in coexistence with*,

$$E \approx -mc^2, \tag{7c}$$

which, of course, *drastically violates* the ostensibly "plausible" supposition of Eq. (7b).

The *conceptually most fundamental problem* with the Dirac Hamiltonian was Dirac's false idea that space-time symmetry of the Schrödinger equation can supplant the requirement of Lorentz covariance of the Hamiltonian operator's associated energy-momentum operator. Consider the generic single-particle Schrödinger equation in configuration representation,

$$i\hbar\partial\psi/\partial t = H(\mathbf{r}, \mathbf{P})\psi. \tag{8a}$$

If this Schrödinger equation *actually accords* with special relativity, its Hamiltonian operator $H(\mathbf{r}, \mathbf{P})$ *perforce* is such that its associated energy-momentum operator $H^\mu = (H(\mathbf{r}, \mathbf{P}), c\mathbf{P})$ *is Lorentz-covariant*. It *also turns out* that *if* this associated energy-momentum operator of the Hamiltonian operator of such a generic Schrödinger equation *is* Lorentz-covariant, then that Schrödinger equation *is the time component of a Lorentz-covariant four-vector equation system* whose *three space components* follow from *just* the familiar configuration-representation fact that,

$$\mathbf{P}\psi = -i\hbar\nabla_{\mathbf{r}}\psi. \tag{8b}$$

To demonstrate this, we *first* point out that the Eq. (8a) generic single-particle Schrödinger equation in configuration representation *together with* Eq. (8b) yields the four-equation system,

$$i\hbar c\,\partial\psi/\partial x_\mu = H^\mu\psi, \tag{8c}$$

which written out in detail is,

$$i\hbar(\partial\psi/\partial t, -c\nabla_{\mathbf{r}}\psi) = (H(\mathbf{r}, \mathbf{P})\psi, c\,\mathbf{P}\psi). \tag{8d}$$

This four-equation system is satisfied because *its time component* is *precisely* the Eq. (8a) generic Schrödinger equation, and *its three space components* are equivalent to,

$$-i\hbar\nabla_{\mathbf{r}}\psi = \mathbf{P}\psi, \tag{8e}$$

which is *precisely* Eq. (8b).

In *addition* to *merely* the straightforward *validity* of the Eq. (8c) four-equation system, it is the case that *since* the space-time differential operator,

$$i\hbar c\,\partial/\partial x_\mu = i\hbar(\partial/\partial t, -c\nabla_{\mathbf{r}}),$$

manifestly is a Lorentz-covariant four-vector operator, *if* the Hamiltonian operator $H(\mathbf{r}, \mathbf{P})$ of the Eq. (8a) generic Schrödinger equation is such that its associated energy-momentum operator $H^\mu = (H(\mathbf{r}, \mathbf{P}), c\mathbf{P})$ *is as well* a Lorentz-covariant four-vector operator, then the Eq. (8c) four-equation system clearly *is a Lorentz-covariant four-vector equation system* whose *time component* of course is the Eq. (8a) generic Schrödinger equation, and whose three space components follow from *just* the familiar configuration-representation fact that Eq. (8b) holds. Therefore *if* the Hamiltonian operator $H(\mathbf{r}, \mathbf{P})$ of the Eq. (8a) generic Schrödinger equation is such that its associated energy-momentum operator $H^\mu = (H(\mathbf{r}, \mathbf{P}), c\mathbf{P})$ *is* a Lorentz-covariant four-vector operator, then the Eq. (8a) generic Schrödinger equation clearly *accords* with special relativity. The converse of this statement is self-evident, so *a necessary and sufficient condition* for the Eq. (8a) generic Schrödinger equation to *accord* with special relativity is that its Hamiltonian operator $H(\mathbf{r}, \mathbf{P})$ is such that its associated energy-momentum operator $H^\mu = (H(\mathbf{r}, \mathbf{P}), c\mathbf{P})$ is a Lorentz-covariant four-vector operator.

Also, since a generic single-particle, configuration-representation Schrödinger equation which *accords* with special relativity is *only the time component* of a Lorentz-covariant four-vector equation system, *it absolutely cannot be space-time symmetric*. Therefore, *making* such a generic Schrödinger equation *space-time symmetric* will *always* produce a *result* which *breaches* some aspect of special relativity, which is the *opposite* of Dirac's *intuitive impression* [1, 2, 3, 4], and provides an explanation *why* the Dirac Hamiltonian breaches special relativity.

We also note the arcane fact that since a generic single-particle, configuration-representation Schrödinger equation which *accords* with special relativity *is only the time component* of a Lorentz-covariant four-vector equation system, *no homogeneously-linear recasting* of such a generic Schrödinger equation that *accords* with special relativity *is Lorentz-transformation form-invariant*. This arcane fact is *only* of interest because the Dirac equation has been ad hoc *retrofitted* with a custom-created *claimed extension* of the Lorentz transformation to Dirac's four-component wave functions under which the Dirac equation multiplied by the Dirac matrix β is *form-invariant* [15]. Since the Dirac equation's multiplication by the Dirac matrix β is *indeed* a homogeneously-linear recasting of the Dirac equation, the above arcane fact tells us that if this ad hoc retrofitted custom-created *claimed extension* of the Lorentz transformation to Dirac's four-component wave functions *really is* the Lorentz transformation that it is *claimed* to be, then *the Dirac equation*, whose multiplication by the Dirac matrix β is *form-invariant* under this presumed Lorentz transformation, *definitely cannot be in accord* with special relativity. On the other hand, if this ad hoc retrofitted custom-created *claimed extension* of the Lorentz transformation to Dirac's four-component wave functions *isn't really* the Lorentz transformation that it is *claimed* to be, then we must, of course, *directly examine* the Dirac equation to *check* whether it accords with special relativity. For example, the *free particle* Dirac Hamiltonian is inhomogeneously *linear* in $\mathbf{p}$, which we have seen in detail above *isn't* in accord with special relativity. The *moral* of this *overlong story* about the ad hoc retrofitted custom-created *claimed extension* of the Lorentz transformation to Dirac's four-component wave functions, under which the Dirac equation's multiplication by the Dirac matrix β is *form-invariant* [15], is that *its existence in absolutely no way* demonstrates that the Dirac equation is in *accord* with special relativity; indeed, if this transformation of Dirac's four-component wave functions *really is* the Lorentz transformation which *it is claimed to be*, then the Dirac equation *definitely isn't in accord* with special relativity.

Turning again briefly to the electromagnetically minimally coupled Dirac Hamiltonian of Eq. (5a), namely,

$$H_D(\mathbf{r},\mathbf{P}) = \vec{\alpha}\cdot(c\mathbf{P} - e\mathbf{A}) + e\phi + \beta mc^2,$$

since it breaches special relativity because its particle speed $|\dot{\mathbf{r}}| = c\sqrt{3}$ always exceeds c, and it also grossly breaches the quantum correspondence principle, it clearly cannot correctly describe single-particle relativistic quantum mechanics.

However, the electromagnetically coupled nonrelativistic Pauli Hamiltonian of Eq. (6a), namely,

$$H = \big(|\mathbf{P} - (e/c)\mathbf{A}|^2/(2m)\big) + e\phi - (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}),$$

is physically unobjectionable in the nonrelativistic regime, namely when,

$$|\dot{\mathbf{r}}| = (|\mathbf{P} - (e/c)\mathbf{A}|/m) \ll c.$$

Since *Lorentz-invariant action functionals* produce Lorentz-covariant dynamical theories and, furthermore, the relativistic physics of a single particle is *identical* to its nonrelativistic physics *when the particle is at rest*, one can render a nonrelativistic single-particle theory relativistic by *specializing the nonrelativistic action functional to zero particle velocity*, and then upgrading that *to become Lorentz invariant*.

Given *a nonrelativistic single-particle Hamiltonian* which is to be upgraded *to its relativistic counterpart*, a great many steps are necessary. One must pass from the nonrelativistic Hamiltonian to the corresponding nonrelativistic Lagrangian, thence to the nonrelativistic action functional, which is specialized *to zero particle velocity*. This is *the base to be upgraded to the Lorentz-invariant action functional, whose integrand then yields the relativistic Lagrangian*, from which one passes to the relativistic Hamiltonian. A *caveat* here is that passages between Lagrangians and Hamiltonians entail solving algebraic equations, *which isn't always feasible in closed analytic form*.

Action-based unique relativistic extension of the Pauli Hamiltonian

In preparation for the relativistic extension of the nonrelativistic Pauli Hamiltonian of Eq. (6a), we add to it the particle's rest-mass energy mc^2,

$$H = mc^2 + \big(|\mathbf{P} - (e/c)\mathbf{A}|^2/(2m)\big) + e\phi - (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}). \tag{9a}$$

Note that the addition of such a constant term to a Hamiltonian in no way changes the quantum Heisenberg or classical Hamiltonian equations of motion.

To obtain the nonrelativistic action S_{nr} which corresponds to the Hamiltonian H of Eq. (9a), we first work out the Lagrangian L which corresponds to that Hamiltonian H. The conversion of such a particle Hamiltonian to a particle Lagrangian requires swapping the Hamiltonian's dependence on the canonical three-momentum $\mathbf{P}$ for the Lagrangian's dependence on the particle's three-velocity $\dot{\mathbf{r}}$. We obtain that particle three-velocity $\dot{\mathbf{r}}$ from the Heisenberg equation of motion (or alternatively, in this case, from the equivalent classical Hamiltonian equation of motion),

$$\dot{\mathbf{r}} = (-i/\hbar)[\mathbf{r}, H] = \nabla_{\mathbf{P}} H = (\mathbf{P} - (e/c)\mathbf{A})/m. \tag{9b}$$

We now *invert* the relation of Eq. (9b) between particle velocity $\dot{\mathbf{r}}$ and canonical momentum $\mathbf{P}$ to read,

$$\mathbf{P} = m\dot{\mathbf{r}} + (e/c)\mathbf{A}, \tag{9c}$$

and insert it into the well-known relationship of the Lagrangian to the Hamiltonian, namely,

$$L = \dot{\mathbf{r}} \cdot \mathbf{P} - H\Big|_{\mathbf{P}=m\dot{\mathbf{r}}+(e/c)\mathbf{A}} = -mc^2 + \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 - e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) + (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}), \tag{9d}$$

from which we immediately obtain the nonrelativistic action,

$$S_{\mathrm{nr}} = \int L dt = \int \left[-mc^2 + \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 - e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) + (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})\right] dt.$$

Of course we don't want the nonrelativistic action S_{nr} itself, but its *specialization* S to the case of *zero particle velocity*, namely $\dot{\mathbf{r}} = \mathbf{0}$,

$$S = \int \left[-mc^2 - e\phi + (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})\right] dt. \tag{9e}$$

We shall undertake the Lorentz-invariant upgrade of the three terms of this action S individually. The first term of S which we tackle is that of the free particle,

$$S^0 = \int (-mc^2) dt. \tag{10a}$$

To make S^0 Lorentz-invariant, we only need to replace the time differential dt by the Lorentz-invariant proper time differential $d\tau$,

$$d\tau = \left((dt)^2 - |d\mathbf{r}/c|^2\right)^{\frac{1}{2}} = \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{\frac{1}{2}} dt. \tag{10b}$$

Therefore,

$$d\tau/dt = \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{\frac{1}{2}}, \tag{10c}$$

and from this it of course follows that,

$$dt/d\tau = \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{-\frac{1}{2}}. \tag{10d}$$

The Lorentz-invariant upgraded S^0 therefore is,

$$S^0_{\mathrm{rel}} = \int (-mc^2) d\tau. \tag{10f}$$

Eq. (10f), by use of Eq. (10c) can of course also be expressed as,

$$S^0_{\mathrm{rel}} = \int (-mc^2)\left(1 - |\dot{\mathbf{r}}/c|^2\right)^{\frac{1}{2}} dt. \tag{10g}$$

We next tackle the part of the action S which encompasses the interaction of the particle's charge e with the electromagnetic potential ϕ,

$$S^e = \int (-e\phi) dt. \tag{11a}$$

We carry out the Lorentz-invariant upgrade of S^e by replacing the time differential dt in Eq. (11a) by the Lorentz-invariant time differential $d\tau$, and upgrading the $\dot{\mathbf{r}} = \mathbf{0}$ static-limit potential energy $e\phi$ to a dynamic Lorentz-invariant function of $\dot{\mathbf{r}}$. To do so we first rewrite the static potential energy $e\phi$ as the faux Lorentz invariant,

$$e\phi = eU_\mu(\dot{\mathbf{r}} = \mathbf{0})A^\mu, \tag{11b}$$

that has the faux Lorentz-covariant constituent,

$$U_\mu(\dot{\mathbf{r}} = \mathbf{0}) = \delta^0_\mu. \tag{11c}$$

which is valid *only* in the particle's rest frame where the particle's velocity $\dot{\mathbf{r}} = \mathbf{0}$. To upgrade the static faux Lorentz-covariant $U_\mu(\dot{\mathbf{r}} = \mathbf{0})$ to a dynamic true Lorentz-covariant entity $U_\mu(\dot{\mathbf{r}})$, we Lorentz-boost it from the particle's rest frame to the inertial frame where the particle has velocity $\dot{\mathbf{r}}$,

$$U_\mu(\dot{\mathbf{r}}) = U_\alpha(\dot{\mathbf{r}} = \mathbf{0})\Lambda^\alpha_\mu(\dot{\mathbf{r}}) = \delta^0_\alpha\Lambda^\alpha_\mu(\dot{\mathbf{r}}) = \Lambda^0_\mu(\dot{\mathbf{r}}). \tag{11d}$$

Therefore *the dynamic Lorentz-invariant upgrade of the static potential energy* $e\phi$ is,

$$eU_\mu(\dot{\mathbf{r}})A^\mu = e\Lambda^0_\mu(\dot{\mathbf{r}})A^\mu = e\gamma(\dot{\mathbf{r}})\,(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}), \tag{11e}$$

where,

$$\gamma(\dot{\mathbf{r}}) = (1 - (|\dot{\mathbf{r}}|^2/c^2))^{-\frac{1}{2}} = dt/d\tau. \tag{11f}$$

Thus the Lorentz-invariant upgrade of,

$$S^e = \int(-e\phi)dt,$$

is,

$$S^e_{\mathrm{rel}} = \int(-eU_\mu(\dot{\mathbf{r}})A^\mu)d\tau = \int(-e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}))dt. \tag{11g}$$

Finally we tackle the part of the action S that encompasses the interaction of the particle's spin with the magnetic field,

$$S^{\vec{\sigma}} = \int(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})dt. \tag{12a}$$

Again we replace the differential dt by the Lorentz-invariant differential $d\tau$ and upgrade the static potential energy $-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})$, which is valid in the $\dot{\mathbf{r}} = \mathbf{0}$ particle rest frame, to a dynamic Lorentz-invariant function of $\dot{\mathbf{r}}$. Preliminary to the upgrading of the static potential energy $-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})$, we write it as,

$$-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}) = -(e\hbar/(2mc))(\vec{\sigma}\cdot(\nabla\times\mathbf{A})) = (e\hbar/(2mc))\epsilon_{ijk}\sigma^i(\partial^j A^k). \tag{12b}$$

This representation of the static potential energy can be rewritten as the faux Lorentz invariant,

$$(e\hbar/(2mc))\epsilon_{ijk}\sigma^i(\partial^j A^k) = (e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}} = \mathbf{0})(\partial^\mu A^\nu), \tag{12c}$$

that has the faux Lorentz-covariant constituent,

$$\sigma_{\mu\nu}(\dot{\mathbf{r}} = \mathbf{0}) = \begin{cases} 0 & \text{if } \mu = 0 \text{ or } \nu = 0, \\ \epsilon_{ijk}\sigma^i & \text{if } \mu = j \text{ and } \nu = k,\ j,k = 1,2,3, \end{cases} \tag{12d}$$

which is valid *only* in the particle's rest frame where the particle's velocity $\dot{\mathbf{r}} = \mathbf{0}$. Note that $\sigma_{\mu\nu}(\dot{\mathbf{r}} = \mathbf{0})$ is *antisymmetric* under the interchange of its two indices μ and ν. To upgrade the static faux Lorentz-covariant $\sigma_{\mu\nu}(\dot{\mathbf{r}} = \mathbf{0})$ to a dynamic true Lorentz-covariant entity $\sigma_{\mu\nu}(\dot{\mathbf{r}})$, we Lorentz-boost it from the particle's rest frame to the inertial frame where the particle has velocity $\dot{\mathbf{r}}$,

$$\sigma_{\mu\nu}(\dot{\mathbf{r}}) = \sigma_{\alpha\beta}(\dot{\mathbf{r}} = \mathbf{0})\Lambda^\alpha_\mu(\dot{\mathbf{r}})\Lambda^\beta_\nu(\dot{\mathbf{r}}) = \epsilon_{ijk}\sigma^i\Lambda^j_\mu(\dot{\mathbf{r}})\Lambda^k_\nu(\dot{\mathbf{r}}). \tag{12e}$$

It is apparent from Eq. (12e) that the Lorentz-covariant second-rank tensor $\sigma_{\mu\nu}(\dot{\mathbf{r}})$ is *also* antisymmetric under the interchange of its two indices μ and ν. From Eqs. (12b) through (12e) it is clear that *the dynamic Lorentz-invariant upgrade of the static potential energy* $-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})$ is,

$$\begin{gathered}(e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}})(\partial^\mu A^\nu) = (e\hbar/(2mc))\epsilon_{ijk}\sigma^i\Lambda^j_\mu(\dot{\mathbf{r}})\Lambda^k_\nu(\dot{\mathbf{r}})(\partial^\mu A^\nu) = \\ (e\hbar/(2mc))\left(\vec{\sigma}\cdot[(\boldsymbol{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu)\times(\boldsymbol{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu)]\right),\end{gathered} \tag{12f}$$

where,

$$(\boldsymbol{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu)^j \stackrel{\text{def}}{=} \Lambda^j_\mu(\dot{\mathbf{r}})\partial^\mu \text{ and } (\boldsymbol{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu)^k \stackrel{\text{def}}{=} \Lambda^k_\nu(\dot{\mathbf{r}})A^\nu. \tag{12g}$$

The space components of the Lorentz boost of the four-vector partial-derivative operator,

$$\partial^\mu = ((1/c)(\partial/\partial t), -\nabla),$$

from the rest frame of the particle to the inertial frame in which the particle has velocity $\dot{\mathbf{r}}$ are given by,

$$(\boldsymbol{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu) = -\nabla - (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}\dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\nabla) - \gamma(\dot{\mathbf{r}})(\dot{\mathbf{r}}/c)(1/c)(\partial/\partial t), \tag{12h}$$

and the space components of the *same* Lorentz boost of the electromagnetic four-vector potential,

$$A^\mu = (\phi, \mathbf{A}),$$

are given by,

$$(\boldsymbol{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu) = \mathbf{A} + (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}\dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{A}) - \gamma(\dot{\mathbf{r}})(\dot{\mathbf{r}}/c)\phi. \tag{12i}$$

Using Eqs. (12h) and (12i) one can, with tedious effort, verify that,

$$\begin{gathered}(\boldsymbol{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu) \times (\boldsymbol{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu) = -(\nabla\times\mathbf{A}) - \\ (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}\left[\nabla\times(\dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{A})) + (\dot{\mathbf{r}}\cdot\nabla)(\dot{\mathbf{r}}\times\mathbf{A})\right] - \gamma(\dot{\mathbf{r}})\left[(\dot{\mathbf{r}}/c)\times(\dot{\mathbf{A}}/c) - \nabla\times((\dot{\mathbf{r}}/c)\phi)\right] = \\ -(\nabla\times\mathbf{A}) - (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}\left[\dot{\mathbf{r}}\times[-\nabla(\dot{\mathbf{r}}\cdot\mathbf{A}) + (\dot{\mathbf{r}}\cdot\nabla)\mathbf{A}]\right] + \gamma(\dot{\mathbf{r}})\left[(\dot{\mathbf{r}}/c)\times\left[-\nabla\phi - (\dot{\mathbf{A}}/c)\right]\right] = \\ -(\nabla\times\mathbf{A}) - (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}\left[\dot{\mathbf{r}}\times[-\dot{\mathbf{r}}\times(\nabla\times\mathbf{A})]\right] + \gamma(\dot{\mathbf{r}})\left[(\dot{\mathbf{r}}/c)\times\left[-\nabla\phi - (\dot{\mathbf{A}}/c)\right]\right] = \\ -\mathbf{B} - (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}\left[|\dot{\mathbf{r}}|^2\mathbf{B} - \dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{B})\right] + \gamma(\dot{\mathbf{r}})((\dot{\mathbf{r}}/c)\times\mathbf{E}) = \\ -\gamma(\dot{\mathbf{r}})\mathbf{B} + (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}\dot{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{B}) - \gamma(\dot{\mathbf{r}})(\mathbf{E}\times(\dot{\mathbf{r}}/c)).\end{gathered} \tag{12j}$$

From Eqs. (12f) and (12j) one sees that the dynamic Lorentz-invariant upgrade of the static potential energy $-(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})$ is,

$$\begin{gathered}(e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}})(\partial^\mu A^\nu) = (e\hbar/(2mc))\left(\vec{\sigma}\cdot[(\boldsymbol{\Lambda}_\mu(\dot{\mathbf{r}})\partial^\mu)\times(\boldsymbol{\Lambda}_\nu(\dot{\mathbf{r}})A^\nu)]\right) = \\ -(e\hbar/(2mc))\left[\gamma(\dot{\mathbf{r}})(\vec{\sigma}\cdot\mathbf{B}) - (\gamma(\dot{\mathbf{r}}) - 1)|\dot{\mathbf{r}}|^{-2}(\vec{\sigma}\cdot\dot{\mathbf{r}})(\dot{\mathbf{r}}\cdot\mathbf{B}) + \gamma(\dot{\mathbf{r}})(\vec{\sigma}\cdot(\mathbf{E}\times(\dot{\mathbf{r}}/c)))\right],\end{gathered} \tag{12k}$$

and thus the Lorentz-invariant upgrade of the Eq. (12a) spin contribution to the action, namely,

$$S^{\vec{\sigma}} = \int (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})dt.$$

comes out to be,

$$\begin{gathered}S^{\vec{\sigma}}_{\text{rel}} = -\int (e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}})(\partial^\mu A^\nu)d\tau = \\ \int (e\hbar/(2mc))\left[(\vec{\sigma}\cdot\mathbf{B}) - (1 - (\gamma(\dot{\mathbf{r}}))^{-1})|\dot{\mathbf{r}}|^{-2}(\vec{\sigma}\cdot\dot{\mathbf{r}})(\dot{\mathbf{r}}\cdot\mathbf{B}) + (\vec{\sigma}\cdot(\mathbf{E}\times(\dot{\mathbf{r}}/c)))\right]dt = \\ \int (e\hbar/(2mc))\left[(\vec{\sigma}\cdot\mathbf{B}) - (1 + (\gamma(\dot{\mathbf{r}}))^{-1})^{-1}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B}) + (\vec{\sigma}\times\mathbf{E})\cdot(\dot{\mathbf{r}}/c)\right]dt,\end{gathered} \tag{12l}$$

as we see by using Eq. (12k) and the fact that,

$$\gamma(\dot{\mathbf{r}}) = (1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} = dt/d\tau.$$

In the last step of Eq. (12l), we have furthermore interchanged the "dot" $\cdot$ with the "cross" $\times$ in the triple scalar product,

$$(\vec{\sigma}\cdot(\mathbf{E}\times(\dot{\mathbf{r}}/c))),$$

and have as well applied the identity,

$$(1-(\gamma(\dot{\mathbf{r}}))^{-1})|\dot{\mathbf{r}}|^{-2}=(1+(\gamma(\dot{\mathbf{r}}))^{-1})^{-1}c^{-2}.$$

We are now in a position to write down *the Lorentz-invariant upgrade* S_{rel} of the $\dot{\mathbf{r}}=\mathbf{0}$ Pauli action S of Eq. (9e),

$$\begin{gathered}S_{\mathrm{rel}}=S^0_{\mathrm{rel}}+S^e_{\mathrm{rel}}+S^{\vec\sigma}_{\mathrm{rel}}=\int\left[-mc^2-eU_\mu(\dot{\mathbf{r}})A^\mu-(e\hbar/(2mc))\sigma_{\mu\nu}(\dot{\mathbf{r}})(\partial^\mu A^\nu)\right]d\tau=\\ \int\Big[-mc^2(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}-e(\phi-(\dot{\mathbf{r}}/c)\cdot\mathbf{A})+\\ (e\hbar/(2mc))\Big((\vec\sigma\cdot\mathbf{B})-\left(1+(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\vec\sigma\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B})+(\dot{\mathbf{r}}/c)\cdot(\vec\sigma\times\mathbf{E})\Big)\Big]dt\end{gathered}\tag{13a}$$

The *integrand* of this Lorentz-invariant upgrade S_{rel} of the $\dot{\mathbf{r}}=\mathbf{0}$ Pauli action S is of course *the relativistic Pauli Lagrangian* L_{rel},

$$\begin{gathered}L_{\mathrm{rel}}=-mc^2(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}-e(\phi-(\dot{\mathbf{r}}/c)\cdot\mathbf{A})+\\ (e\hbar/(2mc))\Big((\vec\sigma\cdot\mathbf{B})-\left(1+(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\vec\sigma\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B})+(\dot{\mathbf{r}}/c)\cdot(\vec\sigma\times\mathbf{E})\Big),\end{gathered}\tag{13b}$$

where,

$$\mathbf{B}=\nabla\times\mathbf{A}\ \text{ and }\ \mathbf{E}=-\nabla\phi-(\dot{\mathbf{A}}/c).\tag{13c}$$

From Eq. (13b) we calculate *the relativistic Pauli Lagrangian's corresponding canonical momentum*,

$$\begin{gathered}\mathbf{P}=\nabla_{\dot{\mathbf{r}}}L_{\mathrm{rel}}=m\dot{\mathbf{r}}(1-|\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}+(e/c)\mathbf{A}+(e\hbar/(2mc^2))(\vec\sigma\times\mathbf{E})-\\ (e\hbar/(2mc^2))\left(1+(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}\Big[\vec\sigma((\dot{\mathbf{r}}/c)\cdot\mathbf{B})+(\vec\sigma\cdot(\dot{\mathbf{r}}/c))\mathbf{B}+\\ \left(1+(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\dot{\mathbf{r}}/c)(1-|\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}(\vec\sigma\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B})\Big].\end{gathered}\tag{13d}$$

The *last three terms of* Eq. (13d), which all arise from *the relativistic distortion of the magnetic field* $\mathbf{B}$, unfortunately *preclude solving analytically* for the particle's *velocity* $\dot{\mathbf{r}}$ in terms of the system's *canonical momentum* $\mathbf{P}$. For that reason we cannot in general *analytically* parlay the relativistic Pauli system's *energy* E_{rel}, namely,

$$E_{\mathrm{rel}}=\dot{\mathbf{r}}\cdot\mathbf{P}-L_{\mathrm{rel}},\tag{13e}$$

into its relativistic Pauli *Hamiltonian* $H_{\mathrm{rel}}(\mathbf{r},\vec\sigma,\mathbf{P},t)$. However we see from Eq. (13d) that the three offending terms which arise from the relativistic distortion of the magnetic field $\mathbf{B}$ are all *higher-order corrections in powers of* $|\dot{\mathbf{r}}/c|$, so we can easily rewrite Eq. (13d) as *a successive-approximation scheme* for the desired inversion of the canonical momentum $\mathbf{P}$ that is *consonant* with *the systematic carrying out of relativistic corrections*. The scheme *is considerably more transparent*, however, after all occurrences of the particle velocity $\dot{\mathbf{r}}$ on the right-hand side of Eq. (13d) (and as well on the right-hand side of Eq. (13e)) are *replaced* by occurrences *of the free-particle momentum* $\mathbf{p}$, which is,

$$\begin{gathered}\mathbf{p}\stackrel{\mathrm{def}}{=}m\dot{\mathbf{r}}(1-|\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}},\ \text{ and implies,}\\ (\dot{\mathbf{r}}/c)(1-|\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}=\mathbf{p}/(mc),\quad(\dot{\mathbf{r}}/c)=\mathbf{p}(m^2c^2+|\mathbf{p}|^2)^{-\frac{1}{2}},\quad(1-|\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}=mc(m^2c^2+|\mathbf{p}|^2)^{-\frac{1}{2}}.\end{gathered}\tag{13f}$$

Using Eq. (13f) to eliminate all occurrences of the particle velocity $\dot{\mathbf{r}}$ on the right-hand side of Eq. (13d) in favor of the free-particle momentum $\mathbf{p}$ yields,

$$\begin{gathered}\mathbf{P}=\mathbf{p}+(e/c)\mathbf{A}+(e\hbar/(2mc^2))(\vec\sigma\times\mathbf{E})-\\ (e\hbar/(2mc^2))\left(mc+(m^2c^2+|\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1}\times\\ \Big[\vec\sigma(\mathbf{p}\cdot\mathbf{B})+(\vec\sigma\cdot\mathbf{p})\mathbf{B}+\left(mc+(m^2c^2+|\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1}(\mathbf{p}/(mc))(\vec\sigma\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{B})\Big].\end{gathered}\tag{13g}$$

Eq. (13g) can now be readily rewritten *as a successive approximation scheme for the resolution of the free-particle momentum* $\mathbf{p}$ *in terms of the canonical momentum* $\mathbf{P}$,

$$\begin{gathered}\mathbf{p} = \mathbf{P} - (e/c)\mathbf{A} - (e\hbar/(2mc^2))(\vec{\sigma}\times\mathbf{E}) + \\ (e\hbar/(2mc^2))\left(mc + (m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1}\times \\ \left[\vec{\sigma}(\mathbf{p}\cdot\mathbf{B}) + (\vec{\sigma}\cdot\mathbf{p})\mathbf{B} + \left(mc + (m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1}(\mathbf{p}/(mc))(\vec{\sigma}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{B})\right].\end{gathered} \tag{13h}$$

In order for these successive approximations to $\mathbf{p}$ in terms of $\mathbf{P}$ to be able to produce successive approximations to the relativistic Pauli *Hamiltonian* H_{rel}, we must *also* banish all occurrences of the particle velocity $\dot{\mathbf{r}}$ in the system's *energy* E_{rel}, which is given on the right-hand side of Eq. (13e), in favor of the free-particle momentum $\mathbf{p}$.

We shall, however, *first* calculate that relativistic Pauli energy $E_{\mathrm{rel}} = \dot{\mathbf{r}}\cdot\mathbf{P} - L_{\mathrm{rel}}$ of Eq. (13e) *entirely in terms of* $\dot{\mathbf{r}}$ by using the L_{rel} which is given by Eq. (13b) and the $\mathbf{P}$ which is given by Eq. (13d), and *then* use the relations given in Eq. (13f) to eliminate $\dot{\mathbf{r}}$ from E_{rel} in favor of $\mathbf{p}$.

From Eq. (13b) we obtain that,

$$\begin{gathered}-L_{\mathrm{rel}} = mc^2(1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} + e(\phi - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) - \\ (e\hbar/(2mc))\Big((\vec{\sigma}\cdot\mathbf{B}) - \left(1 + (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B}) + (\dot{\mathbf{r}}/c)\cdot(\vec{\sigma}\times\mathbf{E})\Big),\end{gathered} \tag{13i}$$

and from Eq. (13d) we obtain that,

$$\begin{gathered}\dot{\mathbf{r}}\cdot\mathbf{P} = m|\dot{\mathbf{r}}|^2(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} + e(\dot{\mathbf{r}}/c)\cdot\mathbf{A} + (e\hbar/(2mc))(\dot{\mathbf{r}}/c)\cdot(\vec{\sigma}\times\mathbf{E}) - \\ (e\hbar/(2mc))\left(1 + (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B})\times \\ \left[2 + \left(1 + (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}|\dot{\mathbf{r}}/c|^2(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}\right]\end{gathered} \tag{13j}$$

The complicated structure of the last term of Eq. (13j) can be simplified markedly, with the result,

$$\begin{gathered}\dot{\mathbf{r}}\cdot\mathbf{P} = m|\dot{\mathbf{r}}|^2(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} + e(\dot{\mathbf{r}}/c)\cdot\mathbf{A} + (e\hbar/(2mc))(\dot{\mathbf{r}}/c)\cdot(\vec{\sigma}\times\mathbf{E}) - \\ (e\hbar/(2mc))(\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B})(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}\end{gathered} \tag{13k}$$

Putting Eqs. (13i) and (13k) together produces,

$$\begin{gathered}E_{\mathrm{rel}} = \dot{\mathbf{r}}\cdot\mathbf{P} - L_{\mathrm{rel}} = mc^2(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} + e\phi - \\ (e\hbar/(2mc))\left[(\vec{\sigma}\cdot\mathbf{B}) + (\vec{\sigma}\cdot(\dot{\mathbf{r}}/c))((\dot{\mathbf{r}}/c)\cdot\mathbf{B})\left(1 + (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}\right)^{-1}(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}}\right].\end{gathered} \tag{13l}$$

We now use the relations given by Eq. (13f) to express the E_{rel} of Eq. (13l) entirely in terms of free-particle momentum $\mathbf{p}$ instead of in terms of the particle velocity $\dot{\mathbf{r}}$,

$$\begin{gathered}E_{\mathrm{rel}} = (m^2c^4 + |c\mathbf{p}|^2)^{\frac{1}{2}} + e\phi - \\ (e\hbar/(2mc))\left[(\vec{\sigma}\cdot\mathbf{B}) + (\vec{\sigma}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{B})\left(mc + (m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}\right)^{-1}(mc)^{-1}\right].\end{gathered} \tag{13m}$$

Eq. (13m) is to be used in conjunction with the Eq. (13h) successive approximation scheme for obtaining the free-particle momentum $\mathbf{p}$ in terms of the canonical momentum $\mathbf{P}$, in order to generate successive approximations to the relativistic Pauli Hamiltonian H_{rel}.

In those cases where $\mathbf{B} = \mathbf{0}$, Eq. (13h) immediately yields the *exact* relationship of the canonical momentum $\mathbf{P}$ to the free-particle momentum $\mathbf{p}$, namely,

$$\mathbf{p} = \mathbf{P} - (e/c)\mathbf{A} - (e\hbar/(2mc^2))(\vec{\sigma}\times\mathbf{E}), \tag{13n}$$

and for *those* $\mathbf{B} = \mathbf{0}$ cases Eq. (13m) yields the following the *exact* relativistic Pauli Hamiltonian, namely,

$$H_{\rm rel} = \left(m^2c^4 + |c\mathbf{P} - e\mathbf{A} - (e\hbar/(2mc))(\vec{\sigma} \times \mathbf{E})|^2\right)^{\frac{1}{2}} + e\phi. \tag{13o}$$

The relativistically extended Pauli Hamiltonian of Eq. (13o) clearly bears *a very close resemblance* to the *relativistic Lorentz Hamiltonian*, which describes a *spinless* relativistic charged particle interacting with an electromagnetic field. That notwithstanding, the *relativistically extended* Pauli Hamiltonian of Eq. (13o) *also* very clearly *incorporates* the interaction of a *moving* particle's *spin* with *an electric field*, a phenomenon *that is utterly and completely foreign to the the nonrelativistic Pauli Hamiltonian* of Eq. (1), which Eq. (13o) *exactly relativistically extends in those special cases where* $\mathbf{B} = \mathbf{0}$. The *purely relativistic* interaction of a *moving* particle's *spin* with *an electric field* is, of course *the essence* of the *hydrogen atom's spin-orbit interaction*. Thus the Eq. (13o) $\mathbf{B} = \mathbf{0}$ special case of the relativistically extended Pauli Hamiltonian is obviously useful for the hydrogen atom.

The very close resemblance to the physically irreproachable Lorentz Hamiltonian which the Eq. (13o) $\mathbf{B} = \mathbf{0}$ special case of the relativistically extended Pauli Hamiltonian manifests shows that the latter has *none* of the pathologies which are so typical of the Dirac Hamiltonian.

References

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