On Catalan’s Constant: Upgrade 2

Edgar Valdebenito

23-09-2018 16:09:59

abstract

This note presents an integral for Catalan’s constant:

$G = 0.915965...$

Introduction

Catalan´s constant , named after Eugéne Charles Catalan (1814-1894) and usually denoted by G , is defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + ... \tag{1}$$

The numerical value is

$$G = 0.915965594177219015054603514932... \tag{2}$$

It is not known whether G is irrational.

Some representations of Catalan’s constant:

$$G = -\int_0^{\pi/4} \ln(2\sin x)\,dx \tag{3}$$

$$G = 2\int_0^{\pi/4} \ln(2\cos x)\,dx \tag{4}$$

$$G = -\int_0^1 \frac{\ln x}{1+x^2}\,dx \tag{5}$$

$$G = \int_0^{\pi/2} \sinh^{-1}(\sin x)\,dx \tag{6}$$

$$G = \int_0^{\pi/2} \sinh^{-1}(\cos x)\,dx \tag{7}$$

$$G=\frac{1}{4}\int_0^{\pi/2}\ln\left(\frac{1+\cos x}{1-\cos x}\right)dx \qquad (8)$$

$$G=\frac{1}{4}\int_0^{\pi/2}\ln\left(\frac{1+\sin x}{1-\sin x}\right)dx \qquad (9)$$

In this note we give an integral for Catalan's constant.

Integral for Catalan's constant

Let $\alpha=\tan^{-1}\left(\frac{\sqrt{2}\left(\sqrt{3}-1\right)\sqrt[4]{3}}{6}\right)$, then

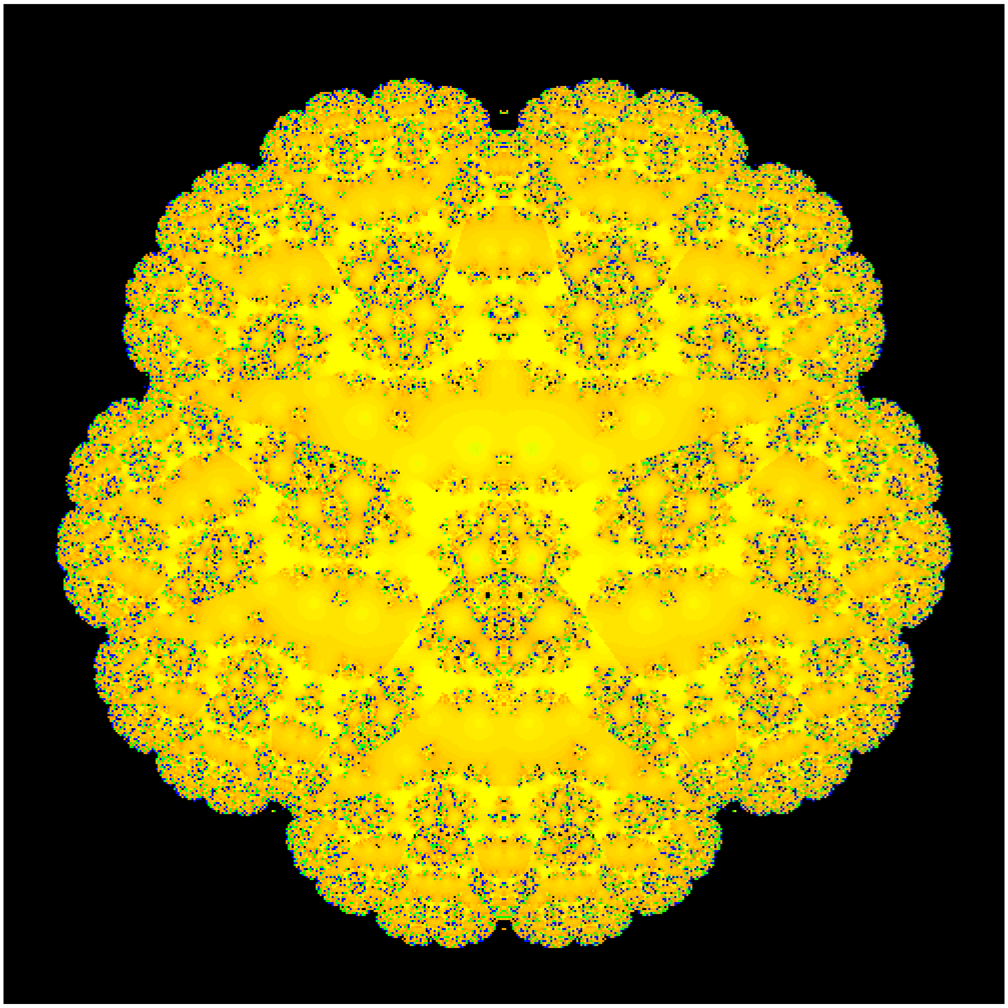
$$\frac{1}{2}G=\int_0^{\alpha}\ln\left(\frac{1+2\sqrt{1-3\tan x}\cos\left(f\left(x\right)\right)}{1-2\sqrt{1-3\tan x}\cos\left(\left(\pi/3\right)+f\left(x\right)\right)}\right)dx \qquad (10)$$

$$f\left(x\right)=\frac{1}{3}\cos^{-1}\left(\frac{2-9\tan x-27\tan^3 x}{2\sqrt{\left(1-3\tan x\right)^3}}\right) \qquad (11)$$

- Change of varibles: $u=\tan x$, $dx=\frac{1}{1+u^2}du$.
- $\beta=\frac{\sqrt{2}\left(\sqrt{3}-1\right)\sqrt[4]{3}}{6}$

$$\frac{1}{2}G=\int_0^{\beta}\frac{1}{1+x^2}\ln\left(\frac{1+2\sqrt{1-3x}\cos\left(F\left(x\right)\right)}{1-2\sqrt{1-3x}\cos\left(\left(\pi/3\right)+F\left(x\right)\right)}\right)dx \qquad (12)$$

$$F\left(x\right)=\frac{1}{3}\cos^{-1}\left(\frac{2-9x-27x^3}{2\sqrt{\left(1-3x\right)^3}}\right) \qquad (13)$$



A Fractal Image

References

1. Adamchik, V.: 32 representations for Catalan's constant. Wolfram Research, Inc. (victor@wolfram.com) , available at website http://www.wolfram.com/~victor/articles/catalan/catalan.html.
2. Berndt, B.C.: Ramanujan's Notebooks: Part I, Springer-Verlag , 1985.
3. Berndt, B.C.: Ramanujan's Notebooks: Part II, Springer-Verlag , 1989.
4. Berndt, B.C.: Ramanujan's Notebooks: Part III, Springer-Verlag , 1991.
5. Boros, G. and Moll, V.H.: Irresistible Integrals, Cambridge Univ. Press, 2004.
6. Borwein, J.M. and Borwein, P.B.: Pi and the AGM , Wiley-Interscience, John Wiley & Sons. , Toronto , 1987 .
7. Bradley, D.M.: Representations of Catalan's constant, 2001 , at http://germain.umemat.maine.edu/faculty/bradley/papers/c1.ps .
8. Fee, G.J.: Computation of Catalan's constant using Ramanujan's formula, ISSAC ' 90 . (Proc. Internat. Symp. Symbolic and Algebraic Computation, Aug. 1990) ACM Press – Addison Wesley , 1990 .
9. Lord, N.: Intriguing integrals: an Euler- inspired odyssey, Math. Gazette 91, 2007, 415-427.
10. Ramanujan, S.: On the integral $\int_0^x \frac{\tan^{-1} t}{t}\,dt$, Journal of the Indian Mathematical Society , VII (1915) , pp. 93 – 96.
11. Valdebenito, E.: Question 445: Catalan's Constant and Some Integration Questions. http://vixra.org/pdf/1804.0016v1pdf .