Existence of solutions for Langevin differential equations involving two fractional orders on the half-line

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Abstract

In this paper, we study the existence and uniqueness of solutions for Langevin differential equations of Riemman-Liouville fractional derivative with boundary value conditions on the half-line. By a classical fixed point theorems, several new existence results of solutions are obtained.

Keywords: fractional derivative; fractional Langevin equation; fixed point theorem.

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1 Introduction

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In this paper, we stady the existence and uniqueness of solutions for the following fractional Langevin equations with boundary conditions

$$\begin{aligned} & D^{\alpha}\left(D^{\beta}+\lambda\right)y\left(t\right)=f\left(t,y\left(t\right)\right), \qquad t\in(0,+\infty), \\ & y(0)=D^{\beta}y\left(0\right)=0,\ \lim_{t\to+\infty}D^{\alpha-1}y\left(t\right)=a,\ \ \lim_{t\to+\infty}D^{\alpha+\beta-1}y\left(t\right)=b, \end{aligned} \tag{1}$$

where $1<\alpha\leqslant 2$ and $0<\beta\leqslant 1$, such that $1<\alpha+\beta\leqslant 2$, with $a,b\in\mathbb{R}$, D^{α} and D^{β} are the Riemman-Liouville fractional derivative. Some new results are obtained by applying standard fixed point theorems.

2 Preliminaries

Definition 1 *[2] The Riemann-Liouville fractional integral of ordre $\alpha \in \mathbb{R}^+$.for a function $f \in L^1[a,b]$ is defined as*

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \tag{2}$$

with Γ is Gamma Euler function.

Definition 2 *[2] Let $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}^*$ where $n-1 < \alpha < n$, The Riemann-Liouville dirivative integral of ordre α.for a function $f \in L^1[a,b]$ is defined as*

$$D_a^\alpha f(t) = D^n I_a^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, \tag{3}$$

with $D^n = \frac{d^n}{dt^n}$.

Properties
Let $\delta > 0$ and $\beta > 0$, for all $f \in L^1[a,b]$, we have

$$I^\delta I^\beta f(t) = I^\beta I^\delta f(t) = I^{\delta+\beta} f(t) \tag{4}$$

$$I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}, \quad \beta > -1. \tag{5}$$

If $\beta > \delta > 0$. Then

$$D^\delta I^\beta f(t) = I^{\beta-\delta} f(t). \tag{6}$$

Lemma 3 *[2]Let $\alpha \in \mathbb{R}^+$ where $n-1 < \alpha \leqslant n$, wiht $n \in \mathbb{N}^*$. Then the differential equation $D^\alpha y(t) = 0$, has this general solution*

$$y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}, \tag{7}$$

where $c_i \in \mathbb{R}$, with $i = 0, 1, 2, ..., n$.

Lemma 4 *[2]Let $\alpha > 0$. Then*

$$I^\alpha D^\alpha y(t) = y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}, \tag{8}$$

where $c_i \in \mathbb{R}$, with $i = 0, 1, 2, ..., n$, and $n-1 < \alpha \leqslant n$.

3 Main results

Lemma 5 *Let $h(t) \in C(\mathbb{R}^+, \mathbb{R})$, $1 < \alpha \leqslant 2$ and $0 < \beta \leqslant 1$, with $1 < \alpha+\beta \leqslant 2$. The following problem*

$$\begin{aligned} &D^\alpha \left(D^\beta + \lambda\right) y(t) = h(t), \qquad t \in (0, +\infty), \\ &y(0) = D^\beta y(0) = 0, \ \lim_{t\to+\infty} D^{\alpha-1} y(t) = a, \quad \lim_{t\to+\infty} D^{\alpha+\beta-1} y(t) = b, \end{aligned} \tag{9}$$

has equivalent to the fractional integral equation

$$\begin{aligned} y(t) &= -\frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} y(s)\, ds + \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} h(s) ds \\ &\quad -\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\int_0^{+\infty} h(s) ds + \frac{a+\lambda b}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1}. \end{aligned} \tag{10}$$

Proof. We applied the operartor I^α on $D^\alpha \left(D^\beta + \lambda\right) y(t) = h(t)$, we get

$$\left(D^\beta + \lambda\right) y(t) = I^\alpha h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \tag{11}$$

where $c_1, c_2 \in \mathbb{R}$,
by the boundary condition $y(0) = 0$ and $D^\beta y(0) = 0$ we have $c_2 = 0$, thus

$$D^\beta y(t) = -\lambda y(t) + I^\alpha h(t) + c_1 t^{\alpha-1}, \tag{12}$$

applied the operartor I^β

$$y(t) = -\lambda I^\beta y(t) + I^{\alpha+\beta} h(t) + c_1 I^\beta t^{\alpha-1} + c_3 t^{\beta-1}, \tag{13}$$

where $c_3 \in \mathbb{R}$
by the boundary condition $y(0) = 0$ we have $c_3 = 0$, therefore

$$y(t) = -\lambda I^\beta y(t) + I^{\alpha+\beta} h(t) + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1}. \tag{14}$$

Applied the operator $D^{\alpha+\beta-1}$ on (14), we get

$$D^{\alpha+\beta-1} y(t) = -\lambda D^{\alpha+\beta-1} I^\beta y(t) + I h(t) + c_1 \Gamma(\alpha). \tag{15}$$

We have

$$\begin{aligned} D^{\alpha+\beta-1} I^\beta y(t) &= \frac{d}{dt} I^{1-(\alpha+\beta-1)} I^\beta y(t) \\ &= \frac{d}{dt} I^{2-\alpha} y(t) \\ &= \frac{d}{dt} I^{1-(\alpha-1)} y(t) \\ &= D^{\alpha-1} y(t). \end{aligned} \tag{16}$$

Substituting (16) into (15)

$$D^{\alpha+\beta-1} y(t) = -\lambda D^{\alpha-1} y(t) + I h(t) + c_1 \Gamma(\alpha), \tag{17}$$

Using the boundary value conditions $\lim_{t\to+\infty} D^{\alpha-1} y(t) = a$ and $\lim_{t\to+\infty} D^{\alpha+\beta-1} y(t) = b$, we get

$$c_1 = \frac{a+\lambda b}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)} \lim_{t\to+\infty} I h(t), \tag{18}$$

Substituting the value of c_1 in (14), we obtain

$$y(t) = -\lambda I^{\beta} y(t) + I^{\alpha+\beta} h(t) - \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \lim_{t\to+\infty} I h(t) + \frac{a+\lambda b}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1}. \quad (19)$$

Therefore

$$\begin{aligned} y(t) &= -\frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds + \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s) ds \\ & -\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \int_0^{+\infty} h(s) ds + \frac{a+\lambda b}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1}. \end{aligned} \quad (20)$$

The proof is complete ■

Consider the space defined by

$$E = \left\{ y \in C(\mathbb{R}^+, \mathbb{R}),\ \sup_{t \geqslant 0} \frac{|y(t)|}{1+t^{\beta+\alpha-1}} \text{ exist} \right\} \quad (21)$$

and with the norm

$$\|y\|_E = \sup_{t\geqslant 0} \frac{|y(t)|}{1+t^{\beta+\alpha-1}}. \quad (22)$$

Lemma 6 *[1] The space $(E, \|.\|_E)$ is Banach space.*

We define the operator $T: E \to E$ by

$$\begin{aligned} Ty(t) &= -\frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds + \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s, y(s))\, ds \\ & -\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \int_0^{+\infty} f(s, y(s))\, ds + \frac{a+\lambda b}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1}. \end{aligned}$$

To be completed.

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[3] To be completed.