A generalized Klein Gordon equation with a closed system condition for the Dirac-current probability/field tensor

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I begin with a short historical analysis of the problem of the electron from Lorentz to Dirac. It is my opinion that this problem has been quasi frozen in time because it has always been formulated within the paradigm of the Minkowski-Laue consensus, the relativistic version of the Maxwell-Lorentz theory. By taking spin away from particles and putting it in the metric, thus following Dirac's vision, I start my attempt to formulate an alternative math-phys language. In the created non-commutative math-phys environment, biquaternion and Clifford algebra related, I formulate an alternative for the Minkowski-Laue consensus. This math-phys environment allows me to formulate a generalization of the Dirac current into a Dirac probability/field tensor with connected closed system condition. This closed system condition includes the Dirac current continuity equation as its time-like part. A generalized Klein Gordon equation that includes this Dirac current probability tensor is formulated and analyzed. The Standard Model's Dirac current based Lagrangians are generalized using this Dirac probability/field tensor. The Lorentz invariance or covariance of the generalized equations and Lagrangians is proven. It is indicated that the Dirac probability/field tensor and its closed system condition closes the gap with General Relativity quite a bit.

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I. A PREVIEW OF THE KEY INNOVATIVE PROPOSAL: IV A 3 THE CLOSED SYSTEM CONDITION FOR THE DIRAC PROBABILITY CURRENT TENSOR

The derivative of the probability density tensor in its closed system condition,

$$\partial_\nu \Phi_\mu{}^\nu \equiv \partial_\nu \Psi^\dagger \gamma_\mu \gamma^\nu \Psi = 0, \tag{1}$$

can be retraced to the Klein Gordon equation on the Dirac level as

$$\partial_\nu \Psi^\dagger \not{V} \not{P} \Psi = \partial_\nu \frac{1}{m_0} \Psi^\dagger \not{P} \not{P} \Psi = \partial_\nu \Psi^\dagger U_0 \not{1} \Psi = U_0 \partial_\nu \Psi^\dagger \Psi = 0. \tag{2}$$

which includes the proof of the closed system condition for the symmetric tensor $\not{T} = \not{V} \not{P}$ as $\partial_\nu \not{T} = 0$. This closed system condition applies to both the Dirac representation as the Weyl representation, as long as it is clear that not only γ_0 but also $\boldsymbol{\alpha}$ and $\overline{\Psi}$ have a Dirac representation and a Weyl representation. The gamma tensor $\gamma_\mu \gamma^\nu$ is given by

$$\gamma_\mu \gamma^\nu = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \begin{bmatrix} \gamma_0\gamma_0 & \gamma_1\gamma_0 & \gamma_2\gamma_0 & \gamma_3\gamma_0 \\ \gamma_0\gamma_1 & \gamma_1\gamma_1 & \gamma_2\gamma_1 & \gamma_3\gamma_1 \\ \gamma_0\gamma_2 & \gamma_1\gamma_2 & \gamma_2\gamma_2 & \gamma_3\gamma_2 \\ \gamma_0\gamma_3 & \gamma_1\gamma_3 & \gamma_2\gamma_3 & \gamma_3\gamma_3 \end{bmatrix} = \begin{bmatrix} \not{1} & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & -\not{1} & -\mathbf{i}\Sigma_3 & \mathbf{i}\Sigma_2 \\ \alpha_2 & \mathbf{i}\Sigma_3 & -\not{1} & -\mathbf{i}\Sigma_1 \\ \alpha_3 & -\mathbf{i}\Sigma_2 & \mathbf{i}\Sigma_1 & -\not{1} \end{bmatrix}. \tag{3}$$

The probability density tensor is then given by

$$\Phi_\mu{}^\nu = \Psi^\dagger \gamma_\mu \gamma^\nu \Psi = \begin{bmatrix} \Psi^\dagger \not{1} \Psi & -\Psi^\dagger \alpha_1 \Psi & -\Psi^\dagger \alpha_2 \Psi & -\Psi^\dagger \alpha_3 \Psi \\ \Psi^\dagger \alpha_1 \Psi & -\Psi^\dagger \not{1} \Psi & -\Psi^\dagger \mathbf{i}\Sigma_3 \Psi & \Psi^\dagger \mathbf{i}\Sigma_2 \Psi \\ \Psi^\dagger \alpha_2 \Psi & \Psi^\dagger \mathbf{i}\Sigma_3 \Psi & -\Psi^\dagger \not{1} \Psi & -\Psi^\dagger \mathbf{i}\Sigma_1 \Psi \\ \Psi^\dagger \alpha_3 \Psi & -\Psi^\dagger \mathbf{i}\Sigma_2 \Psi & \Psi^\dagger \mathbf{i}\Sigma_1 \Psi & -\Psi^\dagger \not{1} \Psi \end{bmatrix}. \tag{4}$$

The time-like part of $\partial_\nu \Phi_\mu{}^\nu = 0$ is given by

$$\frac{1}{c} \partial_t \Psi^\dagger \not{1} \Psi + \nabla_1 \Psi^\dagger \alpha_1 \Psi + \nabla_2 \Psi^\dagger \alpha_2 \Psi + \nabla_3 \Psi^\dagger \alpha_3 \Psi = \frac{1}{c} \partial_t \Psi^\dagger \not{1} \Psi + \boldsymbol{\nabla} \Psi^\dagger \boldsymbol{\alpha} \Psi = 0 \tag{5}$$

This can be abbreviated as the Dirac current continuity equation

$$c \partial_\nu \Psi^\dagger \alpha^\nu \Psi = c \partial_\nu \overline{\Psi} \gamma^\nu \Psi = \partial_\nu J^\nu = 0. \tag{6}$$

This proves that the Klein Gordon equation on the Dirac level includes the continuity equation for the probability current as part of a much stronger closed system condition for the probability density (current-)tensor. That connects the Klein Gordon at Dirac level environment to the Laue closed system condition, which in turn is a basic axiom of or prerequisite for General Relativity's symmetric stress energy density tensors $T = VG$.

The space-like derivatives of $\partial_\nu \Phi_\mu{}^\nu = 0$ can be split into a complex part and a real part. The complex part gives

$$\nabla \times \Psi^\dagger \boldsymbol{\Sigma} \Psi = 0. \tag{7}$$

The real part gives

$$\partial_t \Psi^\dagger \boldsymbol{\alpha} \Psi = c \nabla \Psi^\dagger \mathbb{1} \Psi \tag{8}$$

which can be multiplied by the constants $m_0 c$, and using the Dirac adjoint, to give

$$\partial_t m_0 c \overline{\Psi} \boldsymbol{\gamma} \Psi = \nabla m_0 c^2 \overline{\Psi} \gamma_0 \Psi. \tag{9}$$

The last two conditions show that the closed system condition for the probability density tensor is a stronger condition than the continuity equation on its own. The above two conditions can be connected to the earlier $\nabla \times \boldsymbol{p} = 0$ and the $\partial_t \boldsymbol{p} = -\nabla U_i$ as there probability/field analogues. The first prohibits a probability/field vorticity in the closed system condition, the second implies a conserved force-field condition for the probability/field, connecting the time-rate of change of the current to the space divergence of the related density.

Given the fact that all Lagrangians of the Standard Model's Dirac fields are based upon the Dirac current, the Dirac adjoint and the use of the Dirac equation to prove the continuity equation for the Dirac current, it's generalization into a Dirac probability or field tensor with connected much stronger closed system condition and a prove of its validity based upon the Dirac level Klein Gordon equation should have some impact. The recognition that the Dirac current is just a part of a tensor and that the Dirac current continuity equation is just the time-like part of a space-time closed system condition of that tensor will close the gap with General Relativity considerably, given the relation of both to the Laue closed system condition $\partial_\nu T_\mu{}^\nu = 0$. I propose to use tensor Lagrangians based on

$$\mathscr{L} = \frac{1}{m_0} \Psi^\dagger \hat{\not{p}} \hat{\not{p}} \Psi, \tag{10}$$

which then contain the inertial probability or inertial field tensor

$$m_\mu{}^\nu c^2 = m_0 \Phi_\mu{}^\nu c^2 = m_0 \Psi^\dagger \gamma_\mu \gamma^\nu \Psi c^2, \tag{11}$$

as a relativistic generalization of the usual Dirac current with Dirac adjoint based Lagrangians of the Standard Model.

II. HISTORICAL INTRODUCTION

A. The problem of the electron from its discovery until spin.

The problem of the electron around 1905 is the starting point of this paper. In the early-relativistic or pre-Einstein period, Lorentz, Abraham and Poincaré worked on the electromagnetic electron theory in an attempt to make it compatible with classical mechanics (1), (2). The problem was the nature of the electron as an elementary particle. Was it possible to deduce the electron's mechanical properties from electrostatic and -magnetic principles only, or should the electron be considered as a part mechanical, part electromagnetic in origin? Abraham considered the mechanical mass of the electron to be completely electromagnetic, Lorentz and Poincaré opted for the composed mass, part mechanic and part electromagnetic in origin. The problem of the Coulomb-self-energy of the static electron was the starting point. If the non-moving electron was a static point then its electromagnetic self-energy (or rest-mass) exploded to infinity in that point, in clear contradiction with the measured finite mechanical mass. This problem could be avoided by giving the electron a finite size and a sphere was the simplest model for the calculations. Abraham, Lorentz and Poincaré all three accepted the spherical electron-model. Abraham wanted his model to be rigid by principle because a deformable charged sphere would need a compensation-force to prevent a Coulomb-force explosion of the sphere. This compensation-force should have a non-electromagnetic origin with a mixed model as a result. Lorentz and Poincaré choose for a deformable sphere with the necessity of a compensation-force, and -pressure, introduced by Poincaré in 1905. (3), (4).

The real electron is stable, so if this spherical model was right and its stability was to be explained, not declared, then a reaction-force should be introduced to balance the Coulomb-force. This is what Poincaré did in 1905 in such a way that not only Newton's third law of Coulomb-action and Poincaré-reaction, but also the invariance of the rest-mass was saved. Both sphere and stress had the defect of being pure ad-hoc solutions because measurements on that size of the sphere ($\approx 10^{-15}m$) were not possible and the electron in a space free of external influences could not possibly be connected to some observable Poincaré-mechanism balancing the electrostatic Coulomb-force. Poincaré tried to connect it to the only possible influence he could imagine in free space, gravity, but he failed to come up with a convincing connection between his stresses and a theory of gravity. The Abraham-sphere and the Poincaré-stress were never more than smart inventions to avoid theoretical paradoxes that arose in the simultaneous application of electromagnetics and mechanics on the electron-level. Fundamentally, they were constructions for mass-, or Coulomb-self-energy-, normalizations and struggled with the problem of the origin of the electron-mass. But although the problem of the electron remained unsolved, a consensus resulted from the theoretical debates concerning relativistic dynamics of the spherical electron. Although the Abraham ideal of a completely electromagnetic world view wasn't accepted by Lorentz and Poincaré, all three used the hypothesis that *all forces should behave as if they were of electromagnetic origin during a global translation of the system. [...] If all forces, including inertial forces, transformed like electromagnetic forces; [...] in order to respect the relativity principle all forces had to transform like electromagnetic forces; that is, according to a representation of the Lorentz group* (1), (5).

Einstein's formulation of his kinematic theory of Special Relativity was an event that lifted the problem of the electron's electrostatic rest-energy and the balance of forces, or the conservation of momentum, to a higher level. In the Special Theory of Relativity these two classically separated aspects came together in the energy-momentum four-vector. Very early, physicists like Poincaré, Abraham (6), Born (7), Ehrenfest (8), Nordström (9) and Einstein (10) realized the need to include stresses, and thus the stress-tensor, to treat the paradoxes surrounding the electron. In 1911, Laue's tensordynamics merged the energy and momentum with the stress-tensor and the flow of energy into the mechanical stress-energy tensor (11), (12). His relativistic tensordynamics unified Newton's mechanics and Special Relativity and can be seen as the end result of a hotly discussed topic by the specialists of special relativity at the time, the relativistic mechanics of deformable, stressed bodies ((13), p. 19). Maxwell's electro-magnetics and Special Relativity were fused mainly by the work of Minkowski (14), who was the first to formulate a version of the electromagnetic stress-energy tensor. From then on the problem of the free electron could only be treated on a fundamental level by formulating it as the problem of the divergence of its electromagnetic stress-energy tensor.

In Laue's tensor-dynamics, basically presented in (11) (see (13), p.n 43-53 for a concise description, including the closed system condition; see (15) for Weyl's description of it including the problem of the electron), conservation-law considerations require the stress-energy tensor of a free particle in space to have a zero divergence:

$$\partial_\mu T^{\mu\nu}_{mech} = 0. \tag{12}$$

This ensures action to equal reaction and energy to be conserved. Further more, the conservation of angular momentum for such a free particle requires the total stress energy tensor to be symmetric, implying that the free particle has no net internal stresses or net internal elastic forces. The key sentence in Laue's 1911 paper, confirming the opinion of Lorentz and Poincaré, stated: *Planck and Einstein have already expressed that all ponderomotoric forces should behave under a Lorentz transformation in an equal manner as in electrodynamics. Thus it should be possible in all areas of physics to put the force density together with the power density into a four force density.* This leading principle lead Laue to the general expression for relativistic mechanics

$$\mathscr{F}^{\nu} = -\partial_{\mu} T^{\mu\nu}_{mech}. \tag{13}$$

He then stated the assumption that in every area of physics a stress energy tensor could be formulated who's components had the same significance as their electromagnetic counterparts. So basically, Laue assumed that all forces of physics behaved relativistically like in electromagnetics, thus presuming a Maxwell-Lorentz structure beneath all forces as a basis of his relativistic mechanics.

According to von Laue, the electron in free space is a (quasi-)static system, so the divergence of its stress-energy tensor should be zero. But in Minkowski's relativistical electro-magnetics, the divergence of the stress-energy tensor of the electromagnetic field is zero only in charge-free space and equals the electromagnetic Lorentz four-force when charges are present. Electromagnetically the electron in free space is a charge in its own field, so:

$$\partial_{\mu} T^{\mu\nu}_{em} = F^{\nu}_{em}. \tag{14}$$

The electron in free space feels its own em-four-force, which is not zero and not balanced by a reaction-force. So the electron in free space acts a net em-force and em-power on itself. A free electron with a non-zero power must have energy flowing in or out without compensation and an infinite amount of energy, positive or negative, will be assembled. The net force that this free particle acts on itself will create a runaway situation and its momentum will become infinite. It is obvious that this is an erroneous theory because a real electron in free space has a constant macroscopic energy and momentum.

Up till now, two strategies have been developed to find a way out of the conflict. The first is to add a mechanical tensor to the electro-magnetic field-tensor and to declare the divergence of the sum to be zero. This could be called the Poincaré-Laue strategy or the compensation-method. In the words of Laue: *inside the electron different kind of stresses besides the electromagnetic stresses necessarily prevail*. It was the relativistic tensor extension of Poincaré's method to compensate for the Coulomb-force and as such first formulated by von Laue in 1911. In a modern expression, in which the word "Poincaré" is often replaced by "mechanic", the logic goes like this (see (16), (17), (18), (19), (20),(21), (22)):

We know that for a free charged particle in vacuum

$$\partial_\mu T^{\mu\nu}_{total} = 0, \tag{15}$$

and that for the same particle

$$\partial_\mu T^{\mu\nu}_{em} = F^{\nu}_{em}. \tag{16}$$

Let us assume

$$\partial_\mu T^{\mu\nu}_{Poincare} = F^{\nu}_{Poincare}, \tag{17}$$

so that

$$T^{\mu\nu}_{total} = T^{\mu\nu}_{Poincare} + T^{\mu\nu}_{em} \tag{18}$$

with

$$\partial_\mu \left(T^{\mu\nu}_{Poincare} + T^{\mu\nu}_{em}\right) = \tag{19}$$

$$F^{\nu}_{Poincare} + F^{\nu}_{em} = 0 \tag{20}$$

and

$$F^{\nu}_{em} = -F^{\nu}_{Poincare} \tag{21}$$

then we have solved the conflict between relativistic mechanics and relativistic electromagnetics.

The solution should have worked fine when macroscopic experimental setups are concerned in which all kinds of mechanical or chemical compensations can be found, but it fails utterly once the fundamental problem of the nature of the proposed Poincaré-like mechanism for an electron in free space is concerned. Then the strategy turns out to be a guessing in the dark and can be placed in the category of "æther-theories" because it uses an unobservable entity, a Poincaré-mechanism, to explain away a true paradox and fundamental problem in the present state of our physical theory (19).

The second strategy is to suggest changes in the electro-magnetic stress-energy tensor, or some of its components, in such a way that the problem can be solved within the frame of electromagnetics. This was in accordance with the original vision of Abraham, who had the encompassing ideal of reducing all of mechanics to electromagnetics. This strategy generated its own controversy regarding the correct formulation of the electromagnetic stress energy tensor or the connected field momentum density (23). According to Lopéz-Mariño and Jiménez, this controversy is still unsolved (24).

Pauli discussed the Poincaré-Laue compensation strategy in his 1921 article ”Relativitätstheorie” and reviewed the electron-theories of Mie and Einstein (16). Mie, in his 1912 article ”Grundlagen einer Theorie der Materie”, tried to solve the conflict between mechanics and electromagnetics in the spirit of the second strategy (25). In the words of Pauli, Mie ”*set himself the task to generalize the field equations and the energy-momentum tensor in the Maxwell-Lorentz theory in such a way that the Coulomb repulsive forces in the interior of the electrical elementary particles are held in equilibrium by other, equally electrical, forces, whereas the deviations from ordinary electrodynamics remain undetectable in regions outside the particle*”((16),p. 188). Einstein's 1919 comment on Mie's theory was: ”*His theory is based mainly on the introduction into the energy-tensor of supplementary terms depending on the components of the electro-dynamical potential, in addition to the energy terms of the Maxwell-Lorentz theory. These new terms, which in outside space are unimportant, are nevertheless effective in the interior of the electrons in maintaining equilibrium against the electric forces of repulsion. In spite of the beauty of the formal structure of this theory, as erected by Mie, Hilbert and Weyl, its physical results have hithertho been unsatisfactory*” (26)

After having weighed Mie's attempt to solve the electron problem by adding $T_{\mu\nu} = J_\mu A_\nu$ to Minskowski's EM tensor, Einstein tried to formulate a theory in which the repulsive Coulomb-forces inside the electron are held in equilibrium by a gravitational pressure. This solution was already tried by Poincaré in 1905 but failed, due to the lack of a developed theory of relativity including gravity. Einstein formulated a gravitational stress-energy tensor capable of balancing the non-zero divergence of the electromagnetic energy-tensor. Einstein himself concluded that the attempt to connect the Poincaré-stress to the metric failed because it resulted in the general stability of *every* spherically symmetrical distribution of charge. That of course would imply the total non-appearance of the Coulomb-force in nature, a solution that was clearly to general. Einstein interested himself in the problem of the non-zero divergence of the electromagnetic stress-energy tensor because the divergence of the Einstein equations gives, in the limiting case of the special theory of relativity:

$$\partial_\mu T^{\mu\nu}_{mech} = 0. \tag{22}$$

For that reason, the electromagnetic stress-energy tensor for charged matter could not be used in Einstein's equations. Einstein's equations are only valid for stress-energy tensors with zero divergence. In the words of Einstein: "*Therefore, by equation (1)[the Einstein equations] we cannot arrive at a theory of the electron by restricting ourselves to the electromagnetic components of the Maxwell-Lorentz theory, as has long been known. Thus if we hold to (1) we are driven to the path of Mie's theory*" (26).

We end with a series of quotes, relating the Laue closed system condition $\partial_\nu T^{\mu\nu} = 0$ to the problem of the Lorentz electron in free space and the introduction of the Poincaré stresses as the formal solution to this problem, solutions which have never been verified experimentally. The proposed or supposed Poincaré stresses exist at the nominalistic or mathematic level, not in the realistic experimental plane.

This equation ($\partial_\nu T^{\mu\nu} = 0$) is exactly the differential form of the conservation laws of energy and momentum for the electromagnetic field. But it is valid only in the region $r > a$, i.e., outside the electron where no matter is present. (Rohrlich, 1960, (27).)

The model of this classical charged particle is a sphere of radius a, mass m, and uniformly distributed surface charge e. As a free object it is a closed system. If the entire particle were expressible by means of a field and an associated energy tensor $\Theta^{\mu\nu}$ such a tensor would necessarily have to satisfy $\partial_\alpha \Theta^{\alpha\mu} = 0$ since the system is closed. [..] The particle is however not purely electromagnetic but contains an electromagnetic component (the Coulomb field) and an non electromagnetic one. We shall accept the usual assumption that these two components are additive in the energy tensors, $\Theta^{\mu\nu} = \Theta_e^{\mu\nu} + \Theta_n^{\mu\nu}$. Neither of the two components are separately conserved. (Rohrlich, 1982, (18).) *Instead, we have $\partial_\alpha \Theta_e^{\mu\nu} + \partial_\alpha \Theta_n^{\mu\nu} = 0$* (Rohrlich, 1970, (28).)

A necessary condition for an energy-stress tensor T_ν^μ to yield covariantly conserved expressions for energy and momentum is $T_{\nu,\mu}^\mu = 0$ (3). Condition (3) is of course not satisfied by the Lorentz electron. One has $M_{\nu,\mu}^\mu = J^\mu F_{\nu\mu}$. This is the well known self-force problem. The above discussion brings out the fact that a classical electrodynamics based solely on the electromagnetic field can be covariant only in the absence of charged particles, since we have $M_{\nu,\mu}^\mu = 0$, and the self-stress problem does not arise.(Tangherlini, 1963, (29).)

This equation, $\frac{\partial T_{ik}}{\partial x_k} = -\frac{1}{c} F_{ik} j_i$ $(4-55)$which is the generalization of equation $(4-41)$ $\frac{\partial T_{ik}}{\partial x_k} = 0$, contains the mathematical formulation of the law of conservation of energy and momentum of the electromagnetic field together with the particles present in it. (Landau, 1951, (30)).

We can introduce a stress tensor representing the Poincaré stress [..]. The quantity $\theta_{Poincare}^{\mu\nu}$ is *constructed so that $\partial_\mu \theta^{\mu\nu} = \partial_\mu(\theta^{\mu\nu}_{EM} + \theta^{\mu\nu}_{Poincare}) = 0$, where $\partial_\mu(\theta^{\mu\nu}_{EM}$ is the electromagnetic stress tensor associated with the Coulomb field of a spherical shell. [..] The above solution of solving the $3/4$ problem is clearly rather formal and arbitrary.* (Kim, Sessler, 1999, (21)).

Until now, the origin and nature of the Poincaré stresses have been unknown. [..] Of course the problem of the nature of the Poincaré stresse remains, and since progress here seems very difficult, work on classical theories in this line has been abandoned. By now the phenomenological approach, with its renormalization program, has produced important results in quantum electrodynamics, but not so much in classical electrodynamics. (Campos, 1986, (19)).

[..]the well-known fact that the introduction of Poincaré stresses or the Rohrlich redefinition of energy and momentum are, in fact, equivalent in essence [..] Eq. (41), $T^{\mu}_{\nu,\mu} = 0$, refers to the total energy-momentum tensor, while the electromagnetic part of it is not divergenceless at all (its divergence is, simply, the Lorentz fourforce density) (Saldin, 2007, (31)).

In 1956 Casimir proposed that the zero-point force could be the Poincaré stress stabilizing a semiclassical model of an electron [12]. Unfortunately as Tim Boyer was to discover a decade later after a heroic calculation [13], the Casimir force in this case is repulsive. (Milton, 2001, (32)).

The weak nuclear force is the missing non-electromagnetic force or the Poincaré stress which holds the elementary electric charge together. (Koschmieder, 2006, (33)).

The above relation, because the pressure being negative, corresponds to a repulsive gravitational force. This is identified with the Poincaré stress of cohesive force in nature which is required to maintain stability for the Lorentz extended electron. (Ray, 1993, (34)).

These quotes illustrate the fact that the problem of the structure of the electron in relation to Laue's closed system condition is ongoing in the foundations of physics discussions in the literature. In all the quotes the problem is formulated using the formalism of the Minkowski-Laue consensus and all proposed solutions are withing the realm of this hundred year old paradigm.

B. Revising Laue's relativistic mechanics as an approach to the problem of the electron

The problem of the electron, defined as the discrepancy between the divergence of the Laue mechanical stress-energy tensor and the Minkowski electromagnetic stress energy tensor, can be approach in a third way. Instead of suggesting changes in the Maxwell-Lorentz theory or suggesting Poincaré mechanisms added to Laue's tensor, on can try to reformulate relativistic mechanics, Laue's theory as the fixation of the growing consensus in the relativistic avant-garde in the first decade of the nineteenth century. The tricky side catch of this approach is that Einstein used Laue's closed system condition, so essentially Laue's relativistic mechanics, or the relativistic formulation of the conservation of energy, momentum and angular momentum, as a basis for his theory of gravity ((35), postulate 1 on p. 1250; (13), p. 57).

The pre-GR theories of gravity of Abraham (36), (37), (38), Nordström (39), (40), (41) and Mie (25), (42), (43), (44), (45), were based upon the four force equations

$$F_{\mu}^{gravity} = \partial_{\mu} L, \tag{23}$$

with

$$L = -\frac{1}{\gamma} U_0. \tag{24}$$

or, with the densities,

$$f_{\mu}^{gravity} = \partial_{\mu} \mathscr{L}, \tag{25}$$

with the Lorentz invariant

$$\mathscr{L} = -T^{\nu\nu} = -u_0, \tag{26}$$

where in the latter the trace of Laue's stress energy density tensor $T^{\nu\nu}$ functioned as the source of field. After the 1911 establishment of the 'Laue consensus', they all three tried to make their theories conformable to Laue's closed system condition. In the words of Nordström: *The previous reasonings concerning the condition of matter result quite general from the dynamics of matter as developed by von Laue, which I have considered valid in both my theories of gravity* (40). Remaining within Laue's relativistic mechanics, a new theory of gravity, adapted to the requirements of relativity, could not be formulated in a satisfactory manner. Implicitly, they tried to formulate a theory of gravity within the structures set up by Maxwell-Lorentz' electromagnetism. This, amongst other things, motivated Einstein to reject relativistic gravity as a four-force derived from a Lorentz scalar u_0 (or scalar potential $u_0 = \rho\phi_g$) and he succesfully managed to explain gravity as a curvature of space instead. But in the limit to Special Relativity, Einstein's equations still lead to Laue's closed system condition ((13), p. 58). In the language of Poincaré, Einstein left the Lorentz group confinement in his search for a relativistic theory of gravity. Thus for gravity he rejected the key hypothesis of the Minskowki-Laue consensus *that all forces, including inertial forces, transformed like electromagnetic forces*.

The SR Lorentz electron however didn't fit into the Laue's closed system condition without invoking some mysterious, unobserved Poincaré stress. GR gravity couldn't function as a Poincaré mechanism, so in the limit, Einstein's theory did not cover the Lorentz electron. This was one of the circumstances that motivated Einstein to engage an unsuccessful metric GR-EM unification program. But since Einstein's successful relativistic theory of gravity, it is generally accepted that a scalar potential, four force, non-metric theory of gravity formulated within the confinements of Minkowski flat space-time, cannot be made fully relativistic. As a conclusion, if we try to approach the problem of the electron by reformulating Laue's relativistic mechanics, inevitably we will run into trouble with the SR limit of the Einstein Equations, although our approach is an anachronistic pre-GR program.

C. The problem of the electron with elementary spin.

Quantum mechanics added new confusing dimensions to the already problematic electron in the form of spin and spinor properties. In a previous paper, in which we discussed the connection between the gravitational geodesic precession and the Thomas precession, we gave a short history of electron spin, which we will repeat here (46). In 1925 Uhlenbeck and Goudsmit introduced the concept of electron-spin. With this idea of an electron spinning on its orbit around the nucleus they managed to explain the doublet terms in the Hydrogen atom's spectral lines in the Röntgen region and also the a-normal Zeeman-effect. (47) But they didn't manage to explain the factor 2 difference in the magnitude of the coupling of their electron spin to its intrinsic magnetic momentum needed to explain the correct width of the splitting of spectral lines in the a-normal Zeeman-effect. (48) They send their results to Bohr, who discussed it with Kramers. Thomas joined the discussion: *I being a reasonably brash young man in the presence of Bohr said, "Why doesn't someone work it out relativistically." Kramers [...] said to me "It would be a very small relativistic correction. You can work it out, I won't."* (49) Thomas did work out the idea of a relativistic precession of the orbit of the electron and found what we now call the Thomas precession, that produced effects that had to be added to the precession of the electron–spin in its own rest–system. (50) He showed that this extra orbital precession of the electron as a gyroscope was a consequence of relativistic velocity addition applied to rotations, two successive Lorentz boosts added up to one Lorentz boost and a rotation. Thomas explained the factor-two difference in the coupling constant α in the Uhlenbeck and Goudsmit approach as a consequence of the kinematics involved, according to which the relativistic precession of the orbit of the electron had to be added to the precession of the spin in the electrons own reference frame. (51)

The old quantum theory (52) as embodied by Bohr and Sommerfeld around 1924 was the context in which the discovery of electron spin and relativistic precession of the spin axis of the electron in a circular orbit and the relativistic precession of this orbit itself took place. Sommerfeld's book "Atombau und Spektrallinien", translated as "Atomic Structure and Spectral Lines", expressed the old approach which was centered around the model of electrons orbiting a nucleus analogous to the planetary system but subjected to Bohr's restrictive quantum postulates. (53) Part of the problem of the fine structures was solved by the innovative work of Uhlenbeck, Goudsmit and Thomas, who were operating in the context of the Bohr–Sommerfeld approach. In the words of Pais: *the discovery of spin, made after Heisenberg had already published the first paper on quantum mechanics, is an advance in the spirit of the old quantum theory, that wonderfully bizarre mixture of classical reasoning supplemented by ad hoc quantum rules.* (54) In the new Quantum Mechanics of Heisenberg, Schrödinger, Born, Pauli and Dirac, the atomic theory based on the model of the semi-classically orbiting electron became outdated and so did the model of the electron with an internal structure of a spinning gyroscope.

But the confusion regarding the relation of the model based Thomas precession with an electron orbiting a nucleus and the math based Dirac spin with the point-like electron did not just go away. Applied to the Poincaré-Lorentz model of the spherical electron, Thomas' electron spin seemed to require a rotating velocity of many times the speed of light, indicating that spin added a new dimension to the older problem of the stability of the spherical electron. At least, for those physicists that still pondered on that issue. In our paper on the geodesic precession we show that the Thomas precession when applied to macroscopic spinning gyroscopes gives correct results, verified by experiments like Gravity Probe B. So in the macroscopic world where the models are a clear representation of reality, Thomas' approach gives correct results. But in the femtoscopic world of the elementary electron, where the electron model leads to paradoxes, Thomas' recipe still functions. The intriguing question is: 'Why is it still functioning?' The Thomas' approach to electron spin has quantum mechanically been superseded by the Dirac's equation and the correspondence principle of Bohr, that heuristic instrument only really understood by Bohr himself, is sometimes invoked to explain the correctness of both theories ((55) , p. 214).

After the theory of Thomas and the one of Dirac, physicists like Frenkel, Kramers and de Broglie tried to understand spin more or less in the relativistic language of anti-symmetric six-vectors, partly in tensor formulation (56), (57), (58), (59). A few decades later, in his 1974 treatise on spin, Tomonaga used a similar six-vector formulation of the relativistic interpretation of Dirac spin ((55), p. 61, p. 207). Six-vectors where introduced by Minkowski in 1908 together with the four-vector notation of space-time as $(\mathbf{r}, ict)$ (14). Four-vectors and six-vectors were given a more profound geometrical interpretation by Sommerfeld in 1910 (60), (61). The most important example of a six-vector was, and still is, the electromagnetic six-vector $\overrightarrow{\mathbb{B}} = \boldsymbol{B} - \mathbf{i}\frac{1}{c}\mathbf{E}$. With the use of this six-vector and the four-vectors for charge-currency and the divergence- and rotation-operators, Minkowski rewrote Maxwell's equations for electromagnetism in a form that was invariant under a Lorentz Transformation. But Minkowski, a mathematician, mostly preferred tensor- and matrix-representation of vectors, whereas Sommerfeld, a physicist, deliberately chose vector-formulations, witch he esteemed more directly connected to the physical world in its three dimensional geometry and for that reason better accessible for the physicists of his day. Due mainly to Laue and Einstein, second-rank tensors became the dominant notation in books and articles on relativism whereas Sommerfeld's six-vector notation got used less and less in the course of the twentieth century but never disappeared.

Frenkel, de Broglie and Kramers started their relativistic spin analysis with an attempt to formulate angular momentum in a relativistic six-vector form analogous to the magnetization-polarization six-vector of electromagnetism. But the relativistic generalization of angular momentum itself was found to lead to a confusion that couldn't be solved. Already with Thomas and Frenkel in 1926 there was a dispute concerning the relativistic generalization of the connected angular frequency. Thomas discussed both options, a four vector with a scalar acceleration and a six-vector with a three vector acceleration. He concluded that in a system where the spinning electron was at rest, the acceleration would be zero anyhow and he showed that both choices lead to the same three relations for the angular velocity. In his own words, using $\mathbf{w}$ for the angular frequency three vector $\boldsymbol{\omega}$: *It makes no difference which of* $\mathbf{w}$, w^{μ}, $w_{\mu\nu}$ *is used to find the secular change.* So in his derivation he didn't have to make a choice between four vector model or six-vector model for the spinning electron, but he nevertheless formulated both options, indicating that he was theorizing within Minkowski-Sommerfeld structures. Thomas then related his result to the spherical model of the electron, the Abraham spinning electron, and arrives at the conclusion that the charges on the surface should rotate with a velocity two hundred times the velocity of light, which he deems absurd. He added: *I think we must look towards the general relativity theory for an adequate solution of the problem of the "structure of the electron"; if indeed this phrase has any meaning at all and if it can be possible to do more than to say how an electron behaves in an external field.* (51)

In the perspective of Thomas, the relativistic generalization of spin angular frequency as a three vector was an acceleration, either as a scalar, resulting in a four vector, or as a three vector, resulting in a six-vector. Thus, with relativistic electron spin frequency as a quantum thing and relativistic electron acceleration as a GR thing, two thus far incompatible math-phys languages seem involved in the problem of the (precession) frequency of the relativistic electron spin. Frenkel argued that the relativistic generalization of spin had to be a six-vector, not a four vector, due to the fact that mechanical electron spin had to be accompanied by a magnetic moment with the Bohr magneton as its size. And in relativistic electromagnetics, it had already been proven that the correlated three vector of magnetic moment was electric polarization. Then the analogy lead to the conclusion that relativistic electron spin had to be a six-vector, with the strange property that its dipole three vector part had to be zero in its own rest system. Frenkel defined the first three components of his angular momentum six-vector as the three components of the 3-dimensional angular momentum **L**, but, curiously enough, in the entire article not a single physical interpretation of the last three components of this six-vector was given. It remained a purely mathematical entity needed for a complete relativistic calculation. The analysis of Frenkel connected the debate concerning the problem of the spinning electron to the already existing confusion of the correct interpretation of the magnetic moment and electric polarization six-vector or anti-symmetric tensor (56). So he added the Minkowski EM math-phys language to the problem of the correct relativistic treatment of the spinning electron.

Kramers in part two of his *Quantum Mechanics*, a translation of *Quantentheorie des Elektrons und der Strahlung* of 1938 started with quoting Frenkels 1926 paper and repeating his six-vector theory of electron spin. After adding the Thomas factor he turns to the Pauli matrix representation and explains that one can add a unit matrix times a constant to the Pauli spin three vector matrix in order to get a most general observable which refers to the electron spin. This in fact looks like the four vector Pauli spin to which de Broglie will refer later on. The result is a combination of a six-vector theory and a matrix four vector, which together combine into the Hamiltonian EM-field spin interaction operator. But in this operator, only the three vectors of the magnetic momentum and the spin angular momentum remain, with the electric polarization and the spin's fourth unit matrix set zero in the rest frame of the electron. To add to the confusion, the foregoing is part of Kramer's non-relativistic treatment of the spinning electron. In the second part, the Dirac spinning electron is presented in the form of a doubling of the Pauli spin structure (57).

De Broglie signaled that Pauli-spin and Dirac-spin defied simple incorporation into the anti-symmetric tensor or six-vector scheme. He formulated relativistic angular momentum as an anti-symmetric tensor and related six-vector but argued that quantum spin should be made relativistic as a density four vector, with a zero time-like part in its own rest system. But later on he connected spin magnetic moment to the usual magnetic moment electric polarization six-vector or anti-symmetric tensor of relativistic electromagnetics. So by relating the three vector spin to both a four vector and to a six-vector generalization without a clear connection between both generalizations, de Broglie puts the reader in a state of confusion regarding the relativistic status of the electron spin that matches Kramers (58), (59).

In Tomonaga's *The story of spin*, the Pauli spin as a three vector is completed by a fourth unit matrix and then this set of four 2x2 matrices is used to produce a six-vector on a Dirac level. But this six-vector, with 2x2 matrices containing the Pauli spin matrices as basic elements, has two three vectors who neither equal the Dirac spin three vector presented earlier in the book. But still, Tomonaga says that we can interpret them *as components of one six-vector, and we can regard them as the relativistic generalization of electron spin*. Later on in the book, the EM spin interaction energy term is given as a linear combination of this six-vector spin and the electromagnetic six-vector (55).

We conclude that the way Thomas, Frenkel, Kramers, de Broglie and Tomonaga presented their attempts towards and understanding of the relativistic spinning electron, just added confusion to an already problematic scene regarding the problem of the electron. But at the same time, elements of their presentations cannot be neglected, as they incorporate experimentally verified quantum aspect and classical aspects of the electron. Individually, they all added insights and provisional partial structures concerning the problematic relation between electron spin and relativistic electrodynamics and mechanics. They invoked two sets of languages to deal with the relativistic problem of the spinning electron, the Minkowski-Laue consensus and Pauli-Dirac spin QM but remained incapable of fusing the concepts and the math-phys of these two paradigmatic approaches.

D. Dirac's return to the pre-quantum theory of the electron

In 1938 Dirac returned to the Lorentz model of the electron in an attempt to find an opening regarding the self-energy problem as it reappeared in quantum mechanics, a problem that prevented the application of quantum mechanics to high-energy radiative processes (62). This problem was eventually solved with the development of QED by Feynman, Schwinger, Tomonaga and Dyson.

But Dirac didn't like the renormalization solution of QED and he continued to try to solve the electron's self-energy problem, related to the Lorentz model, in a more fundamental way, by going back to the pre-quantum theory of relativistic electrodynamics (63).

In an article of 1949 Dirac tried to combine the restricted principle of relativity with the Hamiltonian formulation of (quantum-) dynamics in a paper titled *Forms of Relativistical Dynamics* (64). This lead to the appearance of ten fundamental quantities for each dynamical system, the four-momentum P_μ and the six-vector, in tensor-form, $M_{\mu\nu}$ which had three components equal to the total angular momentum. The remaining three components did not correspond to any such well-known physical quantities, but Dirac considered them equally important in the general scheme. According to Dirac the four-vector P_μ and the six-vector $M_{\mu\nu}$ formed together the ten fundamental quantities or physical variables. In his treatment $M_{\mu\nu}$ was the four-dimensional cross-product of the four-momentum P_μ and the generalized four-coordinate Q_ν with $M_{\mu\nu} = q_\mu p_\nu - q_\nu p_\mu$. In his article, Dirac never gave a physical interpretation of the extra three vector component of his second-rank tensor or six-vector. Dirac mentioned the appearance of ten fundamental quantities in the cross-product of the four-location and the four-momentum as kind of a mixture of the symmetric and the anti-symmetric tensor products. But his attempt to formulate a new relativistic dynamics didn't lead to enduring results.

A few year later, Dirac expressed his intuition regarding the problem of the electron as follows: *Classical electrodynamics is based on Maxwell's equations for the electromagnetic field and Lorentz's equations of motion for electrons. It is an approximate theory [...] and all attempts to make it accurate bring one up against the problem of the structure of the electron, which has not received any satisfactory solution. People hoped at one time that quantum mechanics would remove these difficulties, but this hope has not been fulfilled. To make progress one should therefore re-examine the classical theory of electrons and try to improve on it. (63) We can see now that we may very well have an æther, subject to quantum mechanics and conforming to relativity, [...]. We must make some profound alterations in our theoretical ideas of the vacuum. It is no longer a trivial state, but needs elaborate mathematics for its description.[...] Thus with the new theory of electrodynamics we are rather forced to have and æther. (65) It (the new æther) will probably have to be modified by the introduction of spin variables before a satisfactory quantum theory of electrons can be obtained from it, and only after this has been accomplished will one be able to give a definite answer to the æther question. (66)*

This is the point were we catch on, Dirac's suggestion of introducing spin variables into the vacuum/metric/æther as a necessary step forward in our understanding of the electron. This means that in dealing with the problem of relativistic dynamics regarding the problem of the (spinning) electron, we ignore what was to come afterwards, the Yang-Mills theories of the weak force and the strong force. So not only is our approach anachronistic relative to General Relativity, due to our focus on revising the Minkowski-Laue consensus, but it is also anachronistic relative to the Yang-Mill's theories of the Standard Model. In no way does this mean that we criticize these experimentally strongly tested and verified theories as such. The anachronistic approach is chosen because it greatly simplifies the problem of the electron in the context of revising relativistic dynamics (or replacing the Minkowski-Laue paradigm). The hope is that the fundamental problems regarding the electron can be dealt with in such an anachronistic way so that it will nevertheless allow us to produce interesting results.

In the foundations of physics discussion, several papers support Dirac's view that QED found a (highly successful) way around the infinite self energy of the electron but without solving the problem itself. In 1986 Jiménez and Campos, discussing the Boyer-Rohrlich controversy, wrote: *[...] questions about the stability of the electron, the nature of the electron mass (totally electromagnetic or partly nonelectromagnetic), and infinite self energy. The clarification of these problems is worthwhile since most of them appear again in quantum electrodynamics* (19). In their paper, the whole problem of around 1905-1915 reappears, unsolved, almost frozen in time. In 1999 they end a paper titled *Models of the classical electron after a century* with the sentence *Thus, after a century, the search for a deeper understanding of the electron continues* (67). But strangely enough, modern discussions of the problem of the electron in the math-phys framework of Laue's relativistic mechanics do not incorporate the problem of electron spin with Pauli of Dirac matrices as it appeared in the treatments of Kramers, de Broglie and Tomonaga.

We believe that three results presented in the following mathematical part might be interesting. First the Lorentz transformation as a basis transformation with invariant coordinates as a result of our way to put spin into the metric. Second the relativistic mechanics as an alternative to the Minkoski-Laue consensus that results from our math-phys language. Further more, our approach is focused on producing only a rough sketch of a math-phys language that is rich enough in its internal structures in order to comprise the discussions regarding the problem of the (spinning) electron in a better way than was possible in the past. Presenting only a first gross outline of a somewhat new math-phys language we ignore inevitably many, many details of the theories involved. The goal was to create one math-phys language for the electron problem, one that replaced the Minkowski-Laue consensus and contained pre-YM Pauli-Dirac QM and pre-GR gravitation. The first goal has been achieved, the second is still open, but has been closed in upon.

III. THE PAULI SPIN LEVEL

A. A complex quaternion basis for the metric

Quaternions can be represented by the basis $(\hat{\mathbf{1}}, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$. This basis has the properties $\hat{\mathbf{I}}\hat{\mathbf{I}} = \hat{\mathbf{J}}\hat{\mathbf{J}} = \hat{\mathbf{K}}\hat{\mathbf{K}} = -\hat{\mathbf{1}}$ and $\hat{\mathbf{1}}\hat{\mathbf{1}} = \hat{\mathbf{1}}$; $\hat{\mathbf{1}}\hat{\mathbf{K}} = \hat{\mathbf{K}}\hat{\mathbf{1}} = \hat{\mathbf{K}}$ for $\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}$; $\hat{\mathbf{I}}\hat{\mathbf{J}} = -\hat{\mathbf{J}}\hat{\mathbf{I}} = \hat{\mathbf{K}}$; $\hat{\mathbf{J}}\hat{\mathbf{K}} = -\hat{\mathbf{K}}\hat{\mathbf{J}} = \hat{\mathbf{I}}$; $\hat{\mathbf{K}}\hat{\mathbf{I}} = -\hat{\mathbf{I}}\hat{\mathbf{K}} = \hat{\mathbf{J}}$. A quaternion number in its summation representation is given by $A = a_0\hat{\mathbf{1}} + a_1\hat{\mathbf{I}} + a_2\hat{\mathbf{J}} + a_3\hat{\mathbf{K}}$, in which the a_μ are real numbers . Bi-quaternions or complex quaternions are given by

$$\begin{aligned} C = A + \mathbf{i}B = c_0\hat{\mathbf{1}} + c_1\hat{\mathbf{I}} + c_2\hat{\mathbf{J}} + c_3\hat{\mathbf{K}} \\ (a_0 + \mathbf{i}b_0)\hat{\mathbf{1}} + (a_1 + \mathbf{i}b_1)\hat{\mathbf{I}} + (a_2 + \mathbf{i}b_2)\hat{\mathbf{J}} + (a_3 + \mathbf{i}b_3)\hat{\mathbf{K}} = \\ a_0\hat{\mathbf{1}} + a_1\hat{\mathbf{I}} + a_2\hat{\mathbf{J}} + a_3\hat{\mathbf{K}} + \mathbf{i}b_0\hat{\mathbf{1}} + \mathbf{i}b_1\hat{\mathbf{I}} + \mathbf{i}b_2\hat{\mathbf{J}} + \mathbf{i}b_3\hat{\mathbf{K}}, \end{aligned} \tag{27}$$

in which the $c_\mu = a_\mu + \mathbf{i}b_\mu$ are complex numbers and the a_μ and b_μ are real numbers.

The biquaternions can be used to provide a basis for relativistic space-time. One way to do this is by making the time coordinate c_0 complex and the space coordinates c_1, c_2, c_3 real. This however produces confusion regarding the time-like complex number as the physics gets more complex. It also produces language conflicts with almost all of modern physics, that is Quantum Mechanics and Special and General Relativity. For this reason, I choose to insert the time-like complex number of c_0 in the basis instead of in the coordinate. So by using $c_0\hat{\mathbf{1}} = b_0\mathbf{i}\hat{\mathbf{1}} = b_0\hat{\mathbf{T}}$ the space-time basis is then given by $(\hat{\mathbf{T}}, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$. In this way, the coordinates are always a set of real numbers $\in \mathbb{R}$. Spinors however are always given by a set of complex numbers.

A set of four numbers $\in \mathbb{R}$ is given by

$$A^\mu = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}, \tag{28}$$

or by

$$A_\mu = [a_0, a_1, a_2, a_3]\,. \tag{29}$$

The biquaternion basis can be given as a set $\mathbf{K}^\mu$ as

$$\mathbf{K}^\mu = \begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{bmatrix}, \tag{30}$$

Then a biquaternion space-time vector can be written as

$$A = A_\mu \mathbf{K}^\mu = [a_0, a_1, a_2, a_3] \begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{bmatrix} = a_0\hat{\mathbf{T}} + a_1\hat{\mathbf{I}} + a_2\hat{\mathbf{J}} + a_3\hat{\mathbf{K}} \tag{31}$$

I apply this to the space-time four vector of relativistic bi-quaternion 4-space R with the four numbers R^μ as

$$R^\mu = \begin{bmatrix} ct \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix}. \tag{32}$$

so with $r_0, r_1, r_2, r_3 \in \mathbb{R}$. Then we have $R = R_\mu \mathbf{K}^\mu$ or

$$R = R_\mu \hat{\mathbf{K}}^\mu = r_0\hat{\mathbf{T}} + r_1\hat{\mathbf{I}} + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}} = r_0\hat{\mathbf{T}} + \mathbf{r}\cdot\mathbf{K}. \tag{33}$$

We use the three-vector analogue of $R_\mu \mathbf{K}^\mu$ when we write $\mathbf{r}\cdot\mathbf{K}$. In this notation we have

$$R^T = -r_0\hat{\mathbf{T}} + r_1\hat{\mathbf{I}} + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}} = -r_0\hat{\mathbf{T}} + \mathbf{r}\cdot\mathbf{K} \tag{34}$$

for the time reversal operator and

$$R^P = r_0\hat{\mathbf{T}} - r_1\hat{\mathbf{I}} - r_2\hat{\mathbf{J}} - r_3\hat{\mathbf{K}} = r_0\hat{\mathbf{T}} - \mathbf{r}\cdot\mathbf{K} \tag{35}$$

for the space reversal operator. We have $R^P = -R^T$. In this notation, the transpose of a matrix will be given by the suffix 't', so $R^t_\mu = R^\mu$. The complex transpose of spinors is given by the dagger symbol, as in $\psi^\dagger$. The complex conjugate of a spinor is given by ψ^*.

B. Matrix representation of the quaternion basis

Quaternions can be represented by 2x2 matrices. Several representations are possible, but most of those choices result in conflict with the standard approach in physics. My choice is

$$\hat{\mathbf{1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \hat{\mathbf{T}} = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix}, \hat{\mathbf{I}} = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}, \hat{\mathbf{J}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \hat{\mathbf{K}} = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{36}$$

I can compare these to the Pauli spin matrices $\boldsymbol{\sigma}_P = (\sigma_x, \sigma_y, \sigma_z)$.

$$\boldsymbol{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \boldsymbol{\sigma}_y = \begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, \boldsymbol{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{37}$$

If I exchange the σ_x and the σ_z, I get $\mathbf{K} = \mathbf{i}\boldsymbol{\sigma}$ and $\mathbf{K}_\mu = \mathbf{i}(\hat{\mathbf{1}}, \boldsymbol{\sigma})$. So in my use of the Pauli matrices, I use $\boldsymbol{\sigma} \equiv (\sigma_I, \sigma_J, \sigma_K) = (\sigma_z, \sigma_y, \sigma_x)$. So also $\hat{\mathbf{I}} = \hat{\mathbf{T}}\boldsymbol{\sigma}_I, \hat{\mathbf{J}} = \hat{\mathbf{T}}\boldsymbol{\sigma}_J, \hat{\mathbf{K}} = \hat{\mathbf{T}}\boldsymbol{\sigma}_K$ and $\boldsymbol{\sigma}_I = -\hat{\mathbf{T}}\hat{\mathbf{I}}, \boldsymbol{\sigma}_J = -\hat{\mathbf{T}}\hat{\mathbf{J}}, \boldsymbol{\sigma}_K = -\hat{\mathbf{T}}\hat{\mathbf{K}}$.

With this choice of matrices, a four-vector R can be written as

$$R = r_0 \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix} + r_1 \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{38}$$

This can be compacted into a matrix representation of R:

$$R = \begin{bmatrix} r_0\mathbf{i} + \mathbf{i}r_1 & r_2 + \mathbf{i}r_3 \\ -r_2 + \mathbf{i}r_3 & r_0\mathbf{i} - \mathbf{i}r_1 \end{bmatrix} = \begin{bmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{bmatrix} \tag{39}$$

with the numbers $R_{00}, R_{01}, R_{10}, R_{11} \in \mathbb{C}$.

C. Multiplication of vectors as matrix multiplication

In general, multiplication of two vectors A and B follows matrix multiplication, with $A_{ij}, B_{ij}, C_{ij} \in \mathbb{C}$.

$$AB = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix} = C. \tag{40}$$

So we have

$$C = AB = \begin{bmatrix} A_{00}B_{00} + A_{01}B_{10} & A_{00}B_{01} + A_{01}B_{11} \\ A_{10}B_{00} + A_{11}B_{10} & A_{10}B_{01} + A_{11}B_{11} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix}. \tag{41}$$

Of course, vectors A, B and C can be expressed with their a_μ, b_μ, c_μ coordinates $\in \mathbb{R}$ and if we use them, after some elementary but elaborate calculations and rearrangements we arrive at the following result of the multiplication expressed in the a_μ, b_μ and c_μ as

$$\begin{aligned} c_0 &= -a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 \\ c_{1K} &= a_2b_3 - a_3b_2 \\ c_{2K} &= a_3b_1 - a_1b_3 \\ c_{3K} &= a_1b_2 - a_2b_1 \\ c_{1\sigma} &= -a_0b_1 - a_1b_0 \\ c_{2\sigma} &= -a_0b_2 - a_2b_0 \\ c_{3\sigma} &= -a_0b_3 - a_3b_0 \end{aligned} \tag{42}$$

In short, if we use the three-dimensional Euclidean dot and cross products of Euclidean three-vectors in classical physics, this gives for the coordinates

$$c_0 = -a_0b_0 - \mathbf{a}\cdot\mathbf{b}$$

$$\mathbf{c}_K = \mathbf{a}\times\mathbf{b} \tag{43}$$

$$\mathbf{c}_\sigma = -a_0\mathbf{b} - \mathbf{a}b_0 \tag{44}$$

And in the quaternion notation we get

$$C = AB = (-a_0b_0 - \mathbf{a}\cdot\mathbf{b})\hat{\mathbf{1}} + (\mathbf{a}\times\mathbf{b})\cdot\mathbf{K} + (-a_0\mathbf{b} - \mathbf{a}b_0)\cdot\boldsymbol{\sigma} \tag{45}$$

This matrix multiplication, in which I used $\hat{\mathbf{T}}\hat{\mathbf{T}} = -\hat{\mathbf{1}}$ and $\hat{\mathbf{T}}\mathbf{K} = -\boldsymbol{\sigma}$, implies that the space-time basis $(\hat{\mathbf{T}}, \mathbf{K})$ is being duplicated by a spin-norm basis $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$.

The physically relevant multiplications of two four-vectors are all in the form $C = A^TB$. The difference between AB and A^TB is in the sign of a_0. This results in

$$C = A^TB = (a_0b_0 - \mathbf{a}\cdot\mathbf{b})\hat{\mathbf{1}} + (\mathbf{a}\times\mathbf{b})\cdot\mathbf{K} + (a_0\mathbf{b} - \mathbf{a}b_0)\cdot\boldsymbol{\sigma} \tag{46}$$

From this it follows that the physically relevant norm of a four-vector, from a relativistic perspective, is the product A^TA and not the product AA:

$$C = A^TA = (a_0a_0 - \mathbf{a}\cdot\mathbf{a})\hat{\mathbf{1}} + (\mathbf{a}\times\mathbf{a})\cdot\mathbf{K} + (a_0\mathbf{a} - \mathbf{a}a_0)\cdot\boldsymbol{\sigma} = (a_0a_0 - \mathbf{a}\cdot\mathbf{a})\hat{\mathbf{1}} = c^2a_\tau^2\hat{\mathbf{1}}. \tag{47}$$

The main quadratic form of the metric is

$$dR^T dR = (c^2dt^2 - d\mathbf{r}^2)\hat{\mathbf{1}} = c^2d\tau^2\hat{\mathbf{1}} = ds^2\hat{\mathbf{1}} \tag{48}$$

with $ds = cd\tau$. The quadratic giving the distance of a point R to the origin of its reference system is given by

$$R^T R = (c^2t^2 - \mathbf{r}^2)\hat{\mathbf{1}} = c^2\tau^2\hat{\mathbf{1}} = s^2\hat{\mathbf{1}} \tag{49}$$

with $s = c\tau$.

The multiplication of two four vectors can also be arranged as the multiplication of two tensors, a coordinate tensor times a metric tensor using that

$$(A_\mu \mathbf{K}^\mu)^T B_\nu \mathbf{K}^\nu = A_\mu B^\nu (\mathbf{K}_\mu)^T \mathbf{K}^\nu = C_\mu{}^\nu \mathbf{K}_\mu{}^\nu \tag{50}$$

with the metric tensor as

$$\mathbf{K}_\mu{}^\nu = (\mathbf{K}_\mu)^T\mathbf{K}^\nu = \begin{bmatrix} -\hat{\mathbf{T}}, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{bmatrix} = \begin{bmatrix} -\hat{\mathbf{T}}\hat{\mathbf{T}} & \hat{\mathbf{I}}\hat{\mathbf{T}} & \hat{\mathbf{J}}\hat{\mathbf{T}} & \hat{\mathbf{K}}\hat{\mathbf{T}} \\ -\hat{\mathbf{T}}\hat{\mathbf{I}} & \hat{\mathbf{I}}\hat{\mathbf{I}} & \hat{\mathbf{J}}\hat{\mathbf{I}} & \hat{\mathbf{K}}\hat{\mathbf{I}} \\ -\hat{\mathbf{T}}\hat{\mathbf{J}} & \hat{\mathbf{I}}\hat{\mathbf{J}} & \hat{\mathbf{J}}\hat{\mathbf{J}} & \hat{\mathbf{K}}\hat{\mathbf{J}} \\ -\hat{\mathbf{T}}\hat{\mathbf{K}} & \hat{\mathbf{I}}\hat{\mathbf{K}} & \hat{\mathbf{J}}\hat{\mathbf{K}} & \hat{\mathbf{K}}\hat{\mathbf{K}} \end{bmatrix} = \tag{51}$$

$$\begin{bmatrix} \hat{\mathbf{1}} & -\sigma_I & -\sigma_J & -\sigma_K \\ \sigma_I & -\hat{\mathbf{1}} & -\hat{\mathbf{K}} & \hat{\mathbf{J}} \\ \sigma_J & \hat{\mathbf{K}} & -\hat{\mathbf{1}} & -\hat{\mathbf{I}} \\ \sigma_K & -\hat{\mathbf{J}} & \hat{\mathbf{I}} & -\hat{\mathbf{1}} \end{bmatrix}. \tag{52}$$

This multiplication product has a norm $\hat{\mathbf{1}}$ part, a space $\mathbf{K}$ part and a spin $\boldsymbol{\sigma}$ part. So the multiplication of two four vectors $A^T B = C$ has this multiplication matrix. The multiplication combines the properties of symmetric and anti-symmetric in one product.

D. The Lorentz transformation

Usually the Lorentz transformation is given as a coordinate transformation against a Minkowski spacetime background. This spacetime background is an inert, static theater in which the physics of special relativity takes place. Without gravity, this metric is presumed to be flat or inert. A Lorentz transformation acts upon the coordinates, not upon the metric. This is the context of Special Relativity. Quantum mechanics, from Schrödinger's to Dirac's version, is defined in this environment of Special Relativity. The metric of Quantum Theory is Minkowski flat or inert.

A normal Lorentz transformation between two reference frames connected by a relative velocity v in the $x-$ or $\hat{\mathbf{I}}$-direction, with the usual $\gamma = 1/\sqrt{1 - v^2/c^2}$, $\beta = v/c$ and $r_0 = ct$, can be expressed as

$$\begin{bmatrix} r_0' \\ r_1' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} \gamma r_0 - \beta\gamma r_1 \\ \gamma r_1 - \beta\gamma r_0 \end{bmatrix}. \tag{53}$$

We want to connect this to our matrix representation of R as in Eq.(39) which gives

$$R_{00}' = \mathbf{i}r_0' + \mathbf{i}r_1' = \mathbf{i}\gamma r_0 - \mathbf{i}\beta\gamma r_1 + \mathbf{i}\gamma r_1 - \mathbf{i}\beta\gamma r_0 \tag{54}$$

$$R_{11}' = \mathbf{i}r_0' - \mathbf{i}r_1' = \mathbf{i}\gamma r_0 - \mathbf{i}\beta\gamma r_1 - \mathbf{i}\gamma r_1 + \mathbf{i}\beta\gamma r_0. \tag{55}$$

Now we want to introduce rapidity or hyperbolic Special Relativity in order to integrate Lorentz transformations into our matrix metric. In (46) we gave a brief history of rapidity in its relation to the Thomas precession and the geodesic precession. For this paper we only need elementary rapidity definitions. If we use the rapidity ψ as $e^{\psi} = \cosh\psi + \sinh\psi = \gamma + \beta\gamma$, the previous transformations can be rewritten as

$$R_{00}' = \mathbf{i}r_0' + \mathbf{i}r_1' = (\gamma - \beta\gamma)(\mathbf{i}r_0 + \mathbf{i}r_1) = R_{00}e^{-\psi} \tag{56}$$

$$R_{11}' = \mathbf{i}r_0' - \mathbf{i}r_1' = (\gamma + \beta\gamma)(\mathbf{i}r_0 - \mathbf{i}r_1) = R_{11}e^{\psi}. \tag{57}$$

As a result we have

$$R^L = \begin{bmatrix} R_{00}' & R_{01}' \\ R_{10}' & R_{11}' \end{bmatrix} = \begin{bmatrix} R_{00}e^{-\psi} & R_{01} \\ R_{10} & R_{11}e^{\psi} \end{bmatrix} = U^{-1}RU^{-1}. \tag{58}$$

In the expression $R^L = U^{-1}RU^{-1}$ we used the matrix U as

$$U = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix}. \tag{59}$$

But this means that we can write the result of a Lorentz transformation on R with a Lorentz velocity in the $\hat{\mathbf{I}}$-direction between the two reference systems as

$$R^L = r_0 \begin{bmatrix} \mathbf{i}e^{-\psi} & 0 \\ 0 & \mathbf{i}e^{\psi} \end{bmatrix} + r_1 \begin{bmatrix} \mathbf{i}e^{-\psi} & 0 \\ 0 & -\mathbf{i}e^{\psi} \end{bmatrix} + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{60}$$

This can be written as

$$R^L = r_0U^{-1}\hat{\mathbf{1}}U^{-1} + r_1U^{-1}\hat{\mathbf{I}}U^{-1} + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}} = r_0\hat{\mathbf{1}}^L + r_1\hat{\mathbf{I}}^L + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}}. \tag{61}$$

But because we started with Eq.(53), we now have two equivalent options to express the result of a Lorentz transformation

$$R^L = r_0'\hat{\mathbf{1}} + r_1'\hat{\mathbf{I}} + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}} = r_0\hat{\mathbf{1}}^L + r_1\hat{\mathbf{I}}^L + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}}, \tag{62}$$

either as a coordinate transformation or as a basis transformation.

This shows that we do not need to have an inert metric any more. In our metric, a Lorentz transformation can leave the coordinates invariant and only change or rotate the basis on the level of spin-matrices. Mathematically we can formulate a Lorentz transformation as a matrix internal twist of the quaternion matrix basis, leaving the coordinates unchanged. A Lorentz transformation thus twists the metric. This implies that our metric is not the Minkowski metric of Special Relativity any more, although it remains closely related to it. But is it a metric that can accomodate Quantum Physics?

This result only works for Lorentz transformation between v_x-, v_1- or $\hat{\mathbf{I}}$-aligned reference systems. Reference systems which do not have their relative Lorentz velocity aligned in the $\hat{\mathbf{I}}$-direction will have to be space rotated into such an alignment before the Lorentz transformation in the form $R^L = U^{-1}RU^{-1}$ is applied. In principle, such a rotation in order to achieve the $\hat{\mathbf{I}}$ alignment of the primary reference frame to a secondary reference frame is always possible as an operation prior to a Lorentz transformation.

The interesting thing about the $e^{\psi} = \gamma + \beta\gamma$ term is that it represents a relativistic Doppler-correction applied to the frequency ν of light-signals exchanged between two inertial reference systems.

$$\frac{\nu}{\nu_0} = e^{\psi}. \tag{63}$$

So if we twist the matrix basis internally as to compensate for the relativistic Doppler shift, then the coordinates can remain invariant under a Lorentz transformation. As to the $\hat{\mathbf{I}}$-alignment issue, two reference systems that exchange light signals in order to communicate might as well align their x-axes along the light signal communication direction.

The Lorentz transformation of the coordinates can be written as

$$(R^{\mu})^L = \begin{bmatrix} r'_0 \\ r'_1 \\ r'_2 \\ r'_3 \end{bmatrix} = \Lambda_{\nu}^{\ \mu} R^{\nu} \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \gamma r_0 - \beta\gamma r_1 \\ \gamma r_1 - \beta\gamma r_0 \\ r_2 \\ r_3 \end{bmatrix}$$

So the Lorentz transformation of $R = R_{\mu}\mathbf{K}^{\mu} = \mathbf{K}_{\mu}R^{\mu}$ can be presented as

$$R^L = \mathbf{K}_{\mu}(R^{\mu})^L = \mathbf{K}_{\mu}\Lambda_{\nu}^{\ \mu}R^{\nu} = (\mathbf{K}_{\mu}\Lambda_{\nu}^{\ \mu})R^{\nu} = (\mathbf{K}_{\nu})^L R^{\nu} = U^{-1}\mathbf{K}_{\nu}U^{-1}R^{\nu} = U^{-1}\mathbf{K}_{\nu}R^{\nu}U^{-1} = U^{-1}RU^{-1} \tag{64}$$

This implies the identity $\mathbf{K}_\mu \Lambda_\nu^\mu = U^{-1}\mathbf{K}_\nu U^{-1}$, an identity that isn't possible for the coordinates only. The matrix representation of the basis is key to this identity. As is the $\hat{\mathbf{1}}$ alignment of the two involved reference frames during the Lorentz transformation.

The Lorentz transformation of A^T is also interesting, due to the importance of the product $C = A^T B$ and therefore the Lorentz transformation C^L. Given the inverse Lorentz transformation as

$$A^{L^{-1}} \equiv UAU \tag{65}$$

one can prove

$$\left(A^T\right)^{L^{-1}} = U\left(A^T\right)U = \left(U^{-1}AU^{-1}\right)^T = \left(A^L\right)^T \tag{66}$$

and

$$\left(A^T\right)^L = U^{-1}\left(A^T\right)U^{-1} = (UAU)^T = \left(A^{L^{-1}}\right)^T. \tag{67}$$

Given A and B in reference system S_1 and their product in S_1 as $C = A^T B$. Then in reference system S_2 one has A^L and B^L and their product $C^L = \left(A^L\right)^T B^L$. We then have

$$C^L = \left(A^L\right)^T B^L = \left(A^T\right)^{L^{-1}} B^L = U\left(A^T\right)UU^{-1}BU^{-1} = UA^T BU^{-1} = UCU^{-1}. \tag{68}$$

As a result, it is easy to prove that the quadratic $A^T A = c^2 a_\tau^2 \hat{\mathbf{1}}$ is Lorentz invariant. We have

$$\begin{aligned}(A^L)^T A^L = (A^T)^{-L} A^L = UA^T UU^{-1}AU^{-1} = UA^T AU^{-1} = \\ U(c^2 a_\tau^2)\hat{\mathbf{1}}U^{-1} = UU^{-1}(c^2 a_\tau^2)\hat{\mathbf{1}} = c^2 a_\tau^2 \hat{\mathbf{1}} = A^T A.\end{aligned} \tag{69}$$

So both quadratics $R^T R$ and $dR^T dR$ are Lorentz invariant scalars, as has been shown for every quadratic of four-vectors.

E. Adding the dynamic vectors

If we want to apply the previous to relativistic electrodynamics and to quantum physics, we need to develop the mathematical language further. We start by adding the most relevant dynamic four vectors. The basic definitions we use are quite common in the formulations of relativistic dynamics, see for example (16). We start with a particle with a given three vector velocity as $\mathbf{v}$, a rest mass as m_0 and an inertial mass $m_i = \gamma m_0$, with the usual $\gamma = (\sqrt{1 - v^2/c^2})^{-1}$. We use the Latin suffixes as abbreviations for words, not for numbers. So m_i stands for inertial mass and U_p for potential energy. The Greek suffixes are used as indicating a summation over the numbers

0, 1, 2 and 3. So P_μ stands for a momentum four-vector coordinate row with components $(p_0 = \frac{1}{c}U_i, p_1, p_2, p_3)$. The momentum three-vector is written as $\mathbf{p}$ and has components (p_1, p_2, p_3).

We define the coordinate velocity four vector as

$$V = V_\mu \mathbf{K}^\mu = \frac{d}{dt} R_\mu \mathbf{K}^\mu = c\hat{\mathbf{T}} + \mathbf{v}\cdot\mathbf{K} = v_0\hat{\mathbf{T}} + \mathbf{v}\cdot\mathbf{K}. \tag{70}$$

The proper velocity four vector on the other hand will be defined using the proper time $\tau = t_0$, with $t = \gamma t_0 = \gamma\tau$, as

$$U = U_\mu \mathbf{K}^\mu = \frac{d}{d\tau} R_\mu \mathbf{K}^\mu = \frac{d}{\frac{1}{\gamma}dt} R^\mu \mathbf{K}_\mu = \gamma V_\mu \mathbf{K}_\mu = u_0\hat{\mathbf{T}} + \mathbf{u}\cdot\mathbf{K}. \tag{71}$$

The momentum four vector will be, at least when we have the symmetry condition $\mathbf{p} = m_i\mathbf{v}$,

$$P = P_\mu \mathbf{K}^\mu = m_i V_\mu \mathbf{K}^\mu = m_i V = m_0 U_\mu \mathbf{K}^\mu = m_0 U. \tag{72}$$

The four vector partial derivative $\partial = \partial_\mu \mathbf{K}^\mu$ will be defined using the coordinate four set

$$\partial_\mu = \left[-\frac{1}{c}\partial_t, \nabla_1, \nabla_2, \nabla_3\right] = [\partial_0, \partial_1, \partial_2, \partial_3] \equiv \frac{\partial}{\partial R_\mu}. \tag{73}$$

The electrodynamic potential four vector $A = A_\mu \mathbf{K}^\mu$ will be defined by the coordinate four set

$$A_\mu = \left[\frac{1}{c}\phi, A_1, A_2, A_3\right] = [A_0, A_1, A_2, A_3] \tag{74}$$

The electric four current density vector $J = J_\mu \mathbf{K}^\mu$ will be defined by the coordinate four set

$$J_\mu = [c\rho_e, J_1, J_2, J_3] = [J_0, J_1, J_2, J_3], \tag{75}$$

with ρ_e as the electric charge density. The electric four current with a charge q will be also be written as J_μ and the context will indicate which one is used.

Although we defined these fourvectors using the coordinate column notation, we will often use the matrix or summation notation, as for example with $P = P_\mu \mathbf{K}^\mu$, written as

$$\begin{aligned} P &= p_0\hat{\mathbf{T}} + p_1\hat{\mathbf{I}} + p_2\hat{\mathbf{J}} + p_3\hat{\mathbf{K}} = p_0\hat{\mathbf{T}} + \mathbf{p}\cdot\mathbf{K} \\ &= \begin{bmatrix} \mathbf{i}p_0 + \mathbf{i}p_1 & p_2 + \mathbf{i}p_3 \\ -p_2 + \mathbf{i}p_3 & \mathbf{i}p_0 - \mathbf{i}p_1 \end{bmatrix} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}. \end{aligned} \tag{76}$$

F. The EM field in our language

I we apply the matrix multiplication rules to the electromagnetic field with four derivative ∂ and four potential A, with $\partial_0 = -\frac{1}{c}\partial_t$ and $A_0 = \frac{1}{c}\phi$, we get $B = \partial^T A$ as

$$B = \partial^T A = (-\frac{1}{c^2}\partial_t\phi - \boldsymbol{\nabla}\cdot\boldsymbol{A})\hat{\mathbf{I}} + (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\mathbf{K} + \frac{1}{c}(-\partial_t\boldsymbol{A} - \boldsymbol{\nabla}\phi)\cdot\boldsymbol{\sigma}. \tag{77}$$

If we apply the Lorenz gauge $\mathbb{B}_0 = -\frac{1}{c^2}\partial_t\phi - \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ and the usual EM definitions of the fields in terms of the potentials we get

$$B = \partial^T A = \boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}. \tag{78}$$

Using $\boldsymbol{\sigma} = -\hat{\mathbf{I}}\mathbf{K} = -\mathbf{i}\mathbf{K}$, this can also be written as

$$B = \partial^T A = (\boldsymbol{B} - \mathbf{i}\frac{1}{c}\boldsymbol{E})\cdot\mathbf{K} = \overrightarrow{\mathbb{B}}\cdot\mathbf{K}. \tag{79}$$

The use of $\mathbb{B} = \boldsymbol{B} - \mathbf{i}\frac{1}{c}\boldsymbol{E}$ dates back to Minkowski's 1908 treatment of the subject (14).

Using $\mathbb{B}$ we can write B as

$$B = \mathbb{B}_1\hat{\mathbf{I}} + \mathbb{B}_2\hat{\mathbf{J}} + \mathbb{B}_3\hat{\mathbf{K}} = \overrightarrow{\mathbb{B}}\cdot\mathbf{K} = \begin{bmatrix} \mathbf{i}\mathbb{B}_1 & \mathbb{B}_2 + \mathbf{i}\mathbb{B}_3 \\ -\mathbb{B}_2 + \mathbf{i}\mathbb{B}_3 & -\mathbf{i}\mathbb{B}_1 \end{bmatrix} = \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix}. \tag{80}$$

For the Lorentz transformation of B we can apply the result of the previous section to get $B^L = (\partial^L)^T A^L = (\partial^T)^{-L}A^L = U(\partial^T)UU^{-1}AU^{-1} = U(\partial^T A)U^{-1} = UBU^{-1}$, so

$$B^L = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix} \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} \begin{bmatrix} e^{-\frac{\psi}{2}} & 0 \\ 0 & e^{\frac{\psi}{2}} \end{bmatrix} = \begin{bmatrix} B_{00} & B_{01}e^{\psi} \\ B_{10}e^{-\psi} & B_{11} \end{bmatrix} \tag{81}$$

which, when written out with $\boldsymbol{E}$ and $\boldsymbol{B}$ leads to the usual result for the Lorentz transformation of the EM field with the Lorentz velocity in the x-direction. But it can also be written as a transformation of the basis, while leaving the coordinates invariant:

$$\begin{aligned} B^L = UBU^{-1} = \mathbb{B}_1 U\hat{\mathbf{I}}U^{-1} + \mathbb{B}_2 U\hat{\mathbf{J}}U^{-1} + \mathbb{B}_3 U\hat{\mathbf{K}}U^{-1} = \\ \mathbb{B}_1\hat{\mathbf{I}} + \mathbb{B}_2\hat{\mathbf{J}}^L + \mathbb{B}_3\hat{\mathbf{K}}^L = \mathbb{B}_1 \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} + \mathbb{B}_2 \begin{bmatrix} 0 & e^{\psi} \\ -e^{-\psi} & 0 \end{bmatrix} + \mathbb{B}_3 \begin{bmatrix} 0 & \mathbf{i}e^{\psi} \\ \mathbf{i}e^{-\psi} & 0 \end{bmatrix}. \end{aligned} \tag{82}$$

The Lorentz transformation of the EM field can be performed by internally twisting the $(\hat{\mathbf{J}}, \hat{\mathbf{K}})$-surface perpendicular to the Lorentz velocity and in the process leaving the EM-coordinates invariant.

That the above equals the usual Lorentz transformation of the EM field can be shown by going back to the 1908 paper by Minkowski (14), where he wrote the transformation in a form equivalent to

$$\begin{bmatrix} \mathbb{B}'_1 \\ \mathbb{B}'_2 \\ \mathbb{B}'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma & \mathbf{i}\beta\gamma \\ 0 & -\mathbf{i}\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} \mathbb{B}_1 \\ \mathbb{B}_2 \\ \mathbb{B}_3 \end{bmatrix} = \begin{bmatrix} \mathbb{B}_1 \\ \gamma\mathbb{B}_2 + \mathbf{i}\beta\gamma\mathbb{B}_3 \\ \gamma\mathbb{B}_3 - \mathbf{i}\beta\gamma\mathbb{B}_2 \end{bmatrix} \tag{83}$$

So we have

$$B'_{01} = \mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = \gamma\mathbb{B}_2 + \mathbf{i}\beta\gamma\mathbb{B}_3 + \mathbf{i}\gamma\mathbb{B}_3 + \beta\gamma\mathbb{B}_2 \tag{84}$$

and

$$B'_{10} = -\mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = -\gamma\mathbb{B}_2 - \mathbf{i}\beta\gamma\mathbb{B}_3 + \mathbf{i}\gamma\mathbb{B}_3 + \beta\gamma\mathbb{B}_2. \tag{85}$$

If we use the rapidity ψ as $e^{\psi} = \cosh\psi + \sinh\psi = \gamma + \beta\gamma$, this can be rewritten as

$$B'_{01} = \mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = (\gamma + \beta\gamma)(\mathbb{B}_2 + \mathbf{i}\mathbb{B}_3) = B_{01}e^{\psi} \tag{86}$$

and

$$B'_{10} = -\mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = (\gamma - \beta\gamma)(-\mathbb{B}_2 + \mathbf{i}\mathbb{B}_3) = B_{10}e^{-\psi}, \tag{87}$$

which leads to Eqn. (81).

G. The Maxwell Equations and the Lorentz force law

The Maxwell equations in our language can be given as, using $J = \rho V$

$$\partial B = \mu_0 J \tag{88}$$

and the Lorentz force law, with a four force density $\mathscr{F}$, as

$$JB = \mathscr{F}. \tag{89}$$

Maxwell's inhomogeneous wave equations can be written as

$$(-\partial^T\partial)B = -\mu_0\partial^T J \tag{90}$$

and with the Lorentz invariant quadratic derivative,

$$-\partial^T\partial = (\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_t^2)\hat{\mathbf{1}} \tag{91}$$

we get the homogeneous wave equations of the EM field in free space in the familiar form as

$$(-\partial^T \partial)B = \boldsymbol{\nabla}^2 B - \frac{1}{c^2}\partial_t^2 B = 0. \tag{92}$$

I will look at $\partial B = \mu_0 J$ first. The underlying structure then also applies to the Lorentz Force Law and the inhomogeneous part of the wave equation. I start with

$$B = \partial^T A = \boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}. \tag{93}$$

Then ∂B is given by

$$\partial B = \left(-\frac{1}{c}\partial_t \hat{\mathbf{T}} + \boldsymbol{\nabla}\cdot\mathbf{K}\right)\left(\boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}\right) =$$
$$-(\boldsymbol{\nabla}\cdot\boldsymbol{B})\hat{\mathbf{1}} + \frac{1}{c}(\boldsymbol{\nabla}\cdot\boldsymbol{E})\hat{\mathbf{T}} + (\boldsymbol{\nabla}\times\boldsymbol{B} - \frac{1}{c^2}\partial_t\boldsymbol{E})\cdot\mathbf{K} + \frac{1}{c}(\boldsymbol{\nabla}\times\boldsymbol{E} + \partial_t\boldsymbol{B})\cdot\boldsymbol{\sigma} \tag{94}$$

If we interpret this result using the knowledge regarding the inhomogeneous Maxwell equations, we get an interesting result. First of all, the part of the Maxwell Equation with the dimension of the norm $\hat{\mathbf{1}}$ is zero and so is the part with the dimension of spin $\boldsymbol{\sigma}$. The space-time parts $\mathbf{K}$ and $\hat{\mathbf{T}}$ equal the space-time parts of the four current $\mu_0 J$. So we get

$$\partial B = -(\boldsymbol{\nabla}\cdot\boldsymbol{B})\hat{\mathbf{1}} + \frac{1}{c}(\boldsymbol{\nabla}\cdot\boldsymbol{E})\hat{\mathbf{T}} + (\boldsymbol{\nabla}\times\boldsymbol{B} - \frac{1}{c^2}\partial_t\boldsymbol{E})\cdot\mathbf{K} + \frac{1}{c}(\boldsymbol{\nabla}\times\boldsymbol{E} + \partial_t\boldsymbol{B})\cdot\boldsymbol{\sigma} =$$
$$0\hat{\mathbf{1}} + \mu_0 c\rho\hat{\mathbf{T}} + \mu_0\boldsymbol{J}\cdot\mathbf{K} + 0\boldsymbol{\sigma} = \mu_0 J. \tag{95}$$

So the spin-norm part of the Maxwell Equations equals zero and the space-time part equals the space-time four current density times μ_0. In the line of this interpretation, magnetic monopoles and the correlated magnetic monopole current should be searched in the dimensions of spin-norm, not in the dimensions of space-time.

As for the Lorentz covariance of the Maxwell Equations, this can be demonstrated quite easily. Given the four-vectors ∂, A and J in reference system S_1, with the Maxwell Equations as $\partial(\partial^T A) = \mu_0 J$, then in reference system S_2 we have the four-vectors ∂^L, A^L and J^L and the covariant Maxwell Equations given as $\partial^L(\partial^L)^T A^L = \mu_0 J^L$. In S_2 this can be proven through

$$\partial^L(\partial^L)^T A^L = \partial^L(\partial^T)^{L^{-1}} A^L = U^{-1}\partial U^{-1} U(\partial^T) U U^{-1} A U^{-1} =$$
$$U^{-1}\partial(\partial^T) A U^{-1} = U^{-1}\partial B U^{-1} = U^{-1}\mu_0 J U^{-1} = \mu_0 J^L. \tag{96}$$

So if we have $\partial B = \mu_0 J$ in one frame of reference, this transforms as $\partial^L B^L = \mu_0 J^L$ in another frame of reference, which means that the equation maintains its form, it is Lorentz covariant. We have form-invariance of the equations.

I will look at $JB = F$ now, with $J = qV$. The underlying structure for the Lorentz Force Law is the same as for the Maxwell equations. So JB is given by

$$JB = \left(cq\hat{\mathbf{T}} + \boldsymbol{J}\cdot\mathbf{K}\right)\left(\boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}\right) =$$

$$-(\boldsymbol{J}\cdot\boldsymbol{B})\hat{\mathbf{1}} + \frac{1}{c}(\boldsymbol{J}\cdot\boldsymbol{E})\hat{\mathbf{T}} + (\boldsymbol{J}\times\boldsymbol{B} + q\boldsymbol{E})\cdot\mathbf{K} + (\frac{1}{c}\boldsymbol{J}\times\boldsymbol{E} - cq\boldsymbol{B})\cdot\boldsymbol{\sigma} \tag{97}$$

If we interpret this result using the knowledge regarding the Lorentz Force Law, we get an interesting result. First of all, the part of the Lorentz force law with the dimension of the norm $\hat{\mathbf{1}}$ is zero and so is the part with the dimension of spin $\boldsymbol{\sigma}$. The space-time parts $\mathbf{K}$ and $\hat{\mathbf{T}}$ equal the space-time parts of the four force F. Thus we get

$$JB = -(\boldsymbol{J}\cdot\boldsymbol{B})\hat{\mathbf{1}} + \frac{1}{c}(\boldsymbol{J}\cdot\boldsymbol{E})\hat{\mathbf{T}} + (\boldsymbol{J}\times\boldsymbol{B} + q\boldsymbol{E})\cdot\mathbf{K} + (\frac{1}{c}\boldsymbol{J}\times\boldsymbol{E} - cq\boldsymbol{B})\cdot\boldsymbol{\sigma} =$$

$$0\hat{\mathbf{1}} + \frac{1}{c}P\hat{\mathbf{T}} + \boldsymbol{F}\cdot\mathbf{K} + 0\boldsymbol{\sigma} = F. \tag{98}$$

So the spin-norm part of the Lorentz Force Law equals zero and the space-time part equals the space-time four force.

In both cases, ∂B and BJ, we get a dual spin-norm and space-time product, with the spin-norm equal zero and the non-zero space-time leading to the inhomogeneous four-vectors of current and force. Speculations about magnetic monopoles are connected to these spin-norm parts. In my analysis, if spin-norm is the twin dual of space-time and as such an integral aspect of the metric as foreseen by Dirac (65), then searches for magnetic monopoles should focus on this spin-norm aspect of the vacuum.

H. Invariant EM field energies and the generalized Poynting theorem

As for the electromagnetic energy density of a pure EM field, we have the two products BB and B^TB. These product are structurally different from the previous $\partial^T A$ and $\partial B = \partial\partial^T A$ because it now involves the multiplication of four four-vectors as in $BB = \partial^T A \partial^T A$.

For BB the antisymmetric part eliminates and we get the norm-time product

$$BB = \left(\boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}\right)\left(\boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}\right) = (\frac{1}{c^2}\boldsymbol{E}^2 - \boldsymbol{B}^2)\hat{\mathbf{1}} + (2\frac{1}{c}\boldsymbol{B}\cdot\boldsymbol{E})\hat{\mathbf{T}}, \tag{99}$$

which, with a complex $u_{EB} = u_E - u_B + 2\mathbf{i}\sqrt{u_B u_E}$, can be written as

$$\frac{1}{2\mu_0}BB = (u_E - u_B)\hat{\mathbf{1}} + (2\sqrt{u_B u_E})\hat{\mathbf{T}} = u_{EB}\hat{\mathbf{1}}. \tag{100}$$

The fact that the product BB is Lorentz invariant follows from $B^L = UBU^{-1}$ and the fact that BB result in a complex scalar value, so

$$B^L B^L = UBU^{-1}UBU^{-1} = UBBU^{-1} = 2\mu_0 u_{EB} U\hat{\mathbf{1}}U^{-1} = 2\mu_0 u_{EB}\hat{\mathbf{1}} = BB. \quad (101)$$

We also have the interesting product $2\partial u_{EB} = \partial(\frac{1}{\mu_0}BB)$, the four divergence of this Lorentz invariant EM energy related product. Using the Maxwell equations $\partial B = \mu_0 J$ and the Lorentz force density law $JB = \mathscr{F}$, we get

$$\partial u_{EB} = \partial(\frac{1}{2\mu_0}BB) \simeq \frac{2}{2\mu_0}(\partial B)B = JB = \mathscr{F}, \quad (102)$$

resulting in $\partial u_{EB} = \mathscr{F}$.

For the second EM energy related product $B^T B$ the antisymmetric part survives and we get the spin-norm product

$$B^T B = \left(\boldsymbol{B}\cdot\mathbf{K} - \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}\right)\left(\boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}\right) = -(\frac{1}{c^2}\boldsymbol{E}^2 + \boldsymbol{B}^2)\hat{\mathbf{1}} - (2\frac{1}{c}\boldsymbol{E}\times\boldsymbol{B})\cdot\boldsymbol{\sigma} =$$
$$-2\mu_0 u_{EM}\hat{\mathbf{1}} - 2\mu_0\frac{1}{c}\boldsymbol{S}\cdot\boldsymbol{\sigma} = -2\mu_0 c\left(\frac{1}{c}u_{EM}\hat{\mathbf{1}} + \frac{1}{c^2}\boldsymbol{S}\cdot\boldsymbol{\sigma}\right) = -2\mu_0 c\left(\frac{1}{c}u_{EM}\hat{\mathbf{1}} + \boldsymbol{g}\cdot\boldsymbol{\sigma}\right). \quad (103)$$

In the last equation, I used the Poynting vector $\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}$, the EM momentum density $\boldsymbol{g} = \frac{1}{c^2}\boldsymbol{S}$ and the EM energy as $2\mu_0 u_{EM} = \boldsymbol{B}^2 + \frac{1}{c^2}\boldsymbol{E}^2$. The last part can also be written as

$$B^T B = -2\mu_0 c\left(\frac{1}{c}u_{EM}\hat{\mathbf{1}} + \boldsymbol{g}\cdot\boldsymbol{\sigma}\right) = 2\mathbf{i}\mu_0 c\left(\frac{1}{c}u_{EM}\hat{\mathbf{T}} + \boldsymbol{g}\cdot\mathbf{K}\right) = 2\mathbf{i}\mu_0 cG. \quad (104)$$

Thus we get the usual EM four momentum density G and the four EM energy current density S as

$$G = \frac{1}{c^2}S = \frac{-\mathbf{i}}{2\mu_0 c}B^T B = \frac{1}{c}u_{EM}\hat{\mathbf{T}} + \boldsymbol{g}\cdot\mathbf{K}, \quad (105)$$

in which G has the appearance of a good relativistic space-time four vector. But according to our analysis it isn't a space-time four vector but a spin-norm four vector. That makes this product an interesting case for studying the characteristics of the spin-norm dual or twin dimension of space-time, as manifesting aspects of the Dirac vacuum or Dirac Æther.

For the Lorentz transformation of $B^T B$, one has to go to the Lorentz transformation of the primary constituting four vectors. We have $B^T B = (\partial^T A)^T(\partial^T A) = (\partial A^T)(\partial^T A)$. The Lorentz transformation of $B^T B$ then results in

$$\partial^L (A^L)^T (\partial^L)^T A^L = \partial^L (A^T)^{L^{-1}} (\partial^T)^{L^{-1}} A^L = U^{-1}\partial U^{-1} U (A^T) UU (\partial^T) UU^{-1} A U^{-1} =$$
$$U^{-1}(\partial A^T) UU (\partial^T A) U^{-1} = U^{-1} B^T UUBU^{-1} = (B^T)^{L^{-1}} B^L = (B^L)^T B^L \quad (106)$$

This means that we have Lorentz covariance for the equation $G = -\frac{\mathbf{i}}{2\mu_0 c} B^T B$. So in Eqn.(105) the EM four momentum density G and the four EM energy current density S are defined in a Lorentz covariant way.

The product $\partial^T G$ is interesting too, being the divergence of the EM momentum density $B^T B$. It brings us at the level of the product of five original four-vectors. It should give a Maxwell-Lorentz structured complex force. We get

$$\partial^T G = \frac{-\mathbf{i}}{2\mu_0 c}\partial^T B^T B = \frac{-\mathbf{i}}{2\mu_0 c}(\partial B B^T)^T \simeq \frac{-\mathbf{i}}{2\mu_0 c}(2\mu_0 J B^T)^T = \frac{-\mathbf{i}}{c}(J^T B) = \mathscr{F} \tag{107}$$

implying that we returned to a product of three four-vectors with as a necessary result a Maxwell Equation, Lorentz Force Law structured outcome. The main difference is in the appearance of the complex number $\mathbf{i}$, $J^T B = \mathbf{i} c \mathscr{F}$, stemming from $\hat{\mathbf{T}} = \mathbf{i}\hat{\mathbf{1}}$, which turns space-time into spin-norm and vice versa. The second related difference is coming from the time reversal in J in $J^T B$.

Calculating $\frac{-\mathbf{i}}{c} J^T B$ gives

$$\frac{-\mathbf{i}}{c} J^T B = \frac{-\mathbf{i}}{c}\left(-cq\hat{\mathbf{T}} + \boldsymbol{J}\cdot\mathbf{K}\right)\left(\boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}\right) =$$
$$\frac{1}{c}(\boldsymbol{J}\cdot\boldsymbol{B})\hat{\mathbf{T}} + \frac{1}{c^2}(\boldsymbol{J}\cdot\boldsymbol{E})\hat{\mathbf{1}} + \frac{1}{c}(\boldsymbol{J}\times\boldsymbol{B} - q\boldsymbol{E})\cdot\boldsymbol{\sigma} - (\frac{1}{c^2}\boldsymbol{J}\times\boldsymbol{E} + q\boldsymbol{B})\cdot\mathbf{K} \tag{108}$$

We see that the spin-norm and space-time switch places due to $\mathbf{i}$ and that the sign of q changes due to T (not the sign of $\boldsymbol{J}$). The other part $\partial^T G$ leads to

$$\partial^T G = (-\frac{1}{c^2}\partial_t u_{EM} - \boldsymbol{\nabla}\cdot\mathbf{g})\hat{\mathbf{1}} + (\boldsymbol{\nabla}\times\mathbf{g})\cdot\mathbf{K} + \frac{1}{c}(\partial_t\mathbf{g} + \boldsymbol{\nabla}u_{EM})\cdot\boldsymbol{\sigma}. \tag{109}$$

The norm $\hat{\mathbf{1}}$ part of the equation $\partial^T G = \frac{-\mathbf{i}}{c}(J^T B)$ contains the relativistic Poynting's theorem:

$$\frac{1}{c^2}\partial_t u_{EM} + \boldsymbol{\nabla}\cdot\mathbf{g} = -\frac{1}{c^2}\boldsymbol{J}\cdot\boldsymbol{E} \tag{110}$$

so using $\mathbf{S} = c^2\mathbf{g}$ we get the relativistic Poynting theorem for EM energy density conservation

$$\partial_t u_{EM} + \boldsymbol{\nabla}\cdot\mathbf{S} = -\boldsymbol{J}\cdot\boldsymbol{E}. \tag{111}$$

The equation $\partial^T G = -\frac{\mathbf{i}}{c}(J^T B)$ can be perceived as the generalizes Poynting theorem. In the derivation, the step $(\partial B B^T)^T \simeq (2(\partial B)B^T)^T = (2\mu_0 J B^T)^T$ does need further evaluation, but that is a topic for another time. It's details don't influence the result regarding the presented derivation of the Poynting theorem.

Two issues are relevant for the present paper. The first point to make is that the Poynting continuity equation refers to an open system when a charge is moving in an electric field. Without the current one has the EM field energy density continuity equation for a closed system

$$\partial^T_\mu S^\mu = \partial_t u_{EM} + \mathbf{\nabla} \cdot \mathbf{S} = 0. \tag{112}$$

The second issue is that this continuity equation has its origin in the norm-like part of the momentum closed system condition $\partial^T G = 0$ of Eqn.(120)

$$\partial^T_\mu G^\mu = \frac{1}{c^2}\partial_t u_{EM} + \mathbf{\nabla} \cdot \mathbf{g} = 0. \tag{113}$$

The other closed system conditions are the space-like absence of vorticity condition

$$\mathbf{\nabla} \times \mathbf{g} = 0 \tag{114}$$

and the spin-like

$$\partial_t \mathbf{g} + \mathbf{\nabla} u_{EM} = 0. \tag{115}$$

The last part can be written as the spin-like conserved force condition

$$\partial_t \mathbf{g} = -\mathbf{\nabla} u_{EM}. \tag{116}$$

This pattern will repeat itself for the Dirac current. In the third part of this paper, the Dirac current will shown to be a probability/field tensor and the continuity equation for the Dirac current will turn out to be the time-like part of the closed system condition for this probability/field tensor.

At the same time, a charge in an electromagnetic field isn't a closed system condition because we get for the generalized Poynting theorem

$$\partial^T G = \mathscr{F} = \frac{-\mathbf{i}}{c}(J^T B) \tag{117}$$

Only in the absence of charges, so when $J = 0$ do we have the closed system condition $\partial^T G$. The problem of the free electron in it's own field hasn't been solved. The inhomogeneous parts of the generalized Poynting theorem are given as a spin-like part and a space-like part. These parts have a Maxwell-Lorentz structure. The structure appearing here is highly analogous to the formulation of a generalized Dirac current continuity equation in the third part of this paper; the closed system condition for the probability/field tensor.

As for the Poincarè strategy to solve the problem of the free electron in the vacuum, one might reformulate the generalized Poynting theorem as

$$\partial^T G - \mathscr{F} = \partial^T G - \frac{-\mathbf{i}}{c}(J^T B) = 0 \tag{118}$$

and then define a new G' for which one has

$$\partial^T G' = \partial^T G - \mathscr{F} = \partial^T G - \frac{-\mathbf{i}}{c}(J^T B) = 0 \tag{119}$$

and declare the problem solved. I am inclined to rebaptize this as the Ouroboros strategy. Because how to restructure the $J^T B$ term in a ∂G" term otherwise that to write is as the original $\partial^T G$ again?

I. Relativistic mechanics

1. The conserved four momentum condition in relativistic mechanics

In SR and GR, Laue's condition for the conservation of energy-momentum in a closed system is $\partial_\nu T^\nu{}_\mu = 0$. In our language we have a comparable but not identical $\partial^T P = 0$ condition as a starting point of our alternative relativistic mechanics. In the case of electrodynamics, when we have the canonical $P = qA$, we have $\partial^T A = B \neq 0$. So in circumstances analogous to a nonzero anti-symmetric EM field, the condition $\partial^T P = q\partial^T A = qB = 0$ is not fulfilled. In the previous section, we saw other conditions in the EM context where the closed system condition is not satisfied due to charges (moving) in EM fields.

The mechanic condition $\partial^T P = 0$ leads to

$$\partial^T P = (-\frac{1}{c^2}\partial_t U_i - \boldsymbol{\nabla}\cdot\mathbf{p})\hat{\mathbf{1}} + (\boldsymbol{\nabla}\times\mathbf{p})\cdot\mathbf{K} + \frac{1}{c}(\partial_t\mathbf{p} + \boldsymbol{\nabla}U_i)\cdot\boldsymbol{\sigma} = 0. \tag{120}$$

so to three subconditions

$$\frac{1}{c^2}\partial_t U_i + \boldsymbol{\nabla}\cdot\mathbf{p} = 0 \tag{121}$$

$$\boldsymbol{\nabla}\times\mathbf{p} = 0 \tag{122}$$

$$\partial_t\mathbf{p} = -\boldsymbol{\nabla}U_i. \tag{123}$$

The first one is the continuity equation, the second means that we have zero vorticity and the third that the related force field can be connected to a potential energy. Due to the second condition, the time derivative of $\boldsymbol{\nabla}\times\mathbf{p}$ must be zero, giving the secondary conserved force field condition

$$\boldsymbol{\nabla}\times\mathbf{F} = 0. \tag{124}$$

The first condition can also be written as

$$\partial_t m_i + \nabla \cdot (m_i \mathbf{v}) = 0, \tag{125}$$

so the continuity equation for inertial mass.

If we have $\partial^T P = 0$ in one system of reference, then in another system of reference we have

$$(\partial^L)^T P^L = (\partial^T)^{L^{-1}} P^L = U \partial^T U U^{-1} P U^{-1} = U \partial^T P U^{-1} = 0, \tag{126}$$

proving that the condition is Lorentz covariant.

With $\partial^T P = 0$ we have a relativistic condition of a mechanical system representing a central force. It is best characterized as the extended continuity condition, it's relativistic completion: the generalized continuity equation. It has as a norm $\hat{\mathbf{1}}$ condition the continuity equation, as a space $\mathbf{K}$ condition the absence of vorticity and as a spin $\boldsymbol{\sigma}$ condition the conserved force condition. This will become crucial in relativistically extending the conserved Dirac current condition in RQM.

2. The stress energy tensor equivalent

In the Laue condition $\partial_\nu T^\nu{}_\mu = 0$ the stress-energy density tensor is $T^\nu{}_\mu = V^\nu G_\mu$. In our math-phys language we would get the not exact analog $T = V^T G$ and $\partial T = 0$, but that would imply a full homogeneous Maxwell-Lorentz structure with the product $\partial V^T G = 0$. Our stress energy density 'tensor' T is given by

$$T = V^T G = (u_i - \mathbf{v} \cdot \mathbf{g})\hat{\mathbf{1}} + (\mathbf{v} \times \mathbf{g}) \cdot \mathbf{K} + c(\mathbf{g} - \frac{1}{c^2} u_i \mathbf{v}) \cdot \boldsymbol{\sigma}. \tag{127}$$

This tensor analog contains all the elements of $T^\nu{}_\mu = V^\nu G_\mu$, with the difference that the cross product $\mathbf{v} \times \mathbf{g}$ appears directly in our $T = V^T G$ whereas ony half of it is in the usual tensor and the anti-symmetric tensor product is needed to get the full cross product.

In the case of a symmetric situation $\mathbf{v}$ has the same direction as $\mathbf{g}$, resulting in

$$T = (u_i - \mathbf{v} \cdot \mathbf{g})\hat{\mathbf{1}} = u_0 \hat{\mathbf{1}} \tag{128}$$

$$\mathbf{v} \times \mathbf{g} = 0 \tag{129}$$

$$\mathbf{g} = \frac{1}{c^2} u_i \mathbf{v}. \tag{130}$$

The third equation contains the mass-energy density equivalence $u_i = \rho_i c^2$, but it also implies the absence of linear stresses. The second equation implies the absence internal pressures. The first equation equals the scalar Lagrangian density, the trace of the Laue mechanical stress-energy density tensor. A symmetric T can be written as $T = \frac{1}{\rho}G^T G$ in the mass density formulation and as $T = \frac{1}{m_0}P^T P$ in the mass formulation.

The divergence of the symmetric T has the space-like part and the spin-like part equal to zero and only the norm-like part possibly non-zero. This leads to a four force density as

$$\mathscr{F} = -\partial T = -\partial \frac{1}{\rho} G^T G = -\partial u_0 \hat{\mathbf{1}} = \frac{1}{c}\partial_t u_0 \hat{\mathbf{T}} - \nabla u_0 \cdot \mathbf{K}. \tag{131}$$

The direct parallel in electromagnetics would be that $\overrightarrow{\mathbb{B}} = 0$ with a Coulomb gauge for the field and that $T_{EM} = J^T A = \rho_0 \phi_0 \hat{\mathbf{1}}$, with $\mathscr{F} = -\partial \rho_0 \phi_0$. In the rest system this would produce a Coulomb force density and a Coulomb force power, which, for a static potential, would be zero. Thus in our relativistic dynamics, in the symmetric case the electromagnetic parallel would only produce a Coulomb force situation.

Only if $\mathbf{v}$ doesn't have the same direction as $\mathbf{g}$ will there be an anti-symmetric component present that is analog to the structure of the Maxwell-Lorentz electromagnetic field/force situation. The Lorentz force is given as $JB = F$, which can be written as $qV\partial^T A = F$ which, by using $P = qA$, results in the mechanical analog $V\partial^T P = F$. This still isn't the full $\partial V^T P = -F$. The Lorentz force law analog in our relativistic dynamics implies that $\partial^T P \neq 0$, so that $m_0 \partial^T U \neq 0$. If we look closer at $V\partial^T$, we see that it contains the three parts

$$(-\frac{\partial}{\partial t} - \mathbf{v}\cdot\nabla)\hat{\mathbf{1}} \equiv -\frac{d}{dt}\hat{\mathbf{1}} \tag{132}$$

$$\mathbf{v}\times\nabla \tag{133}$$

$$c\nabla + \frac{1}{c}\mathbf{v}\partial_t. \tag{134}$$

So the product $-V\partial^T$ is our variant of the absolute derivative, with $\frac{d}{dt}\hat{\mathbf{1}}$ as the scalar norm $\hat{\mathbf{1}}$ part of it. Thus if we go from $\partial^T P = 0$ to $V\partial^T P = F$, we move in our relativistic mechanics from a pure Coulomb force structure or environment to a Lorentz force one, related to a move from a partial derivative to an absolute derivative. As for gravity, the Newton-Coulomb gravito-electric analogy implies that $\partial^T P = 0$ in the low velocity limit should be able to contain Newtonian gravity. At the same time, we already know from experience that gravity lacks a Lorentz force structure, so we should expect the same relative to the mechanical $V\partial^T P = F$. In density expression, the equation $V\partial^T G = \mathscr{F}$ should be a Lorentz invariant expression.

What is becoming apparent is that we have a lot of highly relevant relativistic mechanics before we arrive at analog of the traditional Laue product in its full, non-symmetric realisation

$\partial V^T G = -\mathscr{F}$. It seems that the divergence of the stress energy density tensor looses its central role, and thereby also its function as a role-model. Especially when we realize that the Minkowski-Laue principle example, $\partial^\nu T^{EM}_{\nu\mu} = -\mathscr{F}_\mu$, is a highly compact, complex expression containing in our translation the energy products BB, BB^T, the equations $\partial B = \mu_0 J$ and $JB = \mathscr{F}$ and the higher complex $\partial^T BB$ and $\partial^T B^T B$. In our math-phys language, the compactified Minkowski-Laue equation's content is spread out over several products and equations existing at different layers of complexity. What remains is that in our language, the math-phys structures contained in our relativistic mechanics products and equations continue to mirror the electromagnetic world. So, as far as the worldview or 'Weltanschauung' of the Abraham-Lorentz generation of physicists is concerned, we only replaced one math-phys language by another without loosing the relativistic EM-mechanics analogy.

Relativistic gravity never truly fitted in the scheme of the EM-'Weltanschauung' of the Minkowski-Laue consensus, and neither did spin quantum mechanics together with the Standard Model parts that are build on top of Dirac equation based relativistic quantum mechanics, so we can expect that the math-phys language developed thus far won't be able to grasp those gravitational an quantum environments either. But we managed to put spin, or spin related matrices, in the metric and still have a somewhat sensible math-phys language and we do seem to have developed a more detailed-rich relativistic mechanics as an alternative to Laue's relativistic mechanics. For the problem of the electron however, we need to relate our math-phys language to Pauli spin and, above all, Dirac spin in order to get rid of the mathematical incommensurability between the Poincaré-Laue, Boyer-Rohrlich discussion of the electron problem and the Frenkel, de Broglie, Kramers, Dirac discussion of the same elementary particle, regardless of (im)possible solutions.

3. Action and angular momentum

But first we can check to what extend we can put some structure into the presentations of De Broglie and Kramers on spin and angular momentum in a semi-relativistic context, using the result up until now. Angular momentum is given by $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ so let's try to generalize it with the four vector action product $R^T P$. We get

$$R^T P = (U_i t - \boldsymbol{r}\cdot\boldsymbol{p})\hat{\mathbf{1}} + (\boldsymbol{r}\times\boldsymbol{p})\cdot\mathbf{K} + (ct\boldsymbol{p} - \frac{1}{c}U_i\boldsymbol{r})\cdot\boldsymbol{\sigma} = S\hat{\mathbf{1}} + \boldsymbol{L}\cdot\mathbf{K} + \boldsymbol{Z}\cdot\boldsymbol{\sigma} = S\hat{\mathbf{1}} + (\boldsymbol{Z} - \mathbf{i}\boldsymbol{L})\cdot\boldsymbol{\sigma}. \quad (135)$$

In this one single product we can recognize the scalar action S, the Pauli-level spin-orbit interaction $\mathbf{i}\boldsymbol{L}\cdot\boldsymbol{\sigma}$, but also the angular momentum six-vector $(\boldsymbol{Z}-\mathbf{i}\boldsymbol{L})$ of Frenkel, de Broglie and Kramer. Clearly $\boldsymbol{Z}=ct\boldsymbol{p}-m_ic\boldsymbol{r}$ represents the barycentric momentum of de Broglie and the sixvector completion of angular momentum with Frenkel, Kramers and Dirac, as a part of the six-vector $\boldsymbol{L}-\mathbf{i}\boldsymbol{Z}$.

In the rest system of the electron this three vector $\boldsymbol{Z}$ is supposed to be zero, leaving us with something like a four vector dot product $-\mathbf{i}S\hat{\mathbf{T}}+\boldsymbol{L}\cdot\mathbf{K}$, so the complex $L_\mu=(-\mathbf{i}S,\boldsymbol{L})$. De Broglie called S some scalar value needed for the completion of spin angular momentum as a four vector, or $\hat{\mathbf{T}}+\mathbf{K}=\mathbf{i}(\hat{\mathbf{1}}+\boldsymbol{\sigma})$ when translated in our language.

Dirac added the momentum four vector P to the six-vector $\boldsymbol{L}+\mathbf{i}\boldsymbol{S}$ to get his ten fundamental values in his 1949 paper.

Frenkel and Kramers used the relation between angular momentum and magnetic momentum $\boldsymbol{\mu}=-\frac{e}{2m}\boldsymbol{L}$, which for spin was $\boldsymbol{\mu}_s=-\frac{e}{m}\boldsymbol{L}_s$, to define intrinsic spin and magnetic moment six-vectors as $\boldsymbol{\mu}_s+\mathbf{i}c\boldsymbol{\pi}_s=-\frac{e}{m}(\boldsymbol{L}_s-\mathbf{i}\boldsymbol{Z}_s)$, without specifying the physics of intrinsic electric polarization $\boldsymbol{\pi}_s$.

For the moment, the product R^TP adds some light to the confusing treatments of Frenkel, de Broglie, Kramers and Dirac (who frequently referred to Frenkel's 1926 paper). Especially de Broglie and Kramers were trying hard but without success to formulate QM Pauli-Dirac electron spin in the formalism of the Minkowski-Laue paradigm. The problem in their approaches is that they didn't go beyond the four-vector and six-vector scheme with either symmetric tensors or anti-symmetric ones and nothing in between. This strict dichotomy between symmetric and anti-symmetric is still with us today in the form of bosons versus leptons; Einstein-Bose statistics or Fermi-Dirac statistics. Some kind of Supersymmetry should overcome this dichotomy as non-fundamental on some deeper level. The non-commutative math-phys language developed here is such an attempt to formulate a synthesis of symmetric tensors as thesis and anti-symmetric tensors as anti-thesis. The product of two four-vectors in this language produces a norm $\hat{\mathbf{1}}$, a space $\mathbf{K}$ and spin $\boldsymbol{\sigma}$ outcome, combining symmetric and anti-symmetric.

J. The Klein-Gordon condition

The basic scalar Klein-Gordon wave equation in Quantum Mechanics is

$$(\boldsymbol{\nabla}^2-\frac{1}{c^2}\partial_t^2)\Psi=0 \tag{136}$$

In our environment it can be written as

$$-\partial^T \partial \Psi = (\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_t^2)\hat{\mathbf{1}}\Psi = 0 \tag{137}$$

but then we have a two column spinor as wave-function

$$\Psi = \begin{bmatrix} \Psi_0 \\ \Psi_1 \end{bmatrix} \tag{138}$$

instead of the scalar spinor of Schrödinger- and standard Klein-Gordon QM. But it would result in two identical equations, so a degenerate situation in which the two valued spinor equation can be reduced to a single one.

Thus far, only Lorentz transformation could act on the matrix internal aspect of our basis. And even then, a coordinate interpretation was always possible, leaving the basis inert. So up until now, the matrix part of the basis has been practical but not essential. Spinors on the Pauli and Dirac level change that situation. Spinor wave functions interact with the internal elements, the matrix aspect, of the metric $(\hat{\mathbf{1}}, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$.

The Klein-Gordon Equation has its roots in the quadratic energy-momentum condition

$$P^T P = (\frac{1}{c^2}U_i^2 - p^2)\hat{\mathbf{1}} = \frac{1}{c^2}U_0^2\hat{\mathbf{1}}, \tag{139}$$

which can be linked to the symmetric energy-momentum matrix

$$T = V^T P = \frac{1}{\gamma m_0}P^T P = \frac{1}{\gamma}U_0\hat{\mathbf{1}} = -L\hat{\mathbf{1}}. \tag{140}$$

If you take the density version, by dividing it by a volume, this volume has one of its lengths Lorentz contracted, which then compensates for the γ in L to produce a Lorentz invariant rest-energy density. In Quantum Mechanics this volume is included in the probability density so $\Psi^\dagger L \Psi = u_0$

In Wave Mechanics this is the basis for the introduction of the eigenvalue wave equation

$$P^T P\Psi = (\frac{1}{c^2}U_i^2 - p^2)\hat{\mathbf{1}}\Psi = \frac{1}{c^2}U_0^2\hat{\mathbf{1}}\Psi. \tag{141}$$

With the operator convention $\hat{P} = -\mathbf{i}\hbar\partial$ we can switch from energy-eigenvalue condition to operator-wave equation

$$\hat{P}^T \hat{P}\Psi = \frac{1}{c^2}U_0^2\hat{\mathbf{1}}\Psi. \tag{142}$$

We can make this canonical by applying the replacement $P \to P + qA$ and $\hat{P} \to \hat{P} + qA$ or $\partial \to D = \partial + \mathbf{i}\frac{q}{\hbar}A$. We get the canonical Klein-Gordon wave equation in a biquaternion metric

$$D^T D\Psi = \frac{U_0^2}{c^2\hbar^2}\hat{\mathbf{1}}\Psi. \tag{143}$$

This equation includes the Pauli-spin EM-field interaction term. One issue with the canonical version is the rest energy term U_0 is the question what it should all include. For the moment that question is ignored. But the issue is related to the open or closed system context. A closed system has constant rest energy and thus it has

$$\partial \frac{1}{m_0} P^T P = 0. \tag{144}$$

An open system doesn't have its divergence equal zero. Electromagnetic fields with moving charges are notoriously open systems. That affects the canonical wave equations of Quantum Mechanics.

The $D^T D\Psi$ part can be expanded as

$$D^T D\Psi = \partial^T \partial\Psi + \mathbf{i}\frac{q}{\hbar}\partial^T A\Psi + \mathbf{i}\frac{q}{\hbar}A^T \partial\Psi - \frac{q^2}{\hbar^2}A^T A\Psi. \tag{145}$$

Now, the first and the last terms give scalar quadratics but the two middle terms must be examined more carefully. By writing out the two matrix products and applying the standard differentiation rule to the scalars in these matrixes, one can show that

$$\partial^T A\Psi + A^T \partial\Psi = B\Psi + 2(\frac{1}{c^2}\phi\partial_t + \mathbf{A}\cdot\mathbf{\nabla})\hat{\mathbf{1}}\Psi \tag{146}$$

This gives us for $D^T D\Psi = \frac{U_0^2}{c^2\hbar^2}\hat{\mathbf{1}}\Psi$ the equation

$$\partial_\mu^T \partial^\mu \hat{\mathbf{1}}\Psi - \frac{q^2}{\hbar^2}A_\mu^T A^\mu \hat{\mathbf{1}}\Psi - 2\mathbf{i}\frac{q}{\hbar}A_\mu^T \partial^\mu \hat{\mathbf{1}}\Psi = -\frac{U_0^2}{c^2\hbar^2}\hat{\mathbf{1}}\Psi + \mathbf{i}\frac{q}{\hbar}B\Psi, \tag{147}$$

with

$$\partial^T \partial = (\frac{1}{c^2}\partial_t^2 - \mathbf{\nabla}^2)\hat{\mathbf{1}} = -\partial_\mu^T \partial^\mu \hat{\mathbf{1}}, \tag{148}$$

$$A^T A = (\frac{1}{c^2}\phi^2 - \mathbf{A}^2)\hat{\mathbf{1}} = -A_\mu^T A^\mu \hat{\mathbf{1}}, \tag{149}$$

$$A_\mu^T \partial^\mu = (\frac{1}{c^2}\phi\partial_t + \mathbf{A}\cdot\mathbf{\nabla}). \tag{150}$$

The only non-degenerate part in this equation is $\mathbf{i}\frac{q}{\hbar}B\Psi$. In our units we have the Bohr magneton $\mu_B = \frac{e\hbar}{2m_0}$ and if we multiply the equation by $\frac{\hbar^2}{2m_0}$ we get the non-degenerate term as $\mathbf{i}\mu_B B\Psi$. This can be written as

$$\mathbf{i}\mu_B B\Psi = \mathbf{i}\mu_b \overrightarrow{\mathbb{B}} \cdot \mathbf{K}\Psi = -\mu_b \overrightarrow{\mathbb{B}} \cdot \boldsymbol{\sigma}\Psi = -\mu_b \mathbf{B} \cdot \boldsymbol{\sigma}\Psi + \mathbf{i}\mu_b \frac{1}{c}\mathbf{E} \cdot \boldsymbol{\sigma}\Psi, \tag{151}$$

with the remark that we exchanged the Pauli σ_x and σ_z, as $\sigma_I = \sigma_z$ and $\sigma_K = \sigma_x$. So by putting spin in the metric we get a canonical Klein Gordon equation that includes Pauli-spin EM field interaction terms. Now, we have the spin magnetic moment $\boldsymbol{\mu}_s = \mu_B \boldsymbol{\sigma}$. We can further interpret the relativistic companion of the intrinsic magnetic moment as the intrinsic zitter-effect polarization $\boldsymbol{\pi}_s = \frac{e\lambda_c}{2}\boldsymbol{\sigma}$, we get

$$\mathbf{i}\mu_B B\Psi = -\mathbf{B} \cdot \boldsymbol{\mu}_s \Psi + \mathbf{i}\mathbf{E} \cdot \boldsymbol{\pi}_s \Psi \tag{152}$$

The complete wave equation on the Pauli-spin spinor-level, in which a spinor consists of two complex variables, will then be

$$-\frac{\hbar^2}{2m_0}\hat{\mathbf{1}}\boldsymbol{\nabla}^2\Psi + \frac{\hbar^2}{2m_0c^2}\hat{\mathbf{1}}\partial_t^2\Psi + \frac{q^2}{2m_0}\mathbf{A}^2\hat{\mathbf{1}}\Psi - \frac{q^2\phi^2}{2m_0c^2}\hat{\mathbf{1}}\Psi \tag{153}$$

$$+\frac{\mathbf{i}\hbar q\phi}{m_0c^2}\hat{\mathbf{1}}\partial_t\Psi + \frac{\mathbf{i}q\hbar}{m_0}\mathbf{A} \cdot \boldsymbol{\nabla}\hat{\mathbf{1}}\Psi \tag{154}$$

$$= \frac{U_0}{2}\hat{\mathbf{1}}\Psi + \mathbf{B} \cdot \boldsymbol{\mu}_s \Psi - \mathbf{i}\mathbf{E} \cdot \boldsymbol{\pi}_s \Psi, \tag{155}$$

The first term with $\boldsymbol{\nabla}^2$ is the kinetic term, the $\boldsymbol{A}^2$ part is know as the diamagnetic part of the Pauli equation, the $\mathbf{A} \cdot \boldsymbol{\nabla}$ part as the paramagnetic part, and with the Coulomb gauge this part can also be rearranged into the orbital or angular momentum term causing the Zeeman effect. The $\mathbf{B} \cdot \boldsymbol{\mu}_s$ term is the spin magnetic moment term connected to the anormal Zeeman effect. The other terms are either simply ignored, as for example the $\mathbf{E} \cdot \boldsymbol{\pi}_s$ term, or somehow reduced to a term for the potential and a term for the constant energy.

It is interesting to observe that we have a quadratic time derivative as is usual in the Klein-Gordon equation, but that we also have a linear time derivative. It is my impression that that linear term, together with the intrinsic zitter polarization term constitutes the relativistic complement of the $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ total angular momentum. The relativistic origin of total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ lies in the two cross-products of the square of the canonical momentum

$$\partial^T A\Psi + A^T \partial\Psi = B\Psi + 2(\frac{1}{c^2}\phi\partial_t + \mathbf{A} \cdot \boldsymbol{\nabla})\hat{\mathbf{1}}\Psi. \tag{156}$$

With the use of $\mathbf{i}\mu_B B\Psi = -\mathbf{B} \cdot \boldsymbol{\mu}_s \Psi + \mathbf{i}\mathbf{E} \cdot \boldsymbol{\pi}_s \Psi$, this can be split into the familiar $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ parts as

$$\mathbf{i}\mathbf{B} \cdot \boldsymbol{\mu}_s \Psi + 2\mu_B \mathbf{A} \cdot \hat{\mathbf{1}}\boldsymbol{\nabla}\Psi. \tag{157}$$

and the ignored part as

$$\mathbf{E}\cdot\boldsymbol{\pi}_s\Psi+\frac{q\phi\hbar}{m_0c^2}\hat{\mathbf{1}}\partial_t\Psi. \tag{158}$$

With the intrinsic zitter-effect polarization $\boldsymbol{\pi}_s=\frac{e\lambda_c}{2}\boldsymbol{\sigma}$ and the orbital zitter-effect Compton-level polarization as $\pi_o=e\lambda_c$ this last term can be written as

$$\mathbf{E}\cdot\boldsymbol{\pi}_s\Psi+\frac{\pi_o\phi}{c}\hat{\mathbf{1}}\partial_t\Psi \tag{159}$$

and then interpreted as the total zitter-effect Compton-level polarization.

This zitter-polarization linear in time derivative might well be the damping part of the canonical Klein-Gordon equation and then be responsible for the quantum jumps. It is also possible that these two terms, scaled to the reduced Compton wavelenght of the electron λ_c, are responsible for the electric counterparts of the normal Zeeman effect and the anormal Zeeman effect, ie linear Stark effect and anormal Stark effect. It seems outdated to just ignore the parts of the equation that one cannot connect to some physical experimental phenomena, as Dirac did with the intrinsic polarization term of his equation. But perhaps some of those terms only appear in this analysis due to the non-commutative character of the math-language used/developed.

It is also possible to interpret

$$\mathbf{E}\cdot\boldsymbol{\pi}_s\Psi+\frac{q\phi\hbar}{m_0c^2}\hat{\mathbf{1}}\partial_t\Psi. \tag{160}$$

for stationary states with constant energy as

$$\mathbf{E}\cdot\boldsymbol{\pi}_s\Psi+V\hat{\mathbf{1}}\partial_t\Psi. \tag{161}$$

with $V=q\phi$ and $\partial_t\Psi=\frac{\mathbf{i}U_0}{\hbar}\Psi$. With this stationary state interpretation, the term with the linear time derivative turns out to produce the standard potential energy term, and its energy levels are then the usual Coulomb energy levels. The $\mathbf{E}\cdot\boldsymbol{\pi}_s\Psi$ term then produces a zitter-like Compton reduced wavelength scale smearing out of the principal orbits. Such an effect has been observed for the most inner S-orbits.

So with the equation $D^TD\Psi=\frac{U_0^2}{c^2\hbar^2}\hat{\mathbf{1}}\Psi$ we are able to treat Pauli spin relativistically, provided that the spinor Ψ Lorentz transforms as $\Psi^L=U\Psi$. That however is only the case for the spinors in the Weyl representation and not for spinors in the Dirac representation. The Lorentz transformation of spinors in the Dirac representation can only be achieved at the Dirac spinor level, so with four variable spinors. A two variable Pauli spinor in the Dirac representation cannot be Lorentz transformed on its own, that is, without its Dirac twin. On the Weyl level, a Lorentz transformation of a Pauli spinor is possible, but the transformation to its Dirac representation is impossible without its Weyl twin spinor. In a modern interpretation, this implies that understanding the intrinsics of a quantum jump as a damping term effect is impossible without introducing anti-particles and the related quantum field interpretation, even in atoms. If so, then we should introduce Feynman diagram like analysis in atomic physics's attempts to grasp the intrinsics of quantum jumps. It is however impossible to prove this at the Pauli spin level. In the context of atomic physics at the Pauli level of two variable spinors, quantum jumps are and will remain a mystery, without proof why that is. Just like line beings will never understand angles and surface restricted beings will never be able to understand volumes.

IV. THE DIRAC SPIN LEVEL

A. The Dirac environment metric matrices

In the nineteen twenties, the quadratic relativistic scalar Klein-Gordon wave equation couldn't be applied to the relativistic electron. Dirac linearized the Klein-Gordon equation by going to four by four matrices instead of the two by two Pauli matrices. In his two seminal 1928 papers he introduces the Clifford four set $(\beta, \boldsymbol{\alpha})$ and, using what were later called the gamma matrices, the covariant Clifford four set $(\beta, \boldsymbol{\gamma})$. The Pauli matrices are incorporated in these matrices. Weyl later found a third covariant Clifford four set, which relates to the Dirac covariant set as low velocity relativistic to high velocity relativistic gamma matrices Clifford four set.

All these matrices can be represented as two by two matrices of the biquaternion basis $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$. But using the biquaternion basis $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ as a basis of the space-time metric is already highly problematic. Duplicating this spin-norm basis by going from the Pauli spinor level to the Dirac spinor level is even more so. As a consequence, using the Clifford four set gamma matrices written as $\gamma_\mu = (\gamma_0, \boldsymbol{\gamma})$, as a basis for the space-time metric or as space-time four vectors is truly questionable. It is my opinion that the $(\hat{\mathbf{T}}, \mathbf{K})$ biquaternion basis will provide a more solid basis for connecting the Clifford four sets of Relativistic Quantum Mechanics to ordinary relativistic space-time.

1. The Dirac and Weyl matrices in dual norm-spin mode

In the following I present the Dirac and Weyl matrices using my reversed order of the Pauli spin matrices, with $\sigma_I = \sigma_z$, $\sigma_J = \sigma_y$, $\sigma_K = \sigma_x$ and $\boldsymbol{\sigma} = (\sigma_I, \sigma_J, \sigma_K)$. This implies that the order of the gamma matrices are reversed correspondingly, with $\gamma_1 = \gamma_I = \gamma_z$, $\gamma_2 = \gamma_J = \gamma_y$, $\gamma_3 = \gamma_K = \gamma_x$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) = (\gamma_I, \gamma_J, \gamma_K)$).

In my $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ norm-spin basis the Dirac set $\alpha_\mu = (\beta, \boldsymbol{\alpha})$ can be represented as

$$\alpha_\mu = (\not{1}, \boldsymbol{\alpha}) = \left(\begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & \hat{\mathbf{1}} \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix} \right). \tag{162}$$

The most straightforward doubling of the Pauli level norm-spin set $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ is the Dirac level norm-spin set $\Sigma_\mu = (\not{1}, \boldsymbol{\Sigma})$ defined as

$$\Sigma_\mu = (\not{1}, \boldsymbol{\Sigma}) = \left(\begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & \hat{\mathbf{1}} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix} \right). \tag{163}$$

The set of gamma matrices in the Dirac representation, $\gamma_\mu = (\beta, \boldsymbol{\gamma}) = (\gamma_0, \boldsymbol{\gamma})$, can be defined as

$$\gamma_\mu = (\beta, \boldsymbol{\gamma}) = (\gamma_0, \boldsymbol{\gamma}) = \left(\begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & -\hat{\mathbf{1}} \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} \right) \tag{164}$$

The set of gamma matrices in the Weyl representation, $\gamma_\mu = (\gamma_0, \boldsymbol{\gamma})$, can be defined as

$$\gamma_\mu = (\gamma_0, \boldsymbol{\gamma}) = \left(\begin{bmatrix} 0 & \hat{\mathbf{1}} \\ \hat{\mathbf{1}} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} \right) \tag{165}$$

The last matrix we need to define in this environment is the γ_5 matrix as

$$\gamma_5 = \begin{bmatrix} 0 & \hat{\mathbf{1}} \\ -\hat{\mathbf{1}} & 0 \end{bmatrix}. \tag{166}$$

The most important product needed to understand the genius of Dirac's two 1928 Dirac equation papers is the Dirac gamma product

$$\gamma_0 \gamma_\mu = (\gamma_0 \gamma_0, \gamma_0 \boldsymbol{\gamma}) = (\hat{\mathbf{1}}, \boldsymbol{\alpha}) = \alpha_\mu. \tag{167}$$

This equation is key towards understanding the Dirac four current and the related continuity equation. This product eventually leads to the definition of the Dirac adjoint as $\overline{\Psi} = \Psi^\dagger \gamma_0$, the Dirac probability current as

$$J_\mu = c\overline{\Psi}\gamma_\mu \Psi = c\Psi^\dagger \gamma_0 \gamma_\mu \Psi = c\Psi^\dagger \alpha_\mu \Psi \tag{168}$$

and the Dirac current continuity equation as

$$\partial_\mu J^\mu = c\partial_\mu \overline{\Psi}\gamma^\mu \Psi = c\partial_\mu \Psi^\dagger \gamma_0 \gamma^\mu \Psi = c\partial_\mu \Psi^\dagger \alpha^\mu \Psi = 0. \tag{169}$$

The main innovative result of the third part of this paper is the conclusion that the elements of this probability current four vector can be interpreted as part of a metric probability tensor and that the continuity equation has its origin in the time like part of the closed system condition of that metric probability tensor, as in

$$\partial_\nu {\Phi_\mu}^\nu \equiv \partial_\nu \Psi^\dagger \gamma_\mu \gamma^\nu \Psi = 0. \tag{170}$$

In order to make this consistent as a space-time metric probability condition, I need to introduce the Dirac and Weyl related matrix representations in the time-space basis $(\hat{\mathbf{T}}, \mathbf{K})$, what I will call the bèta matrices, instead of gamma matrices in the norm-spin basis $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$.

In my treatment of RQM, the Weyl representation in the time-space basis $(\hat{\mathbf{T}}, \mathbf{K})$ will prove to be like Machiavelli's "return to the banner" when coherence is fading. In my context, it is the most simple point of departure possible, from where almost all the rest can be derived. To return to the space-time basis,$(\hat{\mathbf{T}}, \mathbf{K})$ and the related Weyl β_μ as its dual-parity version will prove its strategic worth. But the Dirac representation has proven it's worth for almost all practical, experimental area's of interest, so to understand the operator that switches between them is as important.

2. The transformation from the Dirac to the Weyl representation and vice versa

The transformation from the Weyl to the Dirac representation and vice versa is an operator that is usually written as S. Two possible versions of S are being used. The most common one is

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{\mathbf{1}} & \hat{\mathbf{1}} \\ \hat{\mathbf{1}} & -\hat{\mathbf{1}} \end{bmatrix}$$

and the one I prefer is the less common

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{\mathbf{1}} & \hat{\mathbf{1}} \\ -\hat{\mathbf{1}} & \hat{\mathbf{1}} \end{bmatrix}.$$

The reason I will only use the second version is that it has the property $\gamma_0 S = S^{-1}\gamma_0$ and the directly related $S\gamma_0 = \gamma_0 S^{-1}$.

The switch from the Weyl γ_w^ν to the Dirac γ_d^ν is then given by $\gamma_d^\nu = S\gamma_w^\nu S^{-1}$ and the switch from the Dirac to the Weyl representation by the inverse $\gamma_w^\nu = S^{-1}\gamma_d^\nu S$. This also applies to the α^ν matrices, which are almost always given in their Dirac representation, but who can also be written in the Weyl representation as

$$\alpha_w^\nu = S^{-1}\alpha_d^\nu S = \left(\begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & \hat{\mathbf{1}} \end{bmatrix}, \begin{bmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix} \right). \tag{171}$$

As a logical consequence one has

$$\gamma_0^w \gamma_\nu^w = (\gamma_0^w \gamma_0^w, \gamma_0^w \boldsymbol{\gamma}^w) = (\hat{\mathbf{1}}, \boldsymbol{\alpha}^w) = \alpha_\nu^w. \tag{172}$$

A Weyl adjoint can be defined as $\overline{\Psi}^w = \Psi^{\dagger w}\gamma_0^w$ and a Weyl current as $J_\nu^w = \overline{\Psi}^w \gamma_\nu^w \Psi^w$. This Weyl current is exactly the same as the Dirac current, due to the transformation properties of the spinors under the Dirac to Weyl representation transformation, given as $\Psi_w = S^{-1}\Psi_d$ and $\Psi_w^\dagger = \Psi_d^\dagger S$. On has

$$J_\nu^w = \overline{\Psi}^w \gamma_\nu^w \Psi^w = \Psi^{\dagger w}\gamma_0^w \gamma_\nu^w \Psi^w = \Psi^{\dagger w}\alpha_\nu^w \Psi^w = \Psi^{\dagger d} S S^{-1}\alpha_\nu^d S S^{-1}\Psi^d = \Psi^{\dagger d}\alpha_\nu^d \Psi^d = J_\nu^d. \tag{173}$$

The Dirac level norm-spin set $\Sigma_\mu = (\not{1}, \boldsymbol{\Sigma})$ has it's Weyl representation given by the unchanged

$$\Sigma_\nu^w = S^{-1}(\not{1}, \boldsymbol{\Sigma})S = \left(\begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & \hat{\mathbf{1}} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix} \right). \tag{174}$$

3. The closed system condition for the Dirac probability current tensor

The derivative of the probability density tensor in its closed system condition,

$$\partial_\nu \Phi_\mu{}^\nu \equiv \partial_\nu \Psi^\dagger \gamma_\mu \gamma^\nu \Psi = 0, \tag{175}$$

can be retraced to the Klein Gordon equation on the Dirac level as

$$\partial_\nu \Psi^\dagger \not{V} \not{P} \Psi = \partial_\nu \frac{1}{m_0} \Psi^\dagger \not{P} \not{P} \Psi = \partial_\nu \Psi^\dagger U_0 \not{1} \Psi = U_0 \partial_\nu \Psi^\dagger \Psi = 0. \tag{176}$$

which includes the proof of the closed system condition for the symmetric tensor $\not{T} = \not{V}\not{P}$ as $\partial_\nu \not{T} = 0$. This closed system condition applies to both the Dirac representation as the Weyl representation, as long as it is clear that not only γ_0 but also $\boldsymbol{\alpha}$ and $\overline{\Psi}$ have a Dirac representation and a Weyl representation. The gamma tensor $\gamma_\mu \gamma^\nu$ is given by

$$\gamma_\mu \gamma^\nu = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \begin{bmatrix} \gamma_0\gamma_0 & \gamma_1\gamma_0 & \gamma_2\gamma_0 & \gamma_3\gamma_0 \\ \gamma_0\gamma_1 & \gamma_1\gamma_1 & \gamma_2\gamma_1 & \gamma_3\gamma_1 \\ \gamma_0\gamma_2 & \gamma_1\gamma_2 & \gamma_2\gamma_2 & \gamma_3\gamma_2 \\ \gamma_0\gamma_3 & \gamma_1\gamma_3 & \gamma_2\gamma_3 & \gamma_3\gamma_3 \end{bmatrix} = \begin{bmatrix} \not{1} & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & -\not{1} & -\mathbf{i}\Sigma_3 & \mathbf{i}\Sigma_2 \\ \alpha_2 & \mathbf{i}\Sigma_3 & -\not{1} & -\mathbf{i}\Sigma_1 \\ \alpha_3 & -\mathbf{i}\Sigma_2 & \mathbf{i}\Sigma_1 & -\not{1} \end{bmatrix} \tag{177}$$

The probability density tensor is then given by

$$\Phi_{\mu}{}^{\nu} = \Psi^{\dagger}\gamma_{\mu}\gamma^{\nu}\Psi = \begin{bmatrix} \Psi^{\dagger}\mathbb{1}\Psi & -\Psi^{\dagger}\alpha_1\Psi & -\Psi^{\dagger}\alpha_2\Psi & -\Psi^{\dagger}\alpha_3\Psi \\ \Psi^{\dagger}\alpha_1\Psi & -\Psi^{\dagger}\mathbb{1}\Psi & -\Psi^{\dagger}\mathbf{i}\Sigma_3\Psi & \Psi^{\dagger}\mathbf{i}\Sigma_2\Psi \\ \Psi^{\dagger}\alpha_2\Psi & \Psi^{\dagger}\mathbf{i}\Sigma_3\Psi & -\Psi^{\dagger}\mathbb{1}\Psi & -\Psi^{\dagger}\mathbf{i}\Sigma_1\Psi \\ \Psi^{\dagger}\alpha_3\Psi & -\Psi^{\dagger}\mathbf{i}\Sigma_2\Psi & \Psi^{\dagger}\mathbf{i}\Sigma_1\Psi & -\Psi^{\dagger}\mathbb{1}\Psi \end{bmatrix}. \tag{178}$$

The time-like part of $\partial_{\nu}\Phi_{\mu}{}^{\nu} = 0$ is given by

$$\frac{1}{c}\partial_t\Psi^{\dagger}\mathbb{1}\Psi + \nabla_1\Psi^{\dagger}\alpha_1\Psi + \nabla_2\Psi^{\dagger}\alpha_2\Psi + \nabla_3\Psi^{\dagger}\alpha_3\Psi = \frac{1}{c}\partial_t\Psi^{\dagger}\mathbb{1}\Psi + \boldsymbol{\nabla}\Psi^{\dagger}\boldsymbol{\alpha}\Psi = 0 \tag{179}$$

This can be abbreviated as the Dirac current continuity equation

$$c\partial_{\nu}\Psi^{\dagger}\alpha^{\nu}\Psi = c\partial_{\nu}\overline{\Psi}\gamma^{\nu}\Psi = \partial_{\nu}J^{\nu} = 0. \tag{180}$$

This proves that the Klein Gordon equation on the Dirac level includes the continuity equation for the probability current as part of a much stronger closed system condition for the probability density (current-)tensor. That connects the Klein Gordon at Dirac level environment to the Laue closed system condition, which in turn is a basic axiom of or prerequisite for General Relativity's symmetric stress energy density tensors $T = VG$.

The space-like derivatives of $\partial_{\nu}\Phi_{\mu}{}^{\nu} = 0$ can be split into a complex part and a real part. The complex part gives

$$\boldsymbol{\nabla} \times \Psi^{\dagger}\boldsymbol{\Sigma}\Psi = 0. \tag{181}$$

The real part gives

$$\partial_t\Psi^{\dagger}\boldsymbol{\alpha}\Psi = c\boldsymbol{\nabla}\Psi^{\dagger}\mathbb{1}\Psi \tag{182}$$

which can be multiplied by the constants m_0c, and using the Dirac adjoint, to give

$$\partial_t m_0 c\overline{\Psi}\boldsymbol{\gamma}\Psi = \boldsymbol{\nabla} m_0 c^2\overline{\Psi}\gamma_0\Psi. \tag{183}$$

The last two conditions show that the closed system condition for the probability density tensor is a stronger condition than the continuity equation on its own. The above two conditions can be connected to the earlier $\boldsymbol{\nabla} \times \boldsymbol{p} = 0$ and the $\partial_t\boldsymbol{p} = -\boldsymbol{\nabla}U_i$ as there probability/field analogues. The first prohibits a probability/field vorticity in the closed system condition, the second implies a conserved force-field condition for the probability/field, connecting the time-rate of change of the current to the space divergence of the related density.

Given the fact that all Lagrangians of the Standard Model's Dirac fields are based upon the Dirac current, the Dirac adjoint and the use of the Dirac equation to prove the continuity equation for the Dirac current, it's generalization into a Dirac probability or field tensor with connected much stronger closed system condition and a prove of its validity based upon the Dirac level Klein Gordon equation should have some impact. The recognition that the Dirac current is just a part of a tensor and that the Dirac current continuity equation is just the time-like part of a space-time closed system condition of that tensor will close the gap with General Relativity considerably, given the relation of both to the Laue closed system condition $\partial_\nu T_\mu{}^\nu = 0$. I propose to use tensor Lagrangians based on

$$\mathscr{L} = \frac{1}{m_0}\Psi^\dagger \hat{\not{P}}\hat{\not{P}}\Psi, \tag{184}$$

which then contain the inertial probability or inertial field tensor

$$m_\mu{}^\nu c^2 = m_0 \Phi_\mu{}^\nu c^2 = m_0 \Psi^\dagger \gamma_\mu \gamma^\nu \Psi c^2, \tag{185}$$

as a relativistic generalization of the usual Dirac current with Dirac adjoint based Lagrangians of the Standard Model.

4. The Dirac and Weyl matrices in dual time-space mode as bèta matrices

What is absent in the above treatments is the Lorentz transformation and the check if all relations that are given are Lorentz invariant or at least Lorentz covariant. The Lorentz transformation of the matrices, the four vectors and the spinors are most elementary in the time-space $(\hat{\mathbf{1}}, \mathbf{K})$ Weyl representation. I will call these time-space Weyl-Dirac matrices the bèta matrices.

In my math-phys language and with a Möbius kind of doubling in mind I can define matrices through the application of parity or point reflection P and time reversal or present reflection T as

$$\begin{bmatrix} P & P \\ P^P & P^T \end{bmatrix} = \begin{bmatrix} P & P \\ -P^T & P^T \end{bmatrix} =$$
$$p_0 \begin{bmatrix} \hat{\mathbf{T}} & \hat{\mathbf{T}} \\ \hat{\mathbf{T}} & -\hat{\mathbf{T}} \end{bmatrix} + p_1 \begin{bmatrix} \hat{\mathbf{I}} & \hat{\mathbf{I}} \\ -\hat{\mathbf{I}} & \hat{\mathbf{I}} \end{bmatrix} + p_2 \begin{bmatrix} \hat{\mathbf{J}} & \hat{\mathbf{J}} \\ -\hat{\mathbf{J}} & \hat{\mathbf{J}} \end{bmatrix} + p_3 \begin{bmatrix} \hat{\mathbf{K}} & \hat{\mathbf{K}} \\ -\hat{\mathbf{K}} & \hat{\mathbf{K}} \end{bmatrix} =$$
$$p_0 \begin{bmatrix} \hat{\mathbf{T}} & \hat{\mathbf{T}} \\ \hat{\mathbf{T}} & -\hat{\mathbf{T}} \end{bmatrix} + \mathbf{p} \cdot \begin{bmatrix} \mathbf{K} & \mathbf{K} \\ -\mathbf{K} & \mathbf{K} \end{bmatrix}. \tag{186}$$

The norm of this matrix is simply $2P^T P = 2U_0 \not{1}$.

I split this into $P_\mu\beta^\mu + P_\mu\xi^\mu$ by defining

$$\not{P} = P_\mu\beta^\mu = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} = p_0 \begin{bmatrix} 0 & \hat{\mathbf{T}} \\ \hat{\mathbf{T}} & 0 \end{bmatrix} + \mathbf{p}\cdot\begin{bmatrix} 0 & \mathbf{K} \\ -\mathbf{K} & 0 \end{bmatrix} = p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\beta} =$$
$$p_0\begin{bmatrix} 0 & \hat{\mathbf{T}} \\ \hat{\mathbf{T}} & 0 \end{bmatrix} + p_1\begin{bmatrix} 0 & \hat{\mathbf{I}} \\ -\hat{\mathbf{I}} & 0 \end{bmatrix} + p_2\begin{bmatrix} 0 & \hat{\mathbf{J}} \\ -\hat{\mathbf{J}} & 0 \end{bmatrix} + p_3\begin{bmatrix} 0 & \hat{\mathbf{K}} \\ -\hat{\mathbf{K}} & 0 \end{bmatrix} \tag{187}$$

with $\not{P} = P_\mu\beta^\mu = p_0\beta_0 + p_1\beta_1 + p_2\beta_2 + p_3\beta_3$, and

$$P_\mu\xi^\mu = \begin{bmatrix} P & 0 \\ 0 & P^T \end{bmatrix} = p_0\begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & -\hat{\mathbf{T}} \end{bmatrix} + \mathbf{p}\cdot\begin{bmatrix} \mathbf{K} & 0 \\ 0 & \mathbf{K} \end{bmatrix} = p_0\xi_0 + \mathbf{p}\cdot\boldsymbol{\xi} =$$
$$p_0\begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & -\hat{\mathbf{T}} \end{bmatrix} + p_1\begin{bmatrix} \hat{\mathbf{I}} & 0 \\ 0 & \hat{\mathbf{I}} \end{bmatrix} + p_2\begin{bmatrix} \hat{\mathbf{J}} & 0 \\ 0 & \hat{\mathbf{J}} \end{bmatrix} + p_3\begin{bmatrix} \hat{\mathbf{K}} & 0 \\ 0 & \hat{\mathbf{K}} \end{bmatrix} \tag{188}$$

with $P_\mu\xi^\mu = p_0\xi_0 + p_1\xi_1 + p_2\xi_2 + p_3\xi_3$.

If I use $\hat{\mathbf{T}} = \mathbf{i}\hat{\mathbf{1}}$ and $\mathbf{K} = \mathbf{i}\boldsymbol{\sigma}$ I get

$$\beta_\mu = (\beta_0, \boldsymbol{\beta}) = \left(\begin{bmatrix} 0 & \mathbf{i}\hat{\mathbf{1}} \\ \mathbf{i}\hat{\mathbf{1}} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{i}\boldsymbol{\sigma} \\ -\mathbf{i}\boldsymbol{\sigma} & 0 \end{bmatrix} \right) = (\mathbf{i}\hat{\mathbf{1}}, \mathbf{i}\boldsymbol{\gamma}) = \mathbf{i}\gamma_\mu \tag{189}$$

which relates the parity dual β_μ to the Weyl bèta representation. The Dirac representation mixes the bèta and the xi representation and thus represents a PT dual. I nevertheless, using the gamma tradition, use the bèta and Feynman slash symbols for both representations in the time-space $\hat{\mathbf{T}}, \mathbf{K}$ basis. This gives for the Dirac bèta representation

$$\not{P} = P_\mu\beta^\mu = p_0\begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & -\hat{\mathbf{T}} \end{bmatrix} + \mathbf{p}\cdot\begin{bmatrix} 0 & \mathbf{K} \\ -\mathbf{K} & 0 \end{bmatrix} = p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\beta} =$$
$$p_0\begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & -\hat{\mathbf{T}} \end{bmatrix} + p_1\begin{bmatrix} 0 & \hat{\mathbf{I}} \\ -\hat{\mathbf{I}} & 0 \end{bmatrix} + p_2\begin{bmatrix} 0 & \hat{\mathbf{J}} \\ -\hat{\mathbf{J}} & 0 \end{bmatrix} + p_3\begin{bmatrix} 0 & \hat{\mathbf{K}} \\ -\hat{\mathbf{K}} & 0 \end{bmatrix}. \tag{190}$$

As with the Weyl representation, in the Dirac representation we have $\beta_\mu = \mathbf{i}\gamma_\mu$.

The transformation matrix S remains unchanged. But its interpretation can be enriched. It isn't just a neutral change of representations, it changes a parity only Weyl dual representation of space-time into a combined parity, time reversal Dirac dual representation of space-time (and vice versa). The transformation operation S adds or removes time reversal from the dual, it is a time reversal transformation.

There is one additional matrix needed for the Dirac equation, the split of the square of the eigen time matrix ξ, defined as

$$\xi = \begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & \hat{\mathbf{T}} \end{bmatrix}. \tag{191}$$

B. The Dirac and Weyl equations in the space-time bèta matrices environment

The trick in formulating equations in the Dirac environment is that they have to be reducible to the Klein Gordon energy condition $P^T P = E^2 \hat{\mathbf{1}}$ with $E = \frac{U_0}{c} = m_0 c$. We have three equations that match this demand, but only the first two use a Clifford four set. The third equation uses tricks to compensate for the limitations of a Clifford three set in a 4-D environment. In the Weyl and Dirac equations we can split $-E^2 \not{1}$ using the ξ matrix, as $\not{E}^2 = (E\xi)^2 = -E^2 \not{1}$.

The Weyl or chiral equation stems from the quadratic $\not{P}\not{P} = \not{E}\not{E}$ in the space-time Weyl representation.

$$\not{P}\not{P} = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} = \begin{bmatrix} -PP^T & 0 \\ 0 & -P^T P \end{bmatrix} = \begin{bmatrix} -E^2 \hat{\mathbf{1}} & 0 \\ 0 & -E^2 \hat{\mathbf{1}} \end{bmatrix} = -E^2 \not{1} = \not{E}\not{E} \tag{192}$$

So we have $\not{P}\not{P} - \not{E}\not{E} = 0$. This leads to $(\not{P} - \not{E})(\not{P} + \not{E}) = 0$. If we split this into two equations, $\not{P} - \not{E} = 0$ and $\not{P} + \not{E} = 0$, then only the trivial all zero solution is possible. But if we add the Dirac spinors, then non zero solutions are possible. We get $\Psi^\dagger(\not{P} - \not{E})(\not{P} + \not{E})\Psi = 0$, which can be split into $\Psi^\dagger(\not{P} - \not{E}) = 0$ and $(\not{P} + \not{E})\Psi = 0$. By interpreting the spinors as waves or wave-like fields all the solutions of those equations can be interpreted as eigenvalue solutions of related operators and we get the Weyl wave equations as

$$\hat{\not{P}}\Psi = \not{E}\Psi \tag{193}$$

$$\hat{\not{P}}\Psi = -\not{E}\Psi \tag{194}$$

if we use $\hat{\not{P}} = -\mathbf{i}\hbar\not{\partial}$ and a four column dual spinor Ψ.

The Dirac equation stems from the quadratic $(p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\beta})^2 = -E^2\not{1}$.

$$\not{P}\not{P} = \begin{bmatrix} p_0\hat{\mathbf{T}} & \mathbf{p}\cdot\mathbf{K} \\ -\mathbf{p}\cdot\mathbf{K} & -p_0\hat{\mathbf{T}} \end{bmatrix} \begin{bmatrix} p_0\hat{\mathbf{T}} & \mathbf{p}\cdot\mathbf{K} \\ -\mathbf{p}\cdot\mathbf{K} & -p_0\hat{\mathbf{T}} \end{bmatrix} = \begin{bmatrix} (-p_0^2 + \mathbf{p}^2)\hat{\mathbf{1}} & 0 \\ 0 & (-p_0^2 + \mathbf{p}^2)\hat{\mathbf{1}} \end{bmatrix} = -E^2\not{1} \tag{195}$$

This leads to the two options for the Dirac equations

$$(\hat{p}_0\beta_0 + \hat{\mathbf{p}}\cdot\boldsymbol{\beta})\Psi = E\not{1}\Psi \tag{196}$$

$$(\hat{p}_0\beta_0 + \hat{\mathbf{p}}\cdot\boldsymbol{\beta})\Psi = -E\not{1}\Psi \tag{197}$$

if we use $\hat{P} = -\mathbf{i}\hbar\partial$ and a four column spinor Ψ.

So in the space-time representation we have the Weyl $\not{P}$ as

$$\not{P}_w = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \tag{198}$$

and the Dirac $\not{P}$ as

$$\not{P}_d = \begin{bmatrix} p_0\hat{\mathbf{T}} & \mathbf{p}\cdot\mathbf{K} \\ -\mathbf{p}\cdot\mathbf{K} & -p_0\hat{\mathbf{T}} \end{bmatrix} \tag{199}$$

and the transformation between them as $\not{P}_w = S^{-1}\not{P}_d S$ and $\not{P}_d = S\not{P}_w S^{-1}$.

C. Lorentz transformations in the Dirac and Weyl representation environments

In part III of this paper I developed the Pauli level basis $(\hat{\mathbf{T}}, \mathbf{K})$ relativistic approach. This resulted in the Lorentz transformation of a four vector $P = (p_0\hat{\mathbf{T}}, \boldsymbol{p}\cdot\mathbf{K})$ as $P^L = U^{-1}PU^{-1}$ and the Lorentz transformation of its time reversal P^T as $(P^L)^T = (P^T)^{L^{-1}} = UP^TU$ with U as

$$U = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix} \tag{200}$$

and the rapidity ψ. The Lorentz transformation of its time reversal P^T was $(P^L)^T = (P^T)^{L^{-1}} = UP^TU$. The quadratic P^TP then is automatically a Lorentz invariant scalar $\frac{U_0^2}{c^2}\hat{\mathbf{1}}$ with the dimension of the norm $\hat{\mathbf{1}}$. If in the space-time representation we have the Weyl $\not{P}$ in a reference system S as

$$\not{P} = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \tag{201}$$

then in reference system S' we have P^L and so also the Weyl $\not{P}^L$ as

$$\not{P}^L = \begin{bmatrix} 0 & P^L \\ -(P^L)^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & U^{-1}PU^{-1} \\ -UP^TU & 0 \end{bmatrix} \tag{202}$$

The question then is how to generate this result. The obvious answer is

$$\not{P}_w^L = \Lambda^{-1}\not{P}_w\Lambda = \begin{bmatrix} U^{-1} & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & U^{-1} \end{bmatrix} = \begin{bmatrix} 0 & U^{-1}PU^{-1} \\ -UP^TU & 0 \end{bmatrix} \tag{203}$$

with the Lorentz transformation matrix

$$\Lambda = \begin{bmatrix} U & 0 \\ 0 & U^{-1} \end{bmatrix} \tag{204}$$

and its obvious inverse Λ^{-1}.

The Klein Gordon equation's Lorentz invariance or covariance depends on the products $\not{p}^L\not{p}^L$. Using the previous result, we have for the Lorentz transformation of the product $\not{p}\not{p}$ in the Weyl representation

$$\not{p}^L\not{p}^L = \Lambda^{-1}\not{p}\Lambda\Lambda^{-1}\not{p}\Lambda = \Lambda^{-1}\not{p}\not{p}\Lambda = \Lambda^{-1}\not{E}\not{E}\Lambda = -E^2\not{1}\Lambda^{-1}\Lambda = -E^2\not{1} = \not{p}\not{p}, \tag{205}$$

so a Lorentz invariant product. This proof then included that $\not{E}^L\not{E}^L = \not{E}\not{E}$. This ensures the Lorentz invariance of the Klein Gordon condition $\not{p}\not{p} = \not{E}\not{E}$ in the Weyl representation.

In the Dirac version, where $\not{p} = p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\beta}$, things get more complicated. We have to start with the Dirac $\not{p}_d$ in the primary reference system and we want to end up with $\not{p}_d^L$ in the secondary reference system. We know how to transform between the Dirac and the Weyl representations and we know how to Lorentz transform the Weyl $\not{p}_w$. This means we have to go from Dirac to Weyl in the primary reference system, then Lorentz transform the Weyl four vector to the secondary reference system and then transform back from the Weyl to the Dirac representation, three operations in total. The total result gives

$$\not{p}_d^L = S\Lambda^{-1}S^{-1}\not{p}_d S\Lambda S^{-1}. \tag{206}$$

For the Klein Gordon equation in the Dirac representation, we get the Lorentz invariance through

$$\not{p}_d^L\not{p}_d^L = S\Lambda^{-1}S^{-1}\not{p}_d S\Lambda S^{-1}S\Lambda^{-1}S^{-1}\not{p}_d S\Lambda S^{-1} = S\Lambda^{-1}S^{-1}\not{p}_d S\Lambda\Lambda^{-1}S^{-1}\not{p}_d S\Lambda S^{-1} = \tag{207}$$

$$S\Lambda^{-1}S^{-1}\not{p}_d SS^{-1}\not{p}_d S\Lambda S^{-1} = S\Lambda^{-1}S^{-1}\not{p}_d\not{p}_d S\Lambda S^{-1} = S\Lambda^{-1}S^{-1}\not{E}_d\not{E}_d S\Lambda S^{-1} = \tag{208}$$

$$-E^2\not{1}S\Lambda^{-1}S^{-1}S\Lambda S^{-1} = -E^2\not{1}S\Lambda^{-1}\Lambda S^{-1} = -E^2\not{1}SS^{-1} = -E^2\not{1} = \not{p}_d\not{p}_d. \tag{209}$$

As for the Lorentz transformation of a Weyl 4-spinor, we have the requirement that we want the Lagrangian density element $\mathscr{L} = \frac{1}{m_0}\Psi^\dagger\not{p}\not{p}\Psi$ to be Lorentz invariant. This requirement is met if

$$\mathscr{L}^L = \frac{1}{m_0}(\Psi^\dagger)^L\not{p}^L\not{p}^L\Psi^L = \frac{1}{m_0}(\Psi^\dagger)^L\Lambda^{-1}\not{p}\Lambda\Lambda^{-1}\not{p}\Lambda\Psi^L = \tag{210}$$

$$\frac{1}{m_0}(\Psi^\dagger)^L\Lambda^{-1}\not{p}\not{p}\Lambda\Psi^L = \frac{1}{m_0}(\Psi^\dagger)\not{p}\not{p}\Psi = \mathscr{L}. \tag{211}$$

This implies first that

$$(\Psi^{\dagger})^{L}\Lambda^{-1} = (\Psi^{\dagger}) \tag{212}$$

so

$$(\Psi^{\dagger})^{L} = (\Psi^{\dagger})\Lambda. \tag{213}$$

Secondly we must have

$$\Lambda\Psi^{L} = \Psi \tag{214}$$

so

$$\Psi^{L} = \Lambda^{-1}\Psi. \tag{215}$$

Thus, in the Weyl space-time representation, $(\Psi^{\dagger})^{L}_{w} = (\Psi^{\dagger})_{w}\Lambda$ and $\Psi^{L}_{w} = \Lambda^{-1}\Psi_{w}$. This gives

$$\Psi^{L}_{w} = \Lambda^{-1}\Psi_{w} = \begin{bmatrix} U^{-1} & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} \Psi^{1}_{w} \\ \Psi^{2}_{w} \end{bmatrix} = \begin{bmatrix} U^{-1}\Psi^{1}_{w} \\ U\Psi^{2}_{w} \end{bmatrix}. \tag{216}$$

Important in this last equation is the result that the bispinors Ψ^{1} and Ψ^{2} do not mix in the Lorentz transformation in the space-time Weyl representation.

The same line of reasoning will give us the Lorentz transformation rules for the spinors in the space-time Dirac representation, respectively

$$(\Psi^{\dagger})^{L}_{d} = (\Psi^{\dagger}_{d})S\Lambda S^{-1} \tag{217}$$

and

$$\Psi^{L}_{d} = S\Lambda^{-1}S^{-1}\Psi_{d}. \tag{218}$$

In details, with rapidity ψ, the operator $S\Lambda^{-1}S^{-1}$ is given as

$$S\Lambda^{-1}S^{-1} = \begin{bmatrix} cosh(\frac{\psi}{2})\hat{\mathbf{1}} & sinh(\frac{\psi}{2})\sigma_{I} \\ sinh(\frac{\psi}{2})\sigma_{I} & cosh(\frac{\psi}{2}\hat{\mathbf{1}} \end{bmatrix} \tag{219}$$

and the operator $S\Lambda S^{-1}$ is given as

$$S\Lambda S^{-1} = \begin{bmatrix} cosh(\frac{\psi}{2})\hat{\mathbf{1}} & -sinh(\frac{\psi}{2})\sigma_{I} \\ -sinh(\frac{\psi}{2})\sigma_{I} & cosh(\frac{\psi}{2}\hat{\mathbf{1}} \end{bmatrix}. \tag{220}$$

The operator $S\Lambda^{-1}S^{-1}$ for the Lorentz transformation of the Dirac spinor Ψ exactly matches the one in (69, Darwin, 1928).

The structure of these transformations look familiar. If we define $\gamma' = cosh(\frac{\psi}{2})$ and $\gamma'\beta' = sinh(\frac{\psi}{2})$, we get the Lorentz transformation of Ψ as

$$\Psi^L = \begin{bmatrix} \gamma'\hat{\mathbf{1}} & \gamma'\beta'\sigma_I \\ \gamma'\beta'\sigma_I & \gamma'\hat{\mathbf{1}} \end{bmatrix} \begin{bmatrix} \Psi^1 \\ \Psi^2 \end{bmatrix} = \begin{bmatrix} \gamma'\hat{\mathbf{1}}\Psi^1 + \gamma'\beta'\sigma_I\Psi^2 \\ \gamma'\hat{\mathbf{1}}\Psi^2 + \gamma'\beta'\sigma_I\Psi^1 \end{bmatrix}. \tag{221}$$

In the hyperbolic formulation, the details of the Lorentz transformation of Ψ gives

$$\Psi^L = \begin{bmatrix} (\Psi^1)^L \\ (\Psi^2)^L \end{bmatrix} = \begin{bmatrix} cosh(\frac{\psi}{2})\hat{\mathbf{1}} & sinh(\frac{\psi}{2})\sigma_I \\ sinh(\frac{\psi}{2})\sigma_I & cosh(\frac{\psi}{2}\hat{\mathbf{1}} \end{bmatrix} \begin{bmatrix} \Psi^1 \\ \Psi^2 \end{bmatrix} = \begin{bmatrix} cosh(\frac{\psi}{2})\hat{\mathbf{1}}\Psi^1 + sinh(\frac{\psi}{2})\sigma_I\Psi^2 \\ cosh(\frac{\psi}{2})\hat{\mathbf{1}}\Psi^2 + sinh(\frac{\psi}{2})\sigma_I\Psi^1 \end{bmatrix}. \tag{222}$$

What we see here is that the Lorentz transformation of the Dirac spinor mixes the two twin Pauli spinors Ψ^1 and Ψ^2. As a consequence, one cannot Lorentz transform a single Pauli spinor in the Dirac representation, so a Lorentz transformation of the Pauli equation without the full Dirac twin is impossible. The Pauli equation on its own cannot possibly be relativistic. So where the Pauli equation describes an electron in either spin up or spin down situation, its Dirac twin does the same with the positron in either spin up or spin down. This means that in the Dirac representation giving an electron a relativistic boost necessarily involves the positron. Giving an electron a boost can be done by letting it absorb a photon, thus realizing a quantum jump. So the quantum jump of the electron necessarily involves its antiparticle, the positron. As a consequence, in the Schrödinger and the Pauli environment quantum jumps must remain a mystery. In other words, it is a useless waist of time to try to fully understand and analyze the intrinsic aspects of quantum jumps in the Schrödinger and the Pauli theories. This can only be achieved on the Dirac level, by including both Ψ^1 and Ψ^2 (and A_ν, as for example in the form of a Feynman vertex).

The Lorentz transformation of the Dirac representation momentum four vector goes as

$$\not{P}_d^L = S\Lambda^{-1}S^{-1}\not{P}_d S\Lambda S^{-1}. \tag{223}$$

In this transformation, $\not{P}_d = P_\mu\beta^\mu$. Because the operators only work on the matrix aspect of β^μ the Lorentz transformation can also be written as

$$\not{P}^L = S\Lambda^{-1}S^{-1}P_\mu\beta^\mu S\Lambda S^{-1} = P_\mu S\Lambda^{-1}S^{-1}\beta^\mu S\Lambda S^{-1} \tag{224}$$

and we can focus on

$$(\beta^\mu)^L = S\Lambda^{-1}S^{-1}\beta^\mu S\Lambda S^{-1} \tag{225}$$

thus interpreting the Lorentz transformation as a boost of the metric. As with the Lorentz transformation of the spinors, where we expressed the operator combinations $S\Lambda^{-1}S^{-1}$ and $S\Lambda S^{-1}$ in terms of the rapidity and the hyperbolic trigonometric expressions, we can calculate the result on the bèta matrices of the $S\Lambda^{-1}S^{-1}$ and $S\Lambda S^{-1}$ operators. After some calculations this results in

$$(\beta^\mu)^L = S\Lambda^{-1}S^{-1}\beta^\mu S\Lambda S^{-1} = \Lambda_\mu{}^\nu \beta^\mu = \beta^\nu \tag{226}$$

with, given the usual Lorentz boost $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$,

$$(\beta^\mu)^L = \begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \beta_3\end{bmatrix}^L = \Lambda_\mu^\nu \beta^\mu = \begin{bmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \beta_3\end{bmatrix} = \begin{bmatrix}\gamma\beta_0 - \beta\gamma\beta_1\\ \gamma\beta_1 - \beta\gamma\beta_0\\ \beta_2\\ \beta_3\end{bmatrix} = \beta^\nu.$$

The Lorentz transformation of $\not{P}$ can then be given as

$$\not{P}^L = P_\mu S\Lambda^{-1}S^{-1}\beta^\mu S\Lambda S^{-1} = P_\nu(\Lambda_\mu^\nu \beta^\mu) = P_\nu \beta^\nu. \tag{227}$$

This result allows us to return to the original interpretation of the Lorentz transformation as a change of the coordinates against the background of a fixed metric, because

$$\not{P}^L = P_\nu(\Lambda_\mu^\nu \beta^\mu) = (P_\nu \Lambda_\mu^\nu)\beta^\mu = P_\mu \beta^\mu. \tag{228}$$

In the space-time Weyl representation the results were the same, giving

$$(\beta^\mu)^L = \Lambda^{-1}\beta^\mu \Lambda = \Lambda_\mu{}^\nu \beta^\mu = \beta^\nu \tag{229}$$

For the spinors however, it is impossible to switch between the classical Lorentz transformation with $\Lambda_\mu{}^\nu$ and the matrix Lorentz transformation with $S\Lambda^{-1}S^{-1}$ and $S\Lambda S^{-1}$ because $\Lambda_\mu{}^\nu$ cannot be split in two halves.

D. The Klein Gordon equation in its full potential

Given the general Lagrangian density $\mathscr{L} = \frac{1}{m_0}\Psi^\dagger(\hat{\not{P}}\hat{\not{P}} - \not{E}\not{E})\Psi$ in the space-time Dirac representation one gets the Klein Gordon equation from

$$\frac{\partial \mathscr{L}}{\partial \Psi^\dagger} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial^\mu \Psi^\dagger)}\right) = 0 \tag{230}$$

resulting in

$$\frac{1}{m_0}(\hat{\not{P}}\hat{\not{P}} - \not{E}\not{E})\Psi = 0. \tag{231}$$

In the canonical version we get

$$\frac{1}{m_0}(\hat{\not{P}}+q\not{A})(\hat{\not{P}}+q\not{A})\Psi = \frac{1}{m_0}\not{E}\not{E}\Psi. \tag{232}$$

leading to

$$\frac{1}{m_0}\hat{\not{P}}\hat{\not{P}}\Psi + \frac{q}{m_0}\hat{\not{P}}\not{A}\Psi + \frac{q}{m_0}\not{A}\hat{\not{P}}\Psi + \frac{q^2}{m_0}\not{A}\not{A}\Psi = -U_0\not{1}\Psi \tag{233}$$

and using $\hat{\not{P}} = -\mathbf{i}\hbar\not{\partial}$ we get

$$-\frac{\hbar^2}{m_0}\not{\partial}\not{\partial}\Psi - \frac{\mathbf{i}q\hbar}{m_0}\not{\partial}\not{A}\Psi - \frac{\mathbf{i}q\hbar}{m_0}\not{A}\not{\partial}\Psi + \frac{q^2}{m_0}\not{A}\not{A}\Psi = -U_0\not{1}\Psi \tag{234}$$

and, including a multiplication by a factor $\frac{1}{2}$,

$$-\frac{\hbar^2}{2m_0}\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_t^2\right)\not{1}\Psi - \frac{\mathbf{i}q\hbar}{2m_0}\left(\not{\partial}\not{A}\Psi + \not{A}\not{\partial}\Psi\right) + \frac{q^2}{2m_0}\left(\boldsymbol{A}^2 - \frac{1}{c^2}\phi^2\right)\not{1}\Psi = -\frac{1}{2}U_0\not{1}\Psi. \tag{235}$$

The $\left(\not{\partial}\not{A}\Psi + \not{A}\not{\partial}\Psi\right)$ part of the equation needs detailed examining. Using the chain rule for the derivation, we get

$$\left(\not{\partial}\not{A}\Psi + \not{A}\not{\partial}\Psi\right) = \left((\not{\partial}\not{A})\Psi + \vec{\not{\partial}}\not{A}\Psi + \not{A}\not{\partial}\Psi\right) \tag{236}$$

in which the arrow in $\vec{\not{\partial}}$ means that the derivation skips $\not{A}$ and only applies to Ψ. Due to the non-commutative character of the math, this is the best way to encode the chain rule.

The term $\not{\partial}\not{A}$ produces the electromagnetic field leading to

$$\not{\partial}\not{A} = \begin{bmatrix} -\frac{1}{c}\partial_t\hat{\mathbf{T}} & \boldsymbol{\nabla}\cdot\mathbf{K} \\ -\boldsymbol{\nabla}\cdot\mathbf{K} & \frac{1}{c}\partial_t\hat{\mathbf{T}} \end{bmatrix} \begin{bmatrix} \frac{1}{c}\phi\hat{\mathbf{T}} & \boldsymbol{A}\cdot\mathbf{K} \\ -\boldsymbol{A}\cdot\mathbf{K} & -\frac{1}{c}\phi\hat{\mathbf{T}} \end{bmatrix} = \tag{237}$$

$$\begin{bmatrix} (\frac{1}{c^2}\partial_t\phi + \boldsymbol{\nabla}\cdot\boldsymbol{A})\hat{\mathbf{1}} - (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\mathbf{K} & -\frac{1}{c}\partial_t\boldsymbol{A}\cdot\hat{\mathbf{T}}\mathbf{K} - \frac{1}{c}\boldsymbol{\nabla}\phi\cdot\hat{\mathbf{T}}\mathbf{K} \\ -\frac{1}{c}\partial_t\boldsymbol{A}\cdot\hat{\mathbf{T}}\mathbf{K} - \frac{1}{c}\boldsymbol{\nabla}\phi\cdot\hat{\mathbf{T}}\mathbf{K} & (\frac{1}{c^2}\partial_t\phi + \boldsymbol{\nabla}\cdot\boldsymbol{A})\hat{\mathbf{1}} - (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\mathbf{K} \end{bmatrix} = \tag{238}$$

$$\begin{bmatrix} -\boldsymbol{B}\cdot\mathbf{K} & \frac{1}{c}\boldsymbol{E}\cdot\hat{\mathbf{T}}\mathbf{K} \\ \frac{1}{c}\boldsymbol{E}\cdot\hat{\mathbf{T}}\mathbf{K} & -\boldsymbol{B}\cdot\mathbf{K} \end{bmatrix} = -\boldsymbol{B}\cdot\begin{bmatrix} \mathbf{i}\boldsymbol{\sigma} & 0 \\ 0 & \mathbf{i}\boldsymbol{\sigma} \end{bmatrix} + \frac{1}{c}\boldsymbol{E}\cdot\begin{bmatrix} 0 & -\boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} = -\mathbf{i}\boldsymbol{B}\cdot\boldsymbol{\Sigma} - \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\alpha}. \tag{239}$$

The terms $\vec{\not{\partial}}\not{A}\Psi + \not{A}\not{\partial}\Psi$ leads to a cancellation of all the parts that are anti-commutative and a doubling of the commutative parts. In the above, that would mean cancellation of the EM field and retaining the Lorenz gauge part. It result in the survival of the norm $\hat{\mathbf{1}}$ parts, so to

$$\not{A}\not{\partial} = \begin{bmatrix} \frac{1}{c}\phi\hat{\mathbf{T}} & \boldsymbol{A}\cdot\mathbf{K} \\ -\boldsymbol{A}\cdot\mathbf{K} & -\frac{1}{c}\phi\hat{\mathbf{T}} \end{bmatrix} \begin{bmatrix} -\frac{1}{c}\partial_t\hat{\mathbf{T}} & \boldsymbol{\nabla}\cdot\mathbf{K} \\ -\boldsymbol{\nabla}\cdot\mathbf{K} & \frac{1}{c}\partial_t\hat{\mathbf{T}} \end{bmatrix} = \tag{240}$$

$$\begin{bmatrix} (\frac{1}{c^2}\phi\partial_t + \boldsymbol{A}\cdot\boldsymbol{\nabla})\hat{\mathbf{1}} & 0 \\ 0 & (\frac{1}{c^2}\phi\partial_t + \boldsymbol{A}\cdot\boldsymbol{\nabla})\hat{\mathbf{1}} \end{bmatrix} = \tag{241}$$

and so

$$\vec{\partial}\!\!\!/ A\!\!\!/ \Psi + A\!\!\!/ \partial\!\!\!/ \Psi = 2\frac{1}{c^2}\phi \mathbb{1}\partial_t \Psi + 2\mathbb{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi. \tag{242}$$

The result of the closer analysis is that we have

$$-\frac{\mathbf{i}q\hbar}{2m_0}\left(\partial\!\!\!/ A\!\!\!/ \Psi + A\!\!\!/ \partial\!\!\!/ \Psi\right) = -\frac{\mathbf{i}q\hbar}{2m_0}\left(-\mathbf{i}\boldsymbol{B}\cdot\boldsymbol{\Sigma}\Psi - \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\alpha}\Psi + 2\frac{1}{c^2}\phi\mathbb{1}\partial_t\Psi + 2\mathbb{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi\right) = \tag{243}$$

$$-\frac{q\hbar}{2m_0}\boldsymbol{B}\cdot\boldsymbol{\Sigma}\Psi + \frac{\mathbf{i}q\hbar}{2m_0 c}\boldsymbol{E}\cdot\boldsymbol{\alpha}\Psi - \frac{\mathbf{i}q\hbar}{m_0 c^2}\phi\mathbb{1}\partial_t\Psi - \frac{\mathbf{i}q\hbar}{m_0}\mathbb{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi = \tag{244}$$

$$-\boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi + \mathbf{i}\boldsymbol{E}\cdot\boldsymbol{\pi}_s\Psi - \frac{\mathbf{i}q\hbar}{m_0 c^2}\phi\mathbb{1}\partial_t\Psi - \frac{\mathbf{i}q\hbar}{m_0}\mathbb{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi \tag{245}$$

The complete Klein Gordon equation then results in

$$-\frac{\hbar^2}{2m_0}\mathbb{1}\boldsymbol{\nabla}^2\Psi + \frac{\hbar^2}{2m_0c^2}\mathbb{1}\partial_t^2\Psi - \frac{\mathbf{i}q\hbar}{m_0c^2}\phi\mathbb{1}\partial_t\Psi - \frac{\mathbf{i}q\hbar}{m_0}\mathbb{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi \tag{246}$$

$$+\frac{q^2}{2m_0}\boldsymbol{A}^2\mathbb{1}\Psi - \frac{q^2\phi^2}{2m_0c^2}\mathbb{1}\Psi = -\frac{1}{2}U_0\mathbb{1}\Psi + \boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi - \mathbf{i}\boldsymbol{E}\cdot\boldsymbol{\pi}_s\Psi. \tag{247}$$

This can be rearranged into

$$-\frac{\hbar^2}{2m_0}\mathbb{1}\boldsymbol{\nabla}^2\Psi + \frac{q^2}{2m_0}\boldsymbol{A}^2\mathbb{1}\Psi - \frac{\mathbf{i}q\hbar}{m_0}\mathbb{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi - \boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi = \tag{248}$$

$$\frac{q^2\phi^2}{2m_0c^2}\mathbb{1}\Psi + \frac{\mathbf{i}q\hbar}{m_0c^2}\phi\mathbb{1}\partial_t\Psi - \frac{\hbar^2}{2m_0c^2}\mathbb{1}\partial_t^2\Psi - \frac{1}{2}U_0\mathbb{1}\Psi - \mathbf{i}\boldsymbol{E}\cdot\boldsymbol{\pi}_s\Psi. \tag{249}$$

This wave equation has a probability tensor for which the closed system condition is met, one that includes the Dirac current continuity equation. It has a linear in time derivative damping term, it has a quadratic in time derivative harmonic term and it has a Hooke's law term. In the above arrangement one has the familiar terms on the left and the terms that are ignored, misrepresented or that have never been derived on the right. In case of a stationary state, we get

$$\frac{\mathbf{i}q\hbar}{m_0c^2}\phi\mathbb{1}\partial_t\Psi = -\frac{Uq\phi}{m_0c^2}\mathbb{1}\Psi \simeq -q\phi\mathbb{1}\Psi = -V\mathbb{1}\Psi \tag{250}$$

Using this we can now rearrange the equation into

$$-\frac{\hbar^2}{2m_0}\mathbb{1}\boldsymbol{\nabla}^2\Psi + \frac{q^2}{2m_0}\boldsymbol{A}^2\mathbb{1}\Psi - \frac{\mathbf{i}q\hbar}{m_0}\mathbb{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi - \boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi + V\mathbb{1}\Psi = \tag{251}$$

$$\frac{q^2\phi^2}{2m_0c^2}\mathbb{1}\Psi - \frac{\hbar^2}{2m_0c^2}\mathbb{1}\partial_t^2\Psi - \frac{1}{2}U_0\mathbb{1}\Psi - \mathbf{i}\boldsymbol{E}\cdot\boldsymbol{\pi}_s\Psi. \tag{252}$$

In the classic interpretation, all the terms on the right hand side are reduced to $E\Psi$, giving

$$-\frac{\hbar^2}{2m_0}\mathbb{1}\boldsymbol{\nabla}^2\Psi + \frac{q^2}{2m_0}\boldsymbol{A}^2\mathbb{1}\Psi - \frac{\mathbf{i}q\hbar}{m_0}\mathbb{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi - \boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi + V\mathbb{1}\Psi = E\Psi. \tag{253}$$

The left hand side is then dubbed the Hamiltonian of the system and that abbreviates the equation to $\hat{H}\Psi = E\Psi$.

For stationary states it is also possible to reduce another term as

$$-\frac{\mathbf{i}q\hbar}{m_0}\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi = -\frac{q\boldsymbol{p}}{m_0}\cdot\boldsymbol{A}\not{1}\Psi \simeq -q\boldsymbol{v}\cdot\boldsymbol{A}\not{1}\Psi = -\boldsymbol{J}\cdot\boldsymbol{A}\not{1}\Psi \tag{254}$$

In a stationary magnetic field for which $\boldsymbol{A} = -\frac{1}{2}\boldsymbol{r}\times\boldsymbol{B}$, this term can also be rewritten as

$$-\frac{\mathbf{i}q\hbar}{m_0}\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi = -\frac{q}{2m_0}\not{1}\boldsymbol{L}\cdot\boldsymbol{B}\Psi, \tag{255}$$

see (70, Schwalb, 2007, p. 144).

The two first order derivative terms can be combined into

$$-\frac{\mathbf{i}q\hbar}{m_0c^2}\phi\not{1}\partial_t\Psi - \frac{\mathbf{i}q\hbar}{m_0}\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi = q\phi\not{1}\Psi - \boldsymbol{J}\cdot\boldsymbol{A}\not{1}\Psi = J_\mu A^\mu\not{1}\Psi. \tag{256}$$

These terms are the particle field interaction terms. Together with $\Psi^\dagger$ we get an interaction probability term as

$$\Psi^\dagger J_\mu A^\mu\not{1}\Psi. \tag{257}$$

Together with the $\boldsymbol{B}$ and $\boldsymbol{E}$ terms we have the charge-EM-field interaction terms

$$-\frac{\mathbf{i}q\hbar}{2m_0}\left(\not{\partial}\not{A}\Psi + \not{A}\not{\partial}\Psi\right) = -\boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi + \mathbf{i}\boldsymbol{E}\cdot\boldsymbol{\pi}_s\Psi + q\phi\not{1}\Psi - \boldsymbol{J}\cdot\boldsymbol{A}\not{1}\Psi \tag{258}$$

$$= -\boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi + \mathbf{i}\boldsymbol{E}\cdot\boldsymbol{\pi}_s\Psi + J_\mu A^\mu\not{1}\Psi \tag{259}$$

We have the two terms

$$-\frac{\mathbf{i}q\hbar}{m_0}\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi - \boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi = -\frac{q}{2m_0}\not{1}\boldsymbol{L}\cdot\boldsymbol{B}\Psi - \frac{q}{m_0}\boldsymbol{B}\cdot\boldsymbol{S}\Psi = \tag{260}$$

$$-\not{1}\boldsymbol{\mu}_L\cdot\boldsymbol{B}\Psi - \boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi = -\boldsymbol{B}\cdot(\boldsymbol{\mu}_L\not{1} + \boldsymbol{\mu}_s)\Psi \tag{261}$$

with orbital magnetic momentum $\boldsymbol{\mu}_L$ and spin magnetic momentum $\boldsymbol{\mu}_s$, so with total magnetic momentum $\boldsymbol{\mu}_J = \boldsymbol{\mu}_L\not{1} + \boldsymbol{\mu}_s$. These are the known terms. But parallel to these we have

$$q\phi\not{1}\Psi + \mathbf{i}\boldsymbol{E}\cdot\boldsymbol{\pi}_s\Psi \tag{262}$$

as integral part of the relativistic $\left(\not{\partial}\not{A}\Psi + \not{A}\not{\partial}\Psi\right)$. In the Hydrogen atom, the first term determines the main quantum number n, and the second term should be that radius plus or minus the reduced Compton wavelength. As a two valued zitter variation on the main quantum number. If a constant external electric field is applied, these two terms should be observable as the linear Stark effect.

In the same line of reasoning, the diamagnetic term containing $\boldsymbol{A}^2$ should have its quadratic Stark effect term containing $q^2\phi^2$ as its relativistic companion.

Interestingly, the main quantum number term $V\Psi = q\phi\not{1}\Psi$ was derived from the original damping term $-\frac{\mathbf{i}q\hbar}{m_0c^2}\phi\not{1}\partial_t\Psi$. Quantum jumps might then be connected to $-\Psi^\dagger\frac{\mathbf{i}q\hbar}{m_0c^2}\phi\not{1}\partial_t\Psi$, when interpreted in a Feynman manner. A quantum jump as a relativistic boost of spinors and enclosed four vectors should be analyzed using the complete Klein Gordon Lagrangian at the Dirac level, as $\mathscr{L} = \frac{1}{m_0}\Psi^\dagger(\hat{\not{P}}\hat{\not{P}} - \not{E}\not{E})\Psi$. Such a quantum jump of an electron, even inside the Hydrogen atom, should include the positron at some level. A quantum jump should always be fast enough to allow virtual positrons to participate in the process of emitting or absorbing a photon. It is my opinion that a fusion of relativistic QFT and the usual Schrödinger-Pauli analysis of atomic physics should be realized in order to get a grip on the internal dynamics of quantum jumps. On the Schrödinger-Pauli level of two by two spin matrices and two valued spinors, the intrinsics of quantum jumps will remain a mystery.

Quantum jumps in the Hydrogen atom should be analyzed intrinsically on a relativistic quantum field level using the Lagrangian $\mathscr{L} = \frac{1}{m_0}\Psi^\dagger(\hat{\not{P}}\hat{\not{P}} - \not{E}\not{E})\Psi$ and the related inertial probability/field tensor ${\Phi_\mu}^\nu = \Psi^\dagger\gamma_\mu\gamma^\nu\Psi$ with inertial probability/field closed system condition

$$\partial_\nu{\Phi_\mu}^\nu = \partial_\nu\Psi^\dagger\gamma_\mu\gamma^\nu\Psi = 0. \tag{263}$$

Of course, a quantum jump implies an open system, due to its photon exchange and its inevitable momentary virtual positron appearance and disappearance, a consideration that should temper expectation. A system with a primary electron that includes the photon that is being emitted or absorbed during a time interval in which a positron appears on the scene as well might again be considered closed.

What should be avoided at all times in the fermion domain is to reduce the Klein Gordon equation as derived in this section to a Pauli level equation or a scalar equation on the Schrödinger level. On the Pauli level, the spinors cannot be properly boosted, only stationary states are allowed and the intrinsics of the quantum jumps will be lost.

E. Closing in on gravity

As for the Lorentz transformation of the usual Lagrangian current density element $\mathscr{L} = \overline{\Psi}\not{P}\Psi = \Psi^\dagger\gamma_0\not{P}\Psi$, this is part of the general Lagrangian density element $\mathscr{L} = \frac{1}{m_0}\Psi^\dagger\not{P}\not{P}\Psi$, the Lorentz invariance of which I have demonstrated. Given the Lorentz invariance of the general $\mathscr{L}$, the Lorentz covariance of the closed system condition for the general Lagrangian density $\partial_\nu \mathscr{L} = 0$ is obvious, because ∂_ν is a four-vector in all reference systems. This includes the Lorentz covariance of the continuity equation for the Dirac current as $m_0 c \partial_\nu \overline{\Psi} \gamma^\nu \Psi = 0$, as its time-like part.

The gamma tensor $\gamma_\mu \gamma^\nu$ is given by

$$\gamma_\mu \gamma^\nu = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \begin{bmatrix} \gamma_0\gamma_0 & \gamma_1\gamma_0 & \gamma_2\gamma_0 & \gamma_3\gamma_0 \\ \gamma_0\gamma_1 & \gamma_1\gamma_1 & \gamma_2\gamma_1 & \gamma_3\gamma_1 \\ \gamma_0\gamma_2 & \gamma_1\gamma_2 & \gamma_2\gamma_2 & \gamma_3\gamma_2 \\ \gamma_0\gamma_3 & \gamma_1\gamma_3 & \gamma_2\gamma_3 & \gamma_3\gamma_3 \end{bmatrix} = \begin{bmatrix} \not{1} & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & -\not{1} & -\mathbf{i}\Sigma_3 & \mathbf{i}\Sigma_2 \\ \alpha_2 & \mathbf{i}\Sigma_3 & -\not{1} & -\mathbf{i}\Sigma_1 \\ \alpha_3 & -\mathbf{i}\Sigma_2 & \mathbf{i}\Sigma_1 & -\not{1} \end{bmatrix} \tag{264}$$

The probability density tensor is given by

$$\Phi_\mu{}^\nu = \Psi^\dagger \gamma_\mu \gamma^\nu \Psi = \begin{bmatrix} \Psi^\dagger \not{1} \Psi & -\Psi^\dagger \alpha_1 \Psi & -\Psi^\dagger \alpha_2 \Psi & -\Psi^\dagger \alpha_3 \Psi \\ \Psi^\dagger \alpha_1 \Psi & -\Psi^\dagger \not{1} \Psi & -\Psi^\dagger \mathbf{i}\Sigma_3 \Psi & \Psi^\dagger \mathbf{i}\Sigma_2 \Psi \\ \Psi^\dagger \alpha_2 \Psi & \Psi^\dagger \mathbf{i}\Sigma_3 \Psi & -\Psi^\dagger \not{1} \Psi & -\Psi^\dagger \mathbf{i}\Sigma_1 \Psi \\ \Psi^\dagger \alpha_3 \Psi & -\Psi^\dagger \mathbf{i}\Sigma_2 \Psi & \Psi^\dagger \mathbf{i}\Sigma_1 \Psi & -\Psi^\dagger \not{1} \Psi \end{bmatrix}. \tag{265}$$

In the space-time bèta matrices representation, we have $\beta_\mu = \mathbf{i}\gamma_\mu$, so $\beta_\mu \beta^\nu = \mathbf{i}\gamma_\mu \mathbf{i}\gamma^\nu = -\gamma_\mu \gamma^\nu$. So in the space-time representation, we have

$$\Phi_\mu{}^\nu = \Psi^\dagger \beta_\mu \beta^\nu \Psi = -\Psi^\dagger \gamma_\mu \gamma^\nu \Psi \tag{266}$$

For the proper velocity, we know that $\not{U}\not{U} = -c^2 \not{1}$. Using $\not{U} = U_\mu \beta^\mu$ we can write this as

$$\not{U}\not{U} = U_\mu \beta^\mu U_\nu \beta^\nu = U_\mu U^\nu \beta_\mu \beta^\nu = -U_\mu U^\nu \gamma_\mu \gamma^\nu = -c^2 \not{1} \tag{267}$$

So we have

$$\Psi^\dagger \not{U}\not{U} \Psi = -U_\mu U^\nu \Psi^\dagger \gamma_\mu \gamma^\nu \Psi = -U_\mu U^\nu \Phi_\mu{}^\nu = -c^2 \Psi^\dagger \Psi \tag{268}$$

The Dirac current can be arrived at by using the coordinate velocity's rest system coordinates as V^ν to get

$$J^\nu = \Phi_\mu{}^\nu V^\mu = \begin{bmatrix} \Psi^\dagger \not{1} \Psi & -\Psi^\dagger \alpha_1 \Psi & -\Psi^\dagger \alpha_2 \Psi & -\Psi^\dagger \alpha_3 \Psi \\ \Psi^\dagger \alpha_1 \Psi & -\Psi^\dagger \not{1} \Psi & -\Psi^\dagger \mathbf{i}\Sigma_3 \Psi & \Psi^\dagger \mathbf{i}\Sigma_2 \Psi \\ \Psi^\dagger \alpha_2 \Psi & \Psi^\dagger \mathbf{i}\Sigma_3 \Psi & -\Psi^\dagger \not{1} \Psi & -\Psi^\dagger \mathbf{i}\Sigma_1 \Psi \\ \Psi^\dagger \alpha_3 \Psi & -\Psi^\dagger \mathbf{i}\Sigma_2 \Psi & \Psi^\dagger \mathbf{i}\Sigma_1 \Psi & -\Psi^\dagger \not{1} \Psi \end{bmatrix} \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c\Psi^\dagger \not{1} \Psi \\ c\Psi^\dagger \alpha_1 \Psi \\ c\Psi^\dagger \alpha_2 \Psi \\ c\Psi^\dagger \alpha_3 \Psi \end{bmatrix}. \tag{269}$$

The generalized Lagrangian probability density element $\mathscr{L} = \frac{1}{m_0}\Psi^\dagger \not{P}\not{P}\Psi$ can be written as

$$\mathscr{L} = \frac{1}{m_0}\Psi^\dagger \not{P}\not{P}\Psi = \Psi^\dagger \not{U}\not{P}\Psi = m_0\Psi^\dagger \not{U}\not{U}\Psi = -m_0c^2\Psi^\dagger\Psi \tag{270}$$

but then we also have the stress-energy probability density Lagrangian product

$$\mathscr{L} = \frac{1}{m_i}\Psi^\dagger \not{P}\not{P}\Psi = \Psi^\dagger \not{V}\not{P}\Psi = -\frac{1}{\gamma}m_0c^2\Psi^\dagger\Psi = \Psi^\dagger L\Psi = V_\mu P^\nu {\Phi_\mu}^\nu = {T_\mu}^\nu {\Phi_\mu}^\nu. \tag{271}$$

Because I already proved the Lorentz invariance of this Lagrangian density, the Lorentz invariance of ${\Phi_\mu}^\nu$ is now proven too, and thus also the Lorentz covariance of the closed system condition. We also get

$$\frac{\partial\mathscr{L}}{\partial(P_\mu)} = V_\nu {\Phi_\mu}^\nu, \tag{272}$$

$$\frac{\partial\mathscr{L}}{\partial(V_\nu {\Phi_\mu}^\nu)} = P_\mu \tag{273}$$

and

$$\frac{\partial\mathscr{L}}{\partial{\Phi_\mu}^\nu} = {T_\mu}^\nu. \tag{274}$$

For a system with external forces applied, the last equation also leads to

$$\partial_\nu\left(\frac{\partial\mathscr{L}}{\partial{\Phi_\mu}^\nu}\right) = \partial_\nu {T_\mu}^\nu = F_\mu. \tag{275}$$

And for closed systems we get

$$\partial_\nu\left(\frac{\partial\mathscr{L}}{\partial{\Phi_\mu}^\nu}\right) = \partial_\nu {T_\mu}^\nu = 0. \tag{276}$$

We can reverse the order for closed systems and get

$$\partial_\nu\left(\frac{\partial\mathscr{L}}{\partial{T_\mu}^\nu}\right) = \partial_\nu {\Phi_\mu}^\nu = 0. \tag{277}$$

The created environment, including the closed system condition for the above product, closes in on General Relativity's concepts and basic elements, as the symmetric stress-energy tensor density and its closed system condition is.

The Klein Gordon probability equation in my space-time bèta matrices environment is

$$\Psi^\dagger(\not{P}\not{P} - \not{E}\not{E})\Psi = \Psi^\dagger(\not{P} - \not{E})(\not{P} + \not{E})\Psi = 0 \tag{278}$$

and can be split into two Dirac equations as

$$\Psi^\dagger(\not{P} - \not{E}) = 0 \tag{279}$$

$$(\not{P} + \not{E})\Psi = 0 \tag{280}$$

These two equations have the same solutions, but, in the Weyl representation the roles of the twin spinors are reversed. In terms of the Dirac fields, the role of particle and anti-particle are reversed. If one then goes from the Weyl representation to the Dirac representation using the S operator, the particle and anti-particle fields will be mixed in both Dirac spinor twins. In terms of the space-time basis, the S operator adds a time-reversal to one half of the dual space-time basis. The lower version is the standard Dirac equation. Its Lagrangian then is given as

$$\mathscr{L} = \Psi^{\dagger}\gamma_0(\not{P}+\not{E})\Psi = \overline{\Psi}(\not{P}+\not{E})\Psi \tag{281}$$

with the Dirac adjoint. This Lagrangian is like the primary hub of the Standard Model. By going backwards in this section, this primary hub can be generalized into a Lagrangian that closes in on gravity. In the process, the Dirac current is generalized into a probability/field density tensor and the Dirac current continuity condition is encapsulated in the closed system condition for this tensor.

F. Regarding Dirac's vision

In the canonical Klein Gordon equation we had the term $\not{\partial}\not{A}$. It introduces the electromagnetic field and leads to a $\frac{-\mathbf{i}q\hbar}{m_c}(\not{\partial}\not{A})\Psi$ term in the wave equation, which can be turned into a Lagrangian spin-energy energy probability density $\frac{-\mathbf{i}q\hbar}{m_c}\Psi^{\dagger}(\not{\partial}\not{A})\Psi$. For $\not{\partial}\not{A}$ we found, using the Lorenz gauge,

$$\not{\partial}\not{A} = -\boldsymbol{B}\cdot\mathbf{i}\boldsymbol{\Sigma} - \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\alpha}. \tag{282}$$

The product $\not{\partial}\not{A}$ can also be written as $\partial_\mu A^\nu \beta_\mu \beta^\nu = -\partial_\mu A^\nu \gamma_\mu \gamma^\nu$ so we have

$$\not{\partial}\not{A} = \partial_\mu A^\nu \beta_\mu \beta^\nu = -\partial_\mu A^\nu \begin{bmatrix} \not{1} & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & -\not{1} & -\mathbf{i}\Sigma_3 & \mathbf{i}\Sigma_2 \\ \alpha_2 & \mathbf{i}\Sigma_3 & -\not{1} & -\mathbf{i}\Sigma_1 \\ \alpha_3 & -\mathbf{i}\Sigma_2 & \mathbf{i}\Sigma_1 & -\not{1} \end{bmatrix} = -\boldsymbol{B}\cdot\mathbf{i}\boldsymbol{\Sigma} - \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\alpha} \tag{283}$$

and with the probability/field tensor we get

$$\frac{-\mathbf{i}q\hbar}{m_c}\Psi^{\dagger}\not{\partial}\not{A}\Psi = \frac{-\mathbf{i}q\hbar}{m_c}\partial_\mu A^\nu \Phi_\mu{}^\nu = -\frac{q\hbar}{m_0}\Psi^{\dagger}\boldsymbol{B}\cdot\boldsymbol{\Sigma}\Psi + \frac{q\hbar}{m_0 c}\Psi^{\dagger}\frac{1}{c}\boldsymbol{E}\cdot\mathbf{i}\boldsymbol{\alpha}\Psi \tag{284}$$

so what is called EM-field spin interaction term can also be framed as an interaction between the EM field $\partial_\mu A^\nu$ and the inertial probability/field density tensor $\Phi_\mu{}^\nu$, an interaction also determined by the quanta of charge, rest mass and action. So lets connect the charge quantum to $q\partial_\mu A^\nu$ and the rest mass quantum to $m_0\Phi_\mu{}^\nu$ and the interaction to $\hbar$. The influence of the complex number is to switch between space-time and norm-space representations. Then spin can be interpreted as being an intrinsic element of the Lorentz invariant rest mass probability/field tensor as

$$m_\mu{}^\nu = m_0\Phi_\mu{}^\nu. \tag{285}$$

If we multiply the rest mass tensor with the proper velocity tensor as $U_\mu U^\nu$ we get the general Lagrangian energy density as

$$\mathscr{L} = U_\mu U^\nu m_\mu{}^\nu = m_0 U_\mu U^\nu \Phi_\mu{}^\nu = \Psi^\dagger \not{U} \not{P} \Psi. \tag{286}$$

The closed system condition for the probability/field tensor then can also be interpreted as a closed system condition for rest mass as

$$\partial_\nu m_\mu{}^\nu = m_0 \partial_\nu \Phi_\mu{}^\nu = 0. \tag{287}$$

The Dirac rest-mass current continuity equation is the time-like part of this closed system condition and the rest-mass Dirac current is an intrinsic part of this rest-mass tensor. This probability/field tensor has a space-like part, a spin like part and a norm-like part, which for the EM field is zero due to the Lorenz gauge. It's derivative has time-like, norm-like, space-like parts and spin-like parts, following the Maxwell-Lorentz equations structure. It is my opinion that this probability/field tensor is, for the moment, the closest on can get to a metric tensor in Quantum Mechanics. In this interpretation, spin is an interaction between an elementary particle's rest-mass and this quantum metric probability/field tensor. Or rest-mass might be the result of an interaction between the electron as something elementary and this metric tensor. To interpret $\Phi_\mu{}^\nu$ as the RQM vacuum tensor is in the spirit of Dirac's vision of spin as being part of the vacuum state.

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