Null-cone integral formulation of QED

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Abstract

In these preliminary notes it is shown that the positive(negative) energy solutions of the Dirac equation also solve a specific integral equation over the past(future) null cone. It is shown that this integral equation yields the same scattering amplitudes as in the Feynman propagator picture, except for an intrinsic energy cutoff for emitted photons at $\omega_{max} = m_e$ due to the positive (negative) energy constraint imposed by the past(future) cone geometry. Fermionic self-energy is therefore finite and calculable.

We will employ the conventions used by Feynman *viz.*

$$k^2 \equiv k^\nu k_\nu, \not{A} = \gamma^\mu A_\mu, \not{x} = \gamma^\mu x_\mu, \mathbf{I} = \gamma^0\gamma^0$$

We will denote an infinitesimal element of the past ($t = -r$) null cone by

$$d\Lambda_+ \equiv \delta_+(t^2 - r^2)d^4x = \frac{d^3\mathbf{r}}{t}$$

It can be shown that, for all $k^2 \geq 0$, $k_0 \geq 0$

$$\int_0^\infty e^{-ik_\nu x^\nu} d\Lambda_+ = \frac{1}{k^2} \tag{1}$$

and

$$\int_0^\infty i\not{x} e^{ik_\nu x^\nu} d\Lambda_+ = \frac{\not{k}}{k^4} \tag{2}$$

(2) and (3) taken together imply that positive energy solutions $\Psi_+(x)$ of the Dirac equation for an electron in the presence of an electromagnetic field satisfy:

$$\int_0^\infty [\mathbf{I} + i\not{x}(m + e\not{A})]\Psi_+(x)d\Lambda_+ = 0 \tag{3}$$

First order elastic scattering from an initial momentum state $|p_1\rangle$ by a Coulomb field $A(Q)$ into a new state at a rate Ω is described by

$$\int_0^\infty \big[[\mathbf{I} + i\not{x}(m + e\not{A}e^{-iQ\cdot x})]\,|p_1\rangle + [\mathbf{I} + i\not{x}m][1 - e^{-i\Omega t}]\,|p_2\rangle\big]d\Lambda_+ = 0 \tag{4}$$

Using $[m - \not{p}]\,|p\rangle = 0$ yields:

$$\int_0^\infty \big[i\not{x}e\not{A}e^{-iQ\cdot x}\,|p_1\rangle + [\mathbf{I} + i\not{x}m]e^{-i\Omega t}\,|p_2\rangle\,\big]d\Lambda_+ = 0 \tag{5}$$

and then

$$em^{-3}\gamma_0 \not{A} e^{-iQ\cdot x} |p_1\rangle + m^{-4}[m^2 - m(m+\Omega)] |p_2\rangle = 0 \tag{6}$$

which yields the expected result

$$\Omega = e \langle p_2| \gamma_0 \not{A} |p_1\rangle \tag{7}$$

2nd order processes such as Compton scattering and Bremsstrahlung involve an intermediate off-shell amplitude which we will call Ψ. The first interaction is described by:

$$\int_0^\infty \left[i\not{t} e \not{A}_1 e^{-iq_1\cdot x} |p_1\rangle + [\mathbf{I} + i\not{t} m]\Psi \right] d\Lambda_+ = 0 \tag{8}$$

$$\implies e(\not{p}_1 + \not{q}_1)^{-1} \not{A}_1 |p_1\rangle + [\mathbf{I} - m(\not{p}_1 + \not{q}_1)^{-1}]\Psi \tag{9}$$

$$\implies \Psi = e(\not{p}_1 + \not{q}_1 - m)^{-1} \not{A}_1 |p_1\rangle \tag{10}$$

The 2nd interaction is described by

$$\int_0^\infty ie\not{t} \not{A}_2 e^{-iq_2\cdot x} \Psi + [\mathbf{I} + i\not{t} m][1 - e^{-i\Omega t}] |p_2\rangle] d\Lambda_+ = 0 \tag{11}$$

$$\implies e\not{p}_2^{-3} \not{A}_1 \Psi - m^{-3}\Omega |p_2\rangle \tag{12}$$

$$\implies \Omega = e^2 \langle p_2| \not{A}_2 (\not{p}_1 + \not{q}_1 - m)^{-1} \not{A}_1 |p_1\rangle \tag{13}$$

Allowing for permutation of the two interactions, the result is actually

$$\implies \Omega = e^2 \langle p_2| \left[\not{A}_2 (\not{p}_1 + \not{q}_1 - m)^{-1} \not{A}_1 + \not{A}_1 (\not{p}_1 - \not{q}_2 - m)^{-1} \not{A}_2 \right] |p_1\rangle \tag{14}$$

..which is the expected result. Higher order processes will also yield the same results as the conventional Feynman procedure. However, all integrations over particle momenta must be truncated because Λ_+ only includes $k_0 > 0$ components. This is of particular relevance to self-energy diagrams, where the cutoff will be m. We believe this finding may be of profound significance to the various hierarchy problems that beset the SM.