Generally covariant quantum theory.

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Abstract

This paper represents a concise introduction to the quantum theory of point particles in a time orientable curved spacetime part of which was presented in the DICE conference in Castiglioncello, Italy.

1 Introduction.

There is only one quantum theory on Minkowski and that is the one presented by Weinberg proceeding upon work started by Wigner and Von Neumann. It is axiomatic, starts from a clear definition of a particle and constrains the dynamics as such that the notion of a field becomes useful. Weinberg wrote his summary after all the Evil happened and the beast was babtised "quantum field theory" instead of relativistic particle dynamics. Often, it is useful to attribute the correct name to something as it must reflect its deepest inner workings such as man, woman and hermaphrodite although it is kind of embarrasing to see the last one as a convex combination of the previous extremal cases (it is much more sexy than both of them).

The idea of this paper is to give two distinct proper introductions to RQT (relativistic quantum theory), a Weinbergian one - which we will end with and was not presented on the conference - and a divine one, starting from the most simple of considerations, having nothing to do a priori with probability theory and Hilbert bundles (instead of spaces). Both approaches provide one with a different view on classical and quantum mechanics; they are geometrical and entirely devoid of a coordinatised language as well as symplectic approaches due to globally hyperbolic foliations.

I thank the organizers of the conference for the opportunity to present my viewpoint on the matter -as well as other things- and will return in due time with further elaborations on this work.

2 Foundational arguments.

In this section, I see nature as a communist reflects upon society; the foundational quantity of everything is contained in an action signifying "work" or "rabota". That is, consider $\phi(\gamma(s), p(\gamma(s))) \in \mathbf{B}$ where $\gamma : [a, b] \to \mathcal{M}$

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is a curve joining an event x to an event y in spacetime $\mathcal{M}$ in affine parametrization with respect to a Lorentzian or Riemannian metric where, moreover, p is a field on that line associated with the physical quantity of "momentum". We do not really know yet what momentum is but it represents a kind of weight or importance given to that motion, p must not be proportional to $\dot{\gamma}$ as weight might sometimes be disfavourable to the current motion. Given that rednecks love calculus, $\mathbf{B}$ is a division algebra over the real numbers with standard operations $+, .$, that is $\mathbb{R}, \mathbb{C}$ or $\mathbb{Q}$ disregarding the non-associative octonions.

A frictionless theory is a dreamworld as no waste is produced. Mathematically, this translates as follows: there exists an involution $\dagger$ and operation $\star$ such that $\phi(\gamma(b-s), p(\gamma(b-s))) \star \phi(\gamma(s), p(\gamma(s)) = 1_\star$ and $\phi(\gamma(b-s), p(\gamma(b-s))) = \phi(\gamma(s), p(\gamma(s)))^\dagger$. It is worthwile to comment upon these; the first one means that that reversing the process is arithmetically equivalent to taking the inverse whereas the second one says that the inverse has a metrical significance. This last stance is useful as inverting two processes must preserve the distance between them. No discussion about this is allowed for.

As a consequence, the constant curve $\gamma_e(s) = x = y$ satisfies

$$\phi(\gamma_e(s), p(\gamma_e(s)))^2 = 1_\star$$

which for $\star = +$ and $\mathbf{B} = \mathbb{R}$ gives $\phi(\gamma(b-s), p(\gamma(b-s)) = -\phi(\gamma(s), p(\gamma(s))$ and $\phi(\gamma_e(s), p(\gamma_e(s))) = 0$. These simple observations give rise to the notion of work *and* classical physics. For, $\mathbf{B} = \mathbb{C}$, we have that

$$\phi(\gamma(b-s), p(\gamma(b-s)) = \overline{\phi(\gamma(s), p(\gamma(s))}$$

and $|\phi(\gamma(s), p(\gamma(s))|^2 = 1$ what leads to the $U(1)$ Fourier waves in quantum theory.

2.1 The classical theory.

The idea is to write down a first order differential equation for the quantity of labour. Reparametrization invariance forces $\frac{d}{ds}\phi(\gamma(s), p(\gamma(s))$ to be proportional to $\frac{d}{ds}\gamma(s)$. Furthermore, the reversion property implies

$$\frac{d}{ds}\phi(\gamma(s), p(\gamma(s)) = \frac{d}{ds}\gamma(s) \cdot \mathbf{F}(\gamma(s), p(\gamma(s)))$$

which is the old Newtonian expression with $\mathbf{F}$ having the meaning of force. Note that this is an expression local in spacetime as the change of work does not depend upon its previous history. Theories with radiative effects do *not* satisfy this expression but nevertheless a more complicated one. To complete the dynamics, Newton supposed that $p(\gamma(s))$ must maximally stimulate the direction in which the particle is moving and that, therefore

$$p(\gamma(s)) = m\dot{\gamma}(s)$$

where $m > 0$ and expresses the weight attached to persistance of motion, called *physical mass*. Another observation was of an Einsteinian nature, namely that the change of work should reflect the change in an inherent physical property of the particle and not depend upon external forces at all. This would mean that, in a way, a particle is free infinitesimally; the lowest order, in the derivatives of the worldline, such invariant is given by the momentum squared

$$h(p(\gamma(s)), p(\gamma(s)))$$

which suggests something like

$$\frac{d}{ds}\left(\frac{m}{2}\left(\frac{d}{ds}\gamma(s)\right)^2\right) = \frac{d}{ds}\phi(\gamma(s), \frac{d}{ds}\gamma(s))$$

and bestowes $\phi(\gamma(s), \frac{d}{ds}\gamma(s))$ with the dimension of mass which it should be given that the notion of force must be associated to something intrinsic which is change of momentum

$$\mathbf{F}(\gamma(s), p(\gamma(s))) := \frac{d}{ds}p(\gamma(s)).$$

This is the simpelest idea possible, given that the kinetic term is the lowest order invariant and m can be thought of as some material based constant. This leads to

$$\frac{m}{2}\left(\frac{d}{ds}\gamma(b)\right)^2 - \frac{m}{2}\left(\frac{d}{ds}\gamma(a)\right)^2 = \phi(\gamma(b), p(\gamma(b))) - \phi(\gamma(a), p(\gamma(a)))$$

and in a way generalizes a concerved quantity given that ϕ depends upon the entire path and not just the endpoints in general.

One could make higher derivative theories also in this way and allow for Newtonian laws with third order derivatives. These naturally appear in the context of backreactions in electromagnetism for example and allow for "unphysical" solutions with causality going backwards in time. For example, an electron would accelerate prior to turning on a lightbulb. Note also that the interpretation of γ as the physical path of the particle natually emerges given that Newtons law fixes it entirely given two "initial data".

2.2 Quantum theory.

Now, we derive quantum theory, as well as the probability interpretation, in the same vein. One notices that the *obvious*, but not only, candidate for an equation of motion is given by

$$\hbar\frac{d}{ds}\phi(\gamma(s), p(\gamma(s))) = ig(p(\gamma(s)), \dot{\gamma}(s))\phi(\gamma(s), p(\gamma(s)))$$

where p is the energy momentum vector and $\dot{\gamma}(s)$ dimensionless. Notice that $\hbar$ is needed for dimensional reasons to get a nontrivial theory given that ϕ must be dimensionless as any physical quantity is a real and not unitary number. On flat spacetimes $\phi(\gamma(s), p(\gamma(s)))$ is topological as it just depends upon the homotopy class or winding number. That is,

$$\phi(\gamma(b), p(\gamma(b))) = e^{ip.(y-x)}$$

which is the standard Fourier wave in y with base point, or origin x. Given that $e^{ip.(y-x)}$ provides for a trivial unitary mapping between $e^{ip.(z-y)}$ and $e^{ip.(z-x)}$, the waves are identical up to a momentum dependent constant multiplicative $U(1)$ factor. In traditional RQT, this is precisely the impact of the translation symmetry in Minkowski QFT. To arrive at the QFT propagator for a quantum field on Minkowski, we notice that the total "propagator"

$$D(x,y) = \alpha \int_{\mathbb{R}^4} d^4p\theta(p^0)\delta(g(p,p) - m^2)\phi(\gamma(b), p)$$

is the expression we are looking after. Indeed, in $\phi(x, w, p)$, as well w as p are uncertain, which is kind of logical given that the momentum is *not* necessarily the maximal forward one but is dragged over the curve as to indicate the initial direction of preference. In order to recuperate the classical bi-functional way of thinking, we have to integrate over p on mass-shell. It utters nothing but the Heisenberg uncertainty principle that if the positions x, y are known sharply, then the momentum is totally uncertain apart from the fact that it needs to be forwards pointing in time and have mass energy mc^2. Actually, what this integration says is that all preferences are taken into account democratically unless some higher intelligence, due to a spiritual interaction, desires differently. This is actually a classical Bayesian way of thinking except that the weights or probabilities are here given by complex numbers. In this vein, quantum theory is one of reality w and desire p which do not need to coincide as happens in the classical case; the theory may get more psychological than this by having more complex "momenta" and Einsteinian constraints as happens in (an appropriate version of) string theory. x is here interpreted as the point of birth of a particle and y as the point of dissapearance, death or annihilation. Actually, this is all there is to free QFT on Minkowski; we dispose of no Hamiltonian operator but the Wightman function, from which the Feynman propagator can be uniquely defined, has the standard singularity structure which makes implementation of interactions difficult.

To arrive at the full expression in curved spacetime, we notice that

$$D(x,y) \in \mathbb{R}$$

if x is spacelike to y written by $x \sim y$ and given that $D(x,y) = \overline{D(y,x)}$ in general we arrive at the conclusion that for spacelike separated events, the creation and annihilation processes at x and y can be swapped without altering the "propagator". This is the expression of Bose-Einstein statistics, a desirable property in the general theory. In a general curved spacetime, ϕ depends upon curve and not just the homotopy class due to the existence of local gravitational degrees of freedom. In light of Bose statistics, only geodesics count given that the scalar product is preserved. In the light of constructing regularized propagators with a smooth structure, consider the "Schroedinger" equation,

$$\frac{d}{ds}\phi(x, y = \exp_x(w); w, k, s) =$$

$$(ig(w(s), k(s)) - \kappa\sqrt{h(w(s), w(s))} - \frac{2}{L^2\sigma^3(x, \exp_x(ws))}\sigma_{\alpha'}(x, \exp_x(ws))w^{\alpha'}(s))\phi(x, y = \exp_x(w); w, k, s)$$

where $s \in [0,1]$ and $w, k \in T_x\mathcal{M}$, h is a Riemannian metric delivering an energy μ and L is a huge mass smoothening out the lightcone. Finally,

$$\frac{D}{ds}k(s) = \frac{D}{ds}w(s) = 0.$$

Notice that violation of unitarity occurs by means of κ, L: they represent irreducible *imaginary* friction terms, meaning that every process has a "cost" which cannot be undone -no perfect "Carnot cycle". It endows spacetime with a kind of effective granularity in the metrics determined by h and σ^2. The solution to the equation reads

$$\phi(x, y = \exp_x(w); w, k) := \phi(x, y = \exp_x(w); w, k, 1) = e^{ik.w} e^{-\kappa \int_0^1 \sqrt{h(w(s),w(s))} ds - \frac{1}{L^2 \sigma^2(x, \exp_x(w))}}$$

whivh produces a generalized wave given by

$$\phi(x, y; k) := \sum_{w \in T_x \mathcal{M} : \exp_x(w) = y} \phi(x, y = \exp_x(w); w, k).$$

This calls for the following definition of the propagator

$$D(x, y) = \alpha \sum_{w \in T_x \mathcal{M} : \exp_x(w) = y} \int_{\mathbb{R}^4} d^4 p \theta(p^0)$$

$$e^{-\mu h(p(0),p(0)) - \mu h(p(1),p(1)) - \mu(1 - \theta(g(w,w)))(h(R_{w(0)} p(0), R_{w(0)} p(0)) - h(R_{-w(1)} p(1), R_{-w(1)} p(1)))}$$

$$\delta(g(p, p) - m^2) \phi(x, y; w, p)$$

where the μ terms express that the creation and annihilation processes come at a cost.

Under reasonable conditions, this regularized propagator is smooth everywhere and has exponential falloff behavior towards infinity. The Feynman propagator gets the following universal prescription:

$$\Delta_{F,\mu,\kappa,L}(x, y) = \sum_{w : \exp_x(w) = y \text{ and w is in the future lightcone of x}} D_{\mu,\kappa,L}(x, w) + \sum_{w' : \exp_y(w') = x \text{ and w' is in the future lightcone of y}} D_{\mu,\kappa,L}(y, w') + \sum_{w : \exp_x(w) = y \text{ and w is spacelike at x}} D_{\mu,\kappa,L}(x, w)$$

meaning that all "information" has to travel towards the future which constitutes clearly the right function to study interactions with.

This theory has been worked out in a book [1] published on Amazon. It turns out that all Feynman diagrams are finite in all known interaction theories for particles of spin less than $\frac{5}{2}$. Moreover, they are suitably bounded and show "exponential falloff behavior" even on Riemannian spaces with negative sectional curvature when friction κ is large enough. All theories need a modification of the standard Dyson expression for diagrams with a large number of internal vertices in order to procure analytical results and make the whole power series well defined. The interested reader is referred to that book for an entire elaboration of this theory.

3 Haute Weinbergian cuisine.

Whereas this previous section procured extremely deep connections between different branches of physics from an elementary point of view, this section is somewhat more traditional but no less profound. It is just so that in the end, the same formalism is recovered in all known cases but a different looking avenue is opened up. The latter might be completely isomorphic to the previous one however.

3.1 Classical physics revisited.

Consider a particle moving in a bundle $\mathcal{E}$ over a Lorentzian spacetime $(\mathcal{M}, g)$ where the fibres are equipped with a metric field and the associated connection preserves the total metric (which is usually a product metric). Regard the wordline as an immersion $\gamma : \mathbb{R} \to \mathcal{E}$ and the momentum as its the push forward of ∂_t with equals

$$\frac{D}{dt} := \nabla_{\frac{d}{dt}\gamma(t)}$$

where ∇ is the bundle connection. Given that we shall only work with functions $f : \mathcal{E} \to \mathbb{R}$, the latter expression can be taken for $(\partial_t)_\star$ as an ordinary vectorfield instead of a general derivative operator. To every curve γ and function f we can attach a function $\gamma_f : \mathbb{R} \to \mathbb{R} : t \to f(\gamma(t))$. We can now define a $C^\infty(\mathbb{R})$ algebra of operators $\mathbb{L}$ on the function space $f : \mathcal{E} \to \mathbb{R}$ mappping them to functions from $\mathbb{R}$ to $\mathbb{R}$. Concretely

$$[(\gamma_f)(g)](t) := f(\gamma(t))g(\gamma(t))$$

and

$$[p_\gamma f](t) := \frac{d}{dt} f(\gamma(t)).$$

We have moreover,

$$\gamma_f(g + h) = \gamma_f(g) + \gamma_f(h)$$

and

$$[(\partial_t)(\gamma_f g)](t) := [(\partial_t)_\star f](t)g(\gamma(t)) + f(\gamma(t))[(\partial_t)_\star(g)](t).$$

This suggests to extend the definition of the momentum in this way to functions $\mathbb{R} \to \mathbb{R}$. The same comment holds for γ_f. In this vein,

$$[\gamma_g \gamma_f h](t) = g(\gamma(t))f(\gamma(t))h(\gamma(t))$$

and

$$[p_\gamma \gamma_f h](t) := \partial_t(f(\gamma(t))h(\gamma(t)))$$

as well as

$$[\gamma_f p_\gamma h](t) := f(\gamma(t))\partial_t h(\gamma(t)).$$

Finally,

$$[p_\gamma p_\gamma h](t) = (\partial_t)^2 h(\gamma(t))$$

which induces a real algebra generated by

$$\gamma_g, p_\gamma$$

where γ varies over all immersions. This algebra is represented by means of linear operators on the function algebra

$$\mathcal{B} := C^\infty(\mathbb{R}) \otimes C^\infty(\mathcal{E})$$

which may be given the structure of an Hilbert algebra in the usual L^2 sense by introducing an einbein on the "time line" $\mathbb{R}$. Concretely

$$[\gamma_f, \gamma_h](g) = 0 = [p_\gamma, p_\gamma](g), [p_\gamma, \gamma_f](g) = p_\gamma(f)\gamma_\star(g) = \gamma_{p_\gamma(f)}(g)$$

where $\gamma_\star$ is the pull back defined by the immersion γ. Here, the commutation relations employ the full $\mathcal{B}$ action but are understood to apply on

$f, g, h \in C^\infty(\mathcal{E})$ and result in an element of $C^\infty(\mathbb{R})$.

Covariant dynamics requires dynamics without potential energy terms; therefore, any force has to be implemented in the momentum what explains the bundle $\mathcal{E}$. Moreover, according to Einstein himself, every force, including the gravitational one, can be gauged away in some point so that locally and physically every particle is a free one meaning that the correct equation is the geodesic bundle equation. Therefore, the classical Hamiltonian is a constraint and moreover, commuting it with a vector leaves a covector if it were an invariant energy so that

$$[\mathcal{H}(\gamma_f, p_\gamma), p_\gamma]$$

cannot represent $\frac{D}{dt}p_\gamma$ unless we would make an extra metric contraction. Actually, the whole Hamiltonian edifice is kind of meaningless as we shall see now. Indeed, taking $\mathcal{H}(\gamma_f, p_\gamma)$ to be p_γ with equations of motion given by

$$[\frac{D}{dt}\triangle\gamma_f](g) := [p_\gamma, \gamma_f](g) = \gamma_{p_\gamma(f)}(g)$$

and

$$[\frac{D}{dt}(\partial_t)_\star](g) = [\frac{D}{dt}\triangle p_\gamma](dg) := [p_\gamma, p_\gamma](dg) = [p_\gamma, p_\gamma](g) = 0$$

where

$$[\frac{D}{dt}\triangle\zeta](g) = [\frac{D}{dt}, \zeta](g).$$

There is nothing more to say really apart from the constraint $g(p_\gamma, p_\gamma) = \frac{m^2c^4}{\hbar^2}$ which is the mass energy relation. This is all what is allowed in classical physics of point particles really and we now proceed to quantum theory. Notice that the dynamical content is completely implied by the commutator algebra which constitutes a total unision between dynamics and kinematics. Physically, this is entirely trivial and completely justified given that the momentum just corresponds to the energy in a rest frame. Note also the presence of $\hbar$ in the latter formula which is there for dimensional reasons; alas, it does not do anything else apart from setting a time scale given that the covariant derivative does not depend upon it.

3.2 Relativistic Quantum Theory.

As we have shown in the previous section, the Poisson Bracket really is a commutator and the Hamiltonian formulation is rather void given that the total free momentum, constrained by the quantum mechanical mass formula is the only real quantity of interest. Unlike in classical physics, quantum mechanics cannot use an external time in a sense given that a particle is not specified anymore by a worldline but by a wave. In a way, it is the complex dual of the classical situation where "worldlines" correspond to functions $\psi : \mathcal{E} \to \mathbb{C}$ which are C^∞. The operators γ_f and p_γ are replaced then by x_f and $i\nabla_V$ where V is a real vectorfield over $\mathcal{E}$ and f is a real valued function over $\mathcal{E}$. Here, $[x_f](g)(x) = f(x)g(x)$ and

$$P(V)(g) := i\nabla_V(g) = iV(g).$$

They obey the algebra

$$[x_f, x_h] = 0, [i\nabla_V, i\nabla_W] = -R(V, W)(\cdot) - \nabla_{[V,W]}$$

and finally

$$[i\nabla_V, x_f] = x_{iV(f)}.$$

The momentum commutation relations have been put in this exotic form because the covariant derivative can work on vectorfields and higher objects too. The i is just there to ensure that the momentum operator is real given that the commutator of two real operators is imaginary. The situation here is very different as one cannot just pick a Hamiltonian linear in the momenta given that one would as thus preselect a nondynamical arrow of time. Hence our only choice is given by

$$H = \sum_{i,j=1}^{n} \eta^{ij} \nabla_{E_i} \nabla_{E_j}$$

where the E_i correspond to loical vielbeins and η^{ij} is the inverse of the standard flat metric. In order for this to work ∇ must be extended to the spin connection to digest local boost transformations. Furthermore, one has

$$H = m^2$$

as constraint. It is clear one has no Heisenberg type dynamics here as the vectorfields really are spacetime vectorfields; hence, the entire theory is encapsulated by the constraint and the geometry of the bundle $\mathcal{E}$. It has been shown by Ashtekar and Magnon that this theory only works out fine in stationary spacetimes with Minkowski as the prime example due to the existence of scalar products on leafs of a foliation for which the latter is preserved in "time".

In 2011, I wrote a book about an operational approach to quantum theory with local vacua delineating a Fock-Hilbert bundle $\otimes_{x\in\mathcal{M}}\mathcal{H}_x$ over the space-time manifold $\mathcal{M}$. However, the approach was troublesome and muddled with two "fundamental errors" of mine, not due to a lack of mathematical precision, but being the consequence of a poor understanding of what curved spacetime really signifies. This error found a natural solution in [1] written on generally covariant quantum theory from the point of view of the Feynman series.

Concretely, we assumed $\mathcal{H}_x$ to be constructed by means of a cyclic quasi-free vacuum state $|0\rangle_x$ and multiparticle states showing Bose or Fermi statistics constructed in the Fock way. The dynamical object was a unitary bi-field $U(x, y)$ mapping $\mathcal{H}_y \to \mathcal{H}_x$ and obeying a Schroedinger like differential equation

$$\frac{d}{dt}U(t, s) = iHU(t, s)$$

but then with the times t, s replaced by x, y. The two errors in the book originated from the mathematical implementation of this idea I conceived; first of all $U(t, s) = U(t)U^{\dagger}(s)$ and moreover the only covariant first order differential operator *homogeneous* in the spacetime coordinates is given by the covariant Dirac operator D. The first condition is equivalent to a "cohomology" condition

$$U(x, y)U(y, z)U(z, x) = 1$$

which turns out to hold in Minkowski or any maximally symmetric spacetime only and reflects the absence of local gravitational degrees of freedom. Consequently, the only solution I was able to find of my field equations was free quantum field theory on Minkowski in a way I shall explain later. The Dirac operator gives all sorts of trouble meaning we have to replace the complex numbers by an appropriate Clifford algebra of signature $(1, 3)$ or $(3, 1)$. This gives rise to negative probabilities and huge problems with the spectral theorem even for finite dimensional Clifford bi-modules. The approach was clearly dead as it stood which I realised later on.

3.3 Taking bi-fields seriously.

As pointed out in [1], the idea of a Hilbert bundle *is* adequate, but the correct differential equation for $U(y, x)$ needs to run over geodesics connecting x with y in a fully reparametrization invariant way. The obvious candidate is given by

$$\frac{d}{ds}U(\gamma(s), x) = i\dot{\gamma}(s)^a P_a U(\gamma(s), x)$$

where $\gamma(s)$ is the unique geodesic connecting x with y and P_a equals the free momentum generator, given by the expression

$$P_a = \sum_{\text{particles j, internal degrees } \sigma_j} \int \frac{d^3k}{\sqrt{k_0}} k_a a^{\dagger}_{k;j,\sigma_j} a_{k,j,\sigma_j}$$

at the point z with respect to the dragged vierbein in x along the geodesic. The coincidence limit is fixed by $U(x, x) = 1$; this suggests one to enlarge the notation to $U(y, x; e_a(y), e_b(x))$ as well as a unitary action $T(\Lambda, e_b(z))$ of the orthochronous Lorentz group $\Lambda \in O^+(1, 3)$ on $U(z, x; {e'}_b(z), e_a(x))$ by means of conjugation $TUT^{\dagger}$. All this has been explained in [2]; in order for $T(\Lambda(s), e_a(\gamma(x))$ to shift through $\frac{d}{ds}$ we need a Lorentz covariant derivative and, henceforth, an antihermitian connection $L_{\mu}(z, e_b(z))$ such that

$$\left(\frac{d}{ds} + \dot{\gamma}^{\mu}(s)L_{\mu}(\gamma(s), \Lambda^b_a(s)e_b(\gamma(s)))\right) T(\Lambda(s), e_b(\gamma(s)))U(\gamma(s), x; e_b(\gamma(s)), e_a(x))T^{\dagger}(\Lambda(s), e_b(\gamma(s)))$$

$$-T(\Lambda(s), e_b(\gamma(s)))U(\gamma(s), x; e_b(\gamma(s)), e_a(x))T^{\dagger}(\Lambda(s), e_b(\gamma(s)))\dot{\gamma}^{\mu}(s)L_{\mu}(\gamma(s), \Lambda^b_a(s)e_b(\gamma(s))) =$$

$$T(\Lambda(s), e_b(\gamma(s))) \left[\left(\frac{d}{ds} + \dot{\gamma}^{\mu}(s)L_{\mu}(\gamma(s), e_b(\gamma(s)))\right) U(\gamma(s), x; e_b(\gamma(s)), e_a(x))\right] T^{\dagger}(\Lambda, e_b(\gamma(s)))$$

$$-T(\Lambda, e_b(\gamma(s)))U(\gamma(s), x; e_b(\gamma(s)), e_a(x))\dot{\gamma}^{\mu}(s)L_{\mu}(\gamma(s), e_b(\gamma(s)))T^{\dagger}(\Lambda(s), e_b(\gamma(s))).$$

In case we dispose of multiple geodesics connecting x with y, we just multiply the corresponding unitary operators in the *same* vierbein at y, the order of which does not matter given that all P_a commute and because the action of the Lorentz group acts by boosting the momenta. Therefore, we can just sum up the momenta which can accomodate for topology change of Minkowski into a flat spacelike cylinder giving rise to the correct field picture.

There is however a small caveat here in case multiple geodesics connect x and y in the sense that the gauge field might acquire a nontrivial significance due to multivaluedness of $e_a(y)$, where the latter is the dragged vielbein from x to y. Hence, it is better to replace the argument y by a tangent vector V in $T\mathcal{M}_x$ and take the x perspective where $\exp_x(v) = y$. In that case, we set $L_{\mu}(sv, \Lambda^b_a(s)e_b(\exp_x(sv)))$ to zero in case

$$\frac{D}{ds}e_b(\exp_x(sv)) = 0$$

for $s = 0 \dots 1$. In other words, the vielbein in the warped point in that direction must be the dragged one; this makes both formalisms entirely equivalent what the free theory is concerned. Notice that by construction, $U(y,x) = U^\dagger(x,y)$ due to the minus sign caused by flipping $\dot{\gamma}^\mu(s)$. Given that the connection $L_\mu(v, e_b(\exp_x(v)))$ is a new object defined on

$$T\mathcal{M}_x(v) \times V\mathcal{M}_g(\exp_x(v))$$

where $V\mathcal{M}_g(\exp_x(v))$ is the nonlinear space of g vierbeins over $\mathcal{M}$, which is equivalent to the group manifold $O^+(1,3)$ regarded as a homogeneous space with a hyperbolic Cartan metric of signature $(3,3)$; it might be opportune to make it more dynamical and invent a new type of non abelian Yang-Mills theory over $T\mathcal{M}_x$. This author tried this also in 2011 but failed to recognize the bundle perspective as well was stuck with Clifford modules for replacements of Hilbert spaces. The easiest thing is to see L_μ as $L_{\partial_{v^j} \exp_x(v)}(v, e_b(\exp_x(v)))$ where e_b varies independently and refers to $y = \exp_x(v)$ and subsequently write out a Yang-Mills equation of the kind

$$\left(D_v L_{\partial_{v^j} \exp_x(v)}(v, e_b(\exp_x(v)))\right)_{kl} = \left(\partial_{v^{[k}} - L_{[k}\right) L_{l]} = (dL)_{kl} - (L \wedge L)_{kl} = 0$$

where d is the Hodge operator on flat tangent space. Life could be more exciting as to pick out the zero solution in parallel transport gauge and we leave this new piece of physics for further examination of the bored ones.

So far, we have determined only our quantum connection; now, we develop bi-fields which are nothing but the warps of coincidence fields meaning

$$\Phi(y,x) : \left(\prod_{v \in T\mathcal{M}_x : \exp_x(v) = y} U(y = \exp_x(v), x; (\exp_x(v))_\star e_a(x), e_a(x)) \right) \Phi(x,x)$$
$$\left(\prod_{v \in T\mathcal{M}_x : \exp_x(v) = y} U(y = \exp_x(v), x; (\exp_x(v))_\star e_a(x), e_a(x)) \right)^\dagger .$$

Here,

$$\Phi(x,x) = \sum_{\text{particles j with internal quantum numbers } \sigma_j} \int_{\mathbb{R}^3} \frac{d^3k}{k_0} (v_{\sigma_j} \sqrt{k_0} a^\dagger_{k,j,\sigma_j} + \overline{v}_{\sigma_j} \sqrt{k_0} a_{k,j,\sigma_j})$$

where v_{σ_j} is an internal field vector associated to the internal particle degrees of freedom. They are needed to obtain different physical behaviour, $\frac{d^3k}{k_0}$ is the on shell relativistic measure in Fourier space on Minkowski and finally, $\sqrt{k_0} a_{k,j,\sigma_j}$ is relativistic normalization of the creation annihilation algebra. I leave it as an elementary excercise to find out principles determining v_{σ_j}. So $\Phi(x,x)$ is the proper democratic relativistic expression taking into all matter degrees of freedom in the universe.

3.4 Interaction theory.

So far, we have delineated the free theory from an operational bi-field formalism which reduces in Minkowski to a single field formalism due to the remarkable "cohomology" property

$$U(x,y)U(y,z)U(z,x) = 1$$

where we have dropped the vielbeins and assumed dragging allalong which is logical given that dragging is trivial and hence consistent along closed paths due to the vanishing of the Riemann tensor. The trick now is to work directly into an interaction picture and forget about a closed bi-field equation. That is, we write down spacetime interaction densities of the kind

$$i\lambda \int_{\mathcal{M}} \sqrt{g(y)}\Phi(y,x)\Phi(y,z)\Phi(y,p)\Phi(y,q).$$

This is an obvious excercise leading to a completely equivalent formalism as in the 2016 book.

References

[1] J. Noldus, Generally covariant quantum theory, Amazon 2018

[2] J. Noldus, Foundations of a theory of quantum gravity, Shiny world coorporation 2011 and Arxiv