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Search for New Numbers.

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CHAPTER I

I.1 Introduction.

In 1984, in the first issue for that year of the "Science and Life" magazine, I read the article "Non-Standard Analysis" by V. Uspensky, in which the notion of hyperreal numbers had been introduced. And, already somewhere in June 1984, I thought – "There was a time when they were not known about. And now learned. So maybe there are other numbers that we don't know now, and then we'll find out. And is it possible not to wait for this, but somehow to find out about these unknown now numbers by myself?".

I.2 The help of dialectics.

Dialectics says, that the subject should be studied in development. In this case, the subject is numbers. And their development can be traced back to my own experience - what numbers, and in what order I studied at school.

I.3 Numbers, studied at school.

At first we studied numbers: 1, 2, 3, ... - they are called natural. Then zero and negative numbers. Natural, zero and negative are called integers.

Then there were fractional numbers. Integers and fractional numbers are called rational numbers.

After them we studied the irrational numbers. Rational numbers together with irrational numbers are called real numbers.

They were followed by imaginary numbers. The sum of real number with imaginary number is called by complex number.

I.4 The direct operations, studied at school.

There are many operations, studied at the school. But the main ones are summation, multiplication, exponentiation. They will be called "direct operations". We introduce the following notation: [1] – summation, [2] – multiplication, [3] - exponentiation. Also, we will introduce the conventional symbols of these three "direct operations": [1] – "+", [2] – "*", [3] – "^".

I.5 The first direct operation [1].

Let us take any two natural numbers: **a**, **b**. And apply to them the first direct operation. The result is also a natural number **c**:

$$a [1] b = c; (a + b = c). \quad (I.5.1)$$

[1]	1	2	3	4	b
1	2	3	4	5	
2	3	4	5	6	
3	4	5	6	7	
4	5	6	7	8	
a					c

Table I.1

I.6 The second direct operation [2].

Let us take any two integers: **a**, **b**. And apply to them the second direct operation. The result will be also an integer **c**:

$$a [2] b = c; (a * b = c); \quad (I.6.1)$$

[2]	1	2	3	b
1	1	2	3	
2	2	4	6	
3	3	6	9	
a				c

Table I.2

I.7 The third direct operation [3].

Let us take any two natural numbers: **a**, **b**. And apply to them the third direct operation [3]. The result is also a natural number - **c**:

$$a [3] b = c; (a \wedge b = c); \quad (I.7.1)$$

[3]	1	2	3	B
1	1	1	1	
2	2	4	8	
3	3	9	27	
a				C

Table I.3

I.8 The definition of inverse operations.

Let a direct operation [n] be defined on some set of numbers:

$$a [n] b = c; \quad (I.8.1)$$

(Here a, b are numbers from this set. They are called operands.)

If in (I.8.1) we rearrange the first operand "a" with the result "c", we got:

$$c [n]1] b = a; \quad (I.8.2)$$

This is the definition of the first inverse operation [n]1] for direct operation [n].

If in (I.8.1) we rearrange the second operand "b" with the result "c", we got:

$$a [n]2] c = b; \quad (I.8.3)$$

This is the definition of the second inverse operation [n]2] for direct operation [n].

I.9 The first inverse operation [1]1] for the first direct operation [1].

Now for [1]1] we build Table I.4 from Table I.1 (using (I.8.1) and (I.8.2)) (rearranging "a" and "c" in the Table I.1) :

[1]1]	1	2	3	4	b
2	1				
3	2	1			
4	3	2	1		
5	4	3	2	1	
6		4	3	2	
7			4	3	
8				4	
c					a

Table I.4

We see from Table I.4, that [1]1] coincides with "-" (subtraction). That is:

$$a = c [1]1] b = c - b; \quad (I.9.1)$$

But not all the cells in Table I.4 are filled in yet. Let's try to find such natural numbers **c** and **b**, that **a** too was a natural number. And that's what we got:

[1]1]	1	2	3	4	b
2	1				
3	2	1			
4	3	2	1		
5	4	3	2	1	
6	5	4	3	2	
7	6	5	4	3	
8	7	6	5	4	
c					a

Table I.5

There are no numbers in the upper right corner of Table I.5 among natural numbers. These numbers: $2 - 2$, $3 - 3$, $4 - 4$, $2 - 3$, $3 - 4$, $2 - 4$. That is, operation $[1]1$ is not equal to operation $[1]$ on the set of natural numbers. To achieve equality with operation $[1]$, it is necessary to move to a wider range of numbers than natural numbers. To integers. These are natural numbers, zero, and negative natural numbers. Then any pair of integers c and b for operation $[1]1$ gives an integer a . And for operation $[1]$ any pair of integers a and b also gives an integer c . And then operations $[1]$ and $[1]1$ become equal. And empty cells in Table I.5 can be filled with new numbers: 0 , -1 , -2 . Then we get Table I.6:

[1]1	1	2	3	4	b
2	1	0	-1	-2	
3	2	1	0	-1	
4	3	2	1	0	
5	4	3	2	1	
6	5	4	3	2	
7	6	5	4	3	
8	7	6	5	4	
c					a

Table I.6

I.10 The second inverse operation $[1]2$ for the first direct operation $[1]$.

Now, for $[1]2$, let's build the Table I.7 from the Table I.1 (using (I.8.1) and (I.8.3)) (rearranging "b" and "c" in the Table I.1):

[1]2]	2	3	4	5	6	7	8	c
1	1	2	3	4				
2		1	2	3	4			
3			1	2	3	4		
4				1	2	3	4	
a								b

Table I.7

From (I.8.2), (I.8.3), Table I.4, and Table I.7, we see that

$$b = a [1]2] \quad c = c [1]1] \quad a = c - a; \quad (I.10.1)$$

To populate the Table I.7 completely, you have numbers a , c be taken from the integers. Then b will be also an integer. And operation $[1]2]$ becomes equal with operation $[1]$. And then we get Table I.8:

[1]2]	2	3	4	5	6	7	8	c
1	1	2	3	4	5	6	7	
2	0	1	2	3	4	5	6	
3	-1	0	1	2	3	4	5	
4	-2	-1	0	1	2	3	4	
a								b

Table I.8

We now have:

$$x [1] y = x + y; \quad (I.10.2)$$

$$x [1]1] y = x - y; \quad (I.10.3)$$

$$x [1]2] y = y - x; \quad (I.10.4)$$

I.11 Connection of operation [2] with operation [1].

The operation of summation is taught at school first. Then move on to the study of the second operation - multiplication. Let's look at some examples of connection of these operations.

$$3 + 3 = 3 * 2 = 6; \quad (I.11.1)$$

$$3 + 3 + 3 = 3 * 3 = 9; \quad (I.11.2)$$

$$2 + 2 + 2 + 2 = 2 * 4 = 8; \quad (I.11.3)$$

We rewrite these examples in the new notation:

$$3 [1] 3 = 3 [2] 2 = 6; \quad (I.11.4)$$

$$3 [1] 3 [1] 3 = 3 [2] 3 = 9; \quad (I.11.5)$$

$$2 [1] 2 [1] 2 [1] 2 = 2 [2] 4 = 8; \quad (I.11.6)$$

The multiplication operation is formed as follows: to the left of the multiplication operation, the summand is written, and to the right of it, the number of these summands is written.

I.12 The first inverse operation [2]1 for the second direct operation [2].

Second direct operation [2]:

$$a [2] b = c; \quad (I.12.1)$$

From (I.8.2) for $n = 2$ we obtain a definition for [2]1):

$$c [2]1) b = a; \quad (I.12.2)$$

Using Table I.2, we obtain Table I.9 for [2]1) (changing "a" and "c" in the Table I.2):

[2]1	1	2	3	b
1	1			
2	2	1		
3	3		1	
4		2		
5				
6		3	2	
7				
8				
9			3	
c				a

Table I.9

Table I.9 shows that $[2]1$ is a division “/”:

$$a = c [2]1 \quad b = c / b; \quad (I.12.3)$$

But not all the cells in Table I.9 are filled in yet. Let's look for **c** and **b** among integers so that **a** was also integer. And we get Table I.10:

[2]1	1	2	3	b
1	1			
2	2	1		
3	3		1	
4	4	2		
5	5			
6	6	3	2	
7	7			
8	8	4		
9	9		3	
c				a

Table I.10

For empty cells in Table I.10 there were no numbers among integers. That is, operation $[2]1$ is not equal to operation $[2]$ on the set of integers. Equality of these operations can be achieved by introducing new fractional numbers. Then all empty cells in Table I.10 can be filled with new fractional numbers. And we get Table I.11:

[2]1	1	2	3	b
1	1	1/2	1/3	
2	2	1	2/3	
3	3	3/2	1	
4	4	2	4/3	
5	5	5/2	5/3	
6	6	3	2	
7	7	7/2	7/3	
8	8	4	8/3	
9	9	9/2	3	
c				a

Table I.11

Integers together with fractional numbers are called rational numbers. And for any pair of rational numbers **a, b** in (I.12.1) there will be a rational number **c**. Also for any pair of rational numbers **c** and **b** in (I.12.2) there will be a rational number **a**. That is, operations $[2]$ and $[2]1$ became equal on the set of rational numbers.

I.13 The second inverse operation $[2]2$ for the second direct operation $[2]$.

Second direct operation $[2]$ of (I.8.1) at $n = 2$:

$$a [2] b = c; \tag{I.13.1}$$

From (I.8.3) for $n = 2$ we obtain a definition for $[2]2$:

$$a [2]2] c = b; \tag{I.13.2}$$

Using Table I.2, we obtain (changing “b” and “c” in the Table I.2) Table I.12 for [2]2]:

[2]2]	1	2	3	4	5	6	7	8	9	c
1	1	2	3							
2		1		2		3				
3			1			2			3	
a										b

Table I.12

This Table shows that $b = c / a;$ (I.13.3)

But not all the cells in Table I.12 are filled in yet. Let's look for c and a among integers so, that b is also integer. And we get Table I.13:

[2]2]	1	2	3	4	5	6	7	8	9	c
1	1	2	3	4	5	6	7	8	9	
2		1		2		3		4		
3			1			2			3	
a										b

Table I.13

For empty cells in Table I.13 there were no numbers among integers. That is, operation [2]2] is not equal to operation [2] on the set of integers. Equality of these operations can be achieved by introducing new fractional numbers. Then all empty cells in Table I.13 can be filled with new fractional numbers. And we get Table I.14:

[2]2]	1	2	3	4	5	6	7	8	9	c
1	1	2	3	4	5	6	7	8	9	
2	1/2	1	3/2	2	5/2	3	7/2	4	9/2	
3	1/3	2/3	1	4/3	5/3	2	7/3	8/3	3	
a										b

Table I.14

Integers together with fractional numbers are called rational numbers. And for any pair of rational numbers **a**, **b** in (I.13.1) there will be a rational number **c**. Also for any pair of rational numbers **a** and **c** in (I.13.2) there will be a rational number **b**. That is, operations [2] and [2]2 became equal on the set of rational numbers.

Given (I.12.3), (I.13.2), (I.13.3) we obtain:

$$b = a [2]2] c = c / a = c [2]1] a; \quad (I.13.4)$$

Now we have:

$$x [2] y = x * y; \quad (I.13.5)$$

$$x [2]1] y = x / y; \quad (I.13.6)$$

$$x [2]2] y = y / x; \quad (I.13.7)$$

I.14 Connection of operation [3] with operation [2].

The operation of exponentiation - [3] is studied at school after multiplication – [2]. These operations are closely related to each other. Here are some examples of this connection:

$$3 * 3 = 3 ^ 2 = 9; \quad (I.14.1)$$

$$3 * 3 * 3 = 3 ^ 3 = 27; \quad (I.14.2)$$

$$2 * 2 * 2 * 2 = 2 ^ 4 = 16; \quad (I.14.3)$$

We rewrite these examples in the new notation:

$$3 [2] 3 = 3 [3] 2 = 9; \quad (I.14.4)$$

$$3 [2] 3 [2] 3 = 3 [3] 3 = 27; \quad (I.14.5)$$

$$2 [2] 2 [2] 2 [2] 2 = 2 [3] 4 = 16; \quad (I.14.6)$$

The exponentiation operation is formed as follows: the factor is written to the left of the exponentiation operation sign, and the number of these factors is written to the right of this sign.

I.15 The first reverse operation [3]1 for the third direct operation [3].

Now we will build for [3]1 the Table I.15 from the Table I.3 (using (I.8.1) and (I.8.2)) (rearranging “a” and “c”):

[3]1	1	2	3	b
1	1	1	1	
2	2			
3	3			
4		2		
8			2	
9		3		
27			3	
c				a

Table I.15

Table I.15 shows that [3]1 coincides with the root:

$$\mathbf{a} = \text{root of degree } \mathbf{b} \text{ from } \mathbf{c}; \quad (\text{I.15.1})$$

Not all cells in Table I.15 are occupied. Let's try to occupy them with rational numbers. Then we get Table I.16:

[3]1	1	2	3	b
1	1	1	1	
2	2			
3	3			
4	4	2		
8	8		2	
9	9	3		
27	27		3	
c				a

Table I.16

Not all the cells of the Table I.16 were filled with rational numbers. That is, operation [3]1] is not equal to operation [3] on the set of rational numbers. We have to introduce new numbers here – the square root of 2, the root of the third degree of 2, the square root of 3, the root of the third degree of 3, and so on. These numbers are called irrational. They – together with rational numbers – are called real numbers. And now all the empty cells in Table I.16 are filled.

And now we will build Table I.17:

[3]1]	1	2	3	b
-1	-1		-1	
1	1	1	1	
c				a

Table I.17

One cell in the top row of Table I.17 could not be filled not only with a rational, but even with a real number. That is, operation [3]1] is still unequal to operation [3] even on the set of real numbers.

Let's introduce another number - the root of the second degree from -1. It is called "imaginary unit", and is usually denoted by the letter i:

$$-1 [3]1] 2 = i; \quad (I.15.2)$$

$$i^2 = -1 < 0; \quad (I.15.3)$$

If the imaginary unit is multiplied by the real number b, you get another new imaginary number. That is, if it is raised to the second degree, it will also be a negative number:

$$(i * b)^2 = -(b * b) < 0; \quad (I.15.4)$$

If real numbers are summed with imaginary numbers, then complex numbers will be obtained.

Now we can fill an empty cell in the top row of Table I.17 with the imaginary unit i and get Table I.18:

[3]1	1	2	3	b
-1	-1	i	-1	
1	1	1	1	
c				a

Table I.18

And now – after the introduction of irrational numbers, and imaginary numbers – operation $[3]1$ becomes equal to operation $[3]$: For any complex numbers c, b for operation $[3]1$ there is always a complex number a . And for any complex a, b for operation $[3]$ there is always a complex number c .

I.16 The second reverse operation $[3]2$ for the third direct operation $[3]$.

From (I.8.1), (I.8.3) for $n = 3$ we have definitions of $[3]$ and $[3]2$ operations:

$$a [3] b = c; \quad (I.16.1)$$

$$a [3]2 c = b; \quad (I.16.2)$$

We need to swap "b" and "c" in Table I.3.

Let's trace in detail how the number-string is born. Consider the first row in Table I.3 and leave it in Table I.18.1:

[3]	1	2	3	b
1	1	1	1	c
2				
3				
a				

Table I.18.1

Rearrange the lines "c" and "b". Get Table I.18.2:

[3]2	1	1	1	c
1	1	2	3	b
2				
3				
a				

Table I.18.2

Consider the second line in Table I. 3 and leave it in Table I. 18.3:

[3]	1	2	3	b
1				
2	2	4	8	c
3				
a				

Table I.18.3

Rearrange the lines "c" and "b". Get Table I.18.4

[3]2	2	4	8	c
1				
2	1	2	3	b
3				
a				

Table I.18.4

Consider the third row in Table I.3 and leave it in Table I.18.5:

[3]	1	2	3	b
1				
2				
3	3	9	27	c
a				

Table I.18.5

Rearrange the lines "c" and "b". Get Table I.18.6

[3]2]	3	9	27	c
1				
2				
3	1	2	3	b
a				

Table I.18.6

Let us combine Tables I.18.2, I.18.4, I.18.6 in one Table I.18.7:

[3]2]	1	1	1	2	4	8	3	9	27	c
1	1	2	3							
2				1	2	3				
3							1	2	3	
a										b

Table I.18.7

The column headings must be different, so let's combine the columns with the same headings in Table I. 18.8:

[3]2]	1	2	4	8	3	9	27	c
1	1, 2, 3							
2		1	2	3				
3					1	2	3	
a								b

Table I.18.8

Now let's arrange the columns in ascending order of their numbers. Get the Table I.19:

[3]2	1	2	3	4	8	9	27	c
1	1, 2, 3							
2		1		2	3			
3			1			2	3	
a								b

Table I.19

So we saw how the number-string is born. How to work with such numbers-strings we will describe further.

Table I.19 shows that operation [3]2 coincides with the logarithm:

$$b = \text{logarithm } (c) \text{ for base } a; \quad (I.16.3)$$

Let's try to fill in the empty cells in Table I.19 using complex numbers. The result is presented in Table I.20:

[3]2	1	2	3	4	8	9	27	c
1	1, 2, 3	?	?	?	?	?	?	
2	0	1	+	2	3	+	+	
3	0	+	1	+	+	2	3	
a								b

Table I.20

The "+" sign marks the cells, where the known numbers are located – the finite complex numbers, only the cells are too small to fit these numbers. But zero fit, and it's standing in two cells.

Those cells, where no known finite complex number can stand, are marked with sing "?". It is possible, that there is a new class of numbers.

I.17 Numbers with “?”.

We introduce into consideration the new numbers through Table I.21:

[3]2	1	2	3	4	8	9	27	c
1	1	?2	?3	?4	?8	?9	?27	
2	0	1	+	2	3	+	+	
3	0	+	1	+	+	2	3	
a								b

Table I.21

Numbers ?2,?3,?4, and so on - are defined through operation [3] as follows:

$$1 [3] ?2 = 1 \wedge ?2 = 2; \quad (I.17.1)$$

$$1 [3] ?3 = 1 \wedge ?3 = 3; \quad (I.17.2)$$

$$1 [3] ?4 = 1 \wedge ?4 = 4; \quad (I.17.3)$$

$$1 [3] ?a = 1 \wedge ?a = a; \quad (I.17.4)$$

Here **a** is any complex number. And the number ?a will be called a "question number".

The simplest solution for (I.17.4) is found in III.8:

$$(III.8.8): \quad ?a = N1 * (\ln a);$$

N1 here is infinitely large number. It is described in III.2:

$$(III.2.2): \quad N1 + 1 = N1;$$

$$(III.2.7): \quad n1 = 1 - 1 = 0;$$

(III.2.5): $N1 * n1 = 1;$

More general solution for (I.17.4) is found in III.9:

(III.9.11): $?a = N1 * (\ln a + i * 2 * PI * k);$

(III.9.12): $i * i = -1; PI = 3.14; k = \dots, -2, -1, 0, 1, 2, \dots;$

I.18 Operation [1] for the two question numbers.

Let's define the addition operation [1] for any two question numbers ?a, ?b:

$?a + ?b = ?c;$ (I.18.1)

$1 ^ { (?a + ?b) } = (1 ^ { ?a }) * (1 ^ { ?b }) = a * b = 1 ^ { ?(a * b) };$ (I.18.2)

$?a + ?b = ?(a * b);$ (I.18.3)

$?a [1] ?b = ?(a [2] b) = ?c;$ (I.18.4)

According to this formula, we build a Table I.22:

[1]	?1	?2	?3	b
?1	?1	?2	?3	
?2	?2	?4	?6	
?3	?3	?6	?9	
a				c

Table I.22

I.19 The first inverse operation [1]1 for the first direct operation [1] on the set of question numbers.

For the question numbers ?a, ?b, ?c for operation [1] in Table I.22, we construct operation [1]1 and display it in Table I.23:

[1]1]	?1	?2	?3	b
?1	?1			
?2	?2	?1		
?3	?3		?1	
?4		?2		
?6		?3	?2	
?9			?3	
c				a

Table I.23

Let's define the subtraction operation " - " [1]1] for any two question numbers ?c, ?b:

$$1 \wedge (?c - ?b) = (1 \wedge ?c) * (1 \wedge (-?b)) = c / (1 \wedge ?b) = c / b =$$

$$= 1 \wedge ?(c / b); \tag{I.19.1}$$

$$?c - ?b = ?(c/b); \tag{I.19.2}$$

$$?c [1]1] ?b = ?c - ?b = ?(c / b); \tag{I.19.3}$$

According to this formula for operation " - " [1]1], we fill in the empty cells in Table I.23 and get Table I.24:

[1]1]	?1	?2	?3	b
?1	?1	?(1/2)	?(1/3)	
?2	?2	?1	?(2/3)	
?3	?3	?(3/2)	?1	
?4	?4	?2	?(4/3)	
?6	?6	?3	?2	
?9	?9	?(9/2)	?3	
c				a

Table I.24

I.20 The second inverse operation [1]2] for the first direct operation [1] on the set of question numbers.

According to Table I.22, moving **b** and **c**, we get Table I.25:

[1]2]	?1	?2	?3	?4	?6	?9	c
?1	?1	?2	?3				
?2		?1		?2	?3		
?3			?1		?2	?3	
a							b

Table I.25

From the comparison of Table I.25 with Table I.23 (where the difference is: ?c - ?b = ?a) Table I.25 shows the difference:

$$?c - ?a = ?b; \tag{I.20.1}$$

From (I.19.2) – after replacement ?b on ?a - we get:

$$?c - ?a = ?\{c / a\}; \tag{I.20.2}$$

$$?b = ?a [1]2] ?c = ?c [1]1] ?a = ?c - ?a = ?(c / a); \tag{I.20.3}$$

And now we can fill in the Table I.25 empty cells and get the Table I.26:

[1]2]	?1	?2	?3	?4	?6	?9	?c
?1	?1	?2	?3	?4	?6	?9	
?2	?(1/2)	?1	?(3/2)	?2	?3	?(9/2)	
?3	?(1/3)	?(2/3)	?1	?(4/3)	?2	?3	
?a							?b

Table I.26

So we have:

$$?x [1] ?y = ?(x * y); \tag{I.20.4}$$

$$?x [1]1] ?y = ?(x / y); \tag{I.20.5}$$

$$?x [1]2] ?y = ?(y / x); \quad (I.20.6)$$

I.21 Operation [2] for the two question numbers.

We define the multiplication operation [2] for two any question numbers ?a, ?b:

$$?a [2] ?b = ?a * ?b = ?c; \quad (I.21.1)$$

[2]	?1	?2	?3	?b
?1	?1	?2	?3	
?2	?2 * ?1	?2 * ?2	?2 * ?3	
?3	?3 * ?1	?3 * ?2	?3 * ?3	
?a				?c

Table I.27

For the simplest solution we have:

$$?1 = N1 * (\ln 1) = N1 * n1; \quad (I.21.2)$$

What follows ?1 = 1; (I.21.3)

With this in mind, the Table I.27 becomes the Table I.28:

[2]	?1	?2	?3	?b
?1	?1	?2	?3	
?2	?2	?2 * ?2	?2 * ?3	
?3	?3	?3 * ?2	?3 * ?3	
?a				?c

Table I.28

$$?3 * ?2 = ?2 * ?3; \quad (I.21.4)$$

Apply to the Table I.28 this equation and we obtain Table I.30:

[2]	?1	?2	?3	?b
?1	?1	?2	?3	
?2	?2	?2 * ?2	?2 * ?3	
?3	?3	?2 * ?3	?3 * ?3	
?a				?c

Table I.30

I.22 The first inverse operation [2]1] for the second direct operation [2] on the set of question numbers.

(I.13.6): $x [2]1] y = x / y;$

(I.21.1): $?a [2] ?b = ?c;$

$?a = ?c [2]1] ?b = ?c / ?b;$ (I.22.1)

Let's switch places **?a** and **?c** in Table I.30 and get Table I.31:

[2]1]	?1	?2	?3	?b
?1	?1			
?2	?2	?1		
?3	?3		?1	
?2 * ?2		?2		
?2 * ?3		?3	?2	
?3 * ?3			?3	
?c				?a

Table I.31

Next, consider the three empty cells ($?b = ?1 = 1$):

$?a = ?c / ?1 = ?c / 1 = ?c;$ (I.22.2)

Table I.31 takes the form of Table I.32:

[2]1	?1	?2	?3	?b
?1	1			
?2	?2	1		
?3	?3		1	
?2 * ?2	?2 * ?2	?2		
?2 * ?3	?2 * ?3	?3	?2	
?3 * ?3	?3 * ?3		?3	
?c				?a

Table 32

(I.22.1): $?a = ?c [2]1 ?b = ?c / ?b;$

Substitute (I.22.1) in Table I.32 and get Table I.33:

[2]1	?1	?2	?3	?b
?1	1	1+	2+	
?2	?2	1	3+	
?3	?3	4+	1	
?2 * ?2	?2 * ?2	?2	5+	
?2 * ?3	?2 * ?3	?3	?2	
?3 * ?3	?3 * ?3	6+	?3	
?c				?a

Table I.33

Because of the closeness I had to enter the symbols:

$1+ = ?1 / ?2 = 1 / ?2;$ (I.22.3)

$2+ = ?1 / ?3 = 1 / ?3;$ (I.22.4)

$3+ = ?2 / ?3;$ (I.22.5)

$4+ = ?3 / ?2;$ (I.22.6)

$5+ = ?2 * ?2 / ?3;$ (I.22.7)

$6+ = ?3 * ?3 / ?2;$ (I.22.8)

I.23 The second inverse operation [2] for the second direct operation [2] on the set of question numbers.

$$(I.21.1): \quad ?a [2] ?b = ?c;$$

$$?b = ?a [2] ?c; \quad (I.23.1)$$

$$(I.13.7): \quad x [2] y = y / x;$$

$$?b = ?a [2] ?c = ?c / ?a; \quad (I.23.2)$$

Let's move **?b** and **?c** in Table I.30. Then we get Table I.34:

[2]	?1	?2	?3	?2 * ?2	?2 * ?3	?3 * ?3	?c
?1	?1	?2	?3				
?2		?1		?2	?3		
?3			?1		?2	?3	
?a							?b

Table I.34

Let's look at the three empty cells in the top row of Table I.34:

Substitute in (I.23.2):

$$?a = ?1 = 1; ?c = ?2 * ?2; \text{ and we get: } ?b = ?2 * ?2; \quad (I.23.3)$$

$$?a = ?1 = 1; ?c = ?2 * ?3; \text{ and we get: } ?b = ?2 * ?3; \quad (I.23.4)$$

$$?a = ?1 = 1; ?c = ?3 * ?3; \text{ and we get: } ?b = ?3 * ?3; \quad (I.23.5)$$

Substituting this into the Table I.34 to get a Table I.35:

[2]	?1	?2	?3	?2 * ?2	?2 * ?3	?3 * ?3	?c
?1	?1	?2	?3	?2 * ?2	?2 * ?3	?3 * ?3	
?2		?1		?2	?3		
?3			?1		?2	?3	
?a							?b

Table I.35

The remaining six empty cells in Table I.35 are filled with numbers using (I.23.2), and we get Table I.36:

[2]2]	?1	?2	?3	?2 * ?2	?2 * ?3	?3 * ?3	?c
?1	1	?2	?3	?2 * ?2	?2 * ?3	?3 * ?3	
?2	1+	1	4+	?2	?3	6+	
?3	2+	3+	1	5+	?2	?3	
?a							?b

Table I.36

Here, the numbers that do not fit in these six cells are indicated by the numbers with the plus sign to the right of the number:

(I.23.2): $?b = ?a [2]2] ?c = ?c / ?a;$

$$1+ = 1 / ?2; \quad (I.23.6)$$

$$2+ = 1 / ?3; \quad (I.23.7)$$

$$3+ = ?2 / ?3; \quad (I.23.8)$$

$$4+ = ?3 / ?2; \quad (I.23.9)$$

$$5+ = ?2 * ?2 / ?3; \quad (I.23.10)$$

$$6+ = ?3 * ?3 / ?2; \quad (I.23.11)$$

I.24 Operation [3] for the two question numbers.

$$?a [3] ?b = ?c; \quad (I.24.1)$$

По формуле (I.24.1), с учётом ?1 = 1, построим Таблицу I.38:

[3]	?1	?2	?3	?b
?1	1	2	3	
?2	?2	?2 [3] ?2	?2 [3] ?3	
?3	?3	?3 [3] ?2	?3 [3] ?3	
?a				?c

Table I.38

I.24.a First inverse operation [3]1 for the third direct operation [3] on the set of question numbers.

(I.24.1): $?a [3] ?b = ?c;$

$?a = ?c [3]1 ?b;$ (I.24.a.1)

Let us rearrange ?a and ?c in Table I.38 and obtain Table I.38.3:

[3]1	?1	?2	?3	?b
1	1			
2		1		
3			1	
?2	?2			
?3	?3			
?2 [3] ?2		?2		
?2 [3] ?3			?2	
?3 [3] ?2		?3		
?3 [3] ?3			?3	
?c				?a

Table I.38.3

Operation [3]1 is the root degree ?b of ?c. Therefore, when ?b = ?1 = 1, we have: ?a = root degree 1 of ?c = ?c.

And we can fill in the empty cells in the first column of Table I.38.3. Thus, we obtain Table I.38.4:

[3]1]	?1	?2	?3	?b
1	1			
2	2	1		
3	3		1	
?2	?2			
?3	?3			
?2 [3] ?2	?2 [3] ?2	?2		
?2 [3] ?3	?2 [3] ?3		?2	
?3 [3] ?2	?3 [3] ?2	?3		
?3 [3] ?3	?3 [3] ?3		?3	
?c				?a

Table I.38.4

Let's fill in the empty cells in Table I.38.4 by numbers with "+" and get the Table I.38.5:

[3]1]	?1	?2	?3	?b
1	1	1+	2+	
2	2	1	3+	
3	3	4+	1	
?2	?2	5+	6+	
?3	?3	7+	8+	
?2 [3] ?2	?2 [3] ?2	?2	9+	
?2 [3] ?3	?2 [3] ?3	10+	?2	
?3 [3] ?2	?3 [3] ?2	?3	11+	
?3 [3] ?3	?3 [3] ?3	12+	?3	
?c				?a

Table I.38.5

Now, let us give meaning to these numbers with a plus sign:

1+ = root degree ?2 of 1; (I. 24.a.2)

2+ = root degree ?3 of 1; (I. 24.a.3)

3+ = root degree ?3 of 2; (I. 24.a.4)

- 4+ = root degree ?2 of 3; (I. 24.a.5)
- 5+ = root degree ?2 of ?2; (I. 24.a.6)
- 6+ = root degree ?3 of ?2; (I. 24.a.7)
- 7+ = root degree ?2 of ?3; (I. 24.a.8)
- 8+ = root degree ?3 of ?3; (I. 24.a.9)
- 9+ = ?2 [3] (?2 / ?3); (I. 24.a.10)
- 10+ = ?2 [3] (?3 / ?2); (I. 24.a.11)
- 11+ = ?3 [3] (?2 / ?3); (I. 24.a.12)
- 12+ = ?3 [3] (?3 / ?2); (I. 24.a.13)

I.24.b Second inverse operation [3]2 for the third direct operation [3] on the set of question numbers.

(I.24.1): ?a [3] ?b = ?c;

?b = ?a [3]2 ?c; (I.24.b.1)

Let us rearrange ?b and “c” in Table I.38 and obtain Table I.38.6:

[3]2]	1	2	3	?2	?3	2d2	2d3	3d2	3d3	?c
?1	1	?2	?2							
?2				?1		?2	?3			
?3					?1			?2	?3	
?a										?b

Table I.38.6

Here:

2d2 = ?2 [3] ?2; 2d3 = ?2 [3] ?3; 3d2 = ?3 [3] ?2; 3d3 = ?3 [3] ?3;

(I.24.b.2)

From (I.24.1) at $?a = ?1 = 1$ it follows: $1 [3] ?b = ?c$; (I. 24.b.3)

$?b = ?(?c)$; (I. 24.b.4)

Here follows Table I.38.7:

	1	2	3	?2	?3	2d2	2d3	3d2	3d3	?c
?1	1	?2	?2	??2	??3	?2d2	?2d3	?3d2	?3d3	
?2				1		?2	?3			
?3					1			?2	?3	
?a										?b

Table I.38.7 [3]2]

Let's fill in the empty cells in Table I.38.7 by Latin letters and get the Table I.38.8:

	1	2	3	?2	?3	2d2	2d3	3d2	3d3	?c
?1	1	?2	?2	??2	??3	?2d2	?2d3	?3d2	?3d3	
?2	e	g	i	1	m	?2	?3	q	r	
?3	f	h	j	k	1	n	p	?2	?3	
?a										?b

Table I.38.8 [3]2]

Now we fill these letters with meaning. Operation [3]2] is a logarithm:

$?b = \log (?c)$ on the ground ($?a$); (I.24.b.5)

$e = \log_{(?2)} (1)$ by base ($?2$); (I.24.b.6)

$f = \log_{(?3)} (1)$ by base ($?3$); (I.24.b.7)

$g = \log (2)$ base ($?2$); (I.24.b.8)

$h = \log (2)$ base ($?3$); (I.24.b.9)

$i = \log_{(?2)} (3)$ by base ($?2$); (I.24.b.10)

$$j = \log_{(3)}(3) \text{ by base } (3); \quad (1.24.b.11)$$

$$k = \log_{(2)}(2) \text{ on the basis } (2); \quad (1.24.b.12)$$

$$m = \log_{(3)}(3) \text{ on the basis } (2); \quad (1.24.b.13)$$

$$n = \log_{(2)}(2) \text{ at the base } (3); \quad (1.24.b.14)$$

$$p = \log_{(3)}(3) \text{ by base } (3); \quad (1.24.b.15)$$

$$q = \log_{(2)}(2) \text{ by base } (2); \quad (1.24.b.16)$$

$$r = \log_{(3)}(3) \text{ by base } (2); \quad (1.24.b.17)$$

I.26 Building new direct operations.

So we traced the process of the emergence of new numbers, starting with natural to complex. And found that they arose at the introduction of reverse operations to direct operations. The introduction of subtraction gave zero and negative numbers. Introduction of division gave the fractional numbers. The introduction of the root gave irrational and imaginary numbers. Their summation gave complex numbers. This pattern was confirmed for all numbers. And if someone wants to build new numbers, you can use this pattern. But the fact is that all reverse operations have already been used. And all direct operations have already been used.

If we were given a new direct operation, we would build a pair of inverse operations for it, and then apply the found pattern to build new numbers. Let us turn to dialectics again. It says that the subject should be studied in development. In this case, the subject is direct operations. And their development can be traced back to our own experience - what direct operations, and in what order were studied at school.

At the very beginning, we have already mentioned all the currently known direct operations, and numbered them in the following order: summation – [1], multiplication – [2], exponentiation – [3]. In this order they have been studied at school. We are now interested in the rules of building new direct operations on the basis of previous direct operations.

In point I.11 the construction of multiplication operation - [2] by summation operation – [1] is considered. And at the end of this paragraph, this rule is summarized. Here it is:

"The multiplication operation is formed as follows: to the left of the multiplication operation, the summand is written, and to the right of it, the number of these summands is written."

In paragraph I.14 we consider the construction of the operation of exponentiation – [3] by operation of multiplication [2]. And at the end of this paragraph, this rule is summarized. Here it is:

"The exponentiation operation is formed as follows: the cofactor is written to the left of the exponentiation operation sign, and the number of these cofactors is written to the right of this sign."

These two definitions can be summarized as follows:

Operation [n+1] is formed by operation [n] as follows: operand for operation [n] is written to the left of [n+1], and the number of these operands is written to the right of [n+1].

I.27 Construction of the fourth direct operation - [4].

Let's build some examples to obtain [4] by [3]:

$$2 [4] 2 = 2 [3] 2 = 4; \quad (I.27.1)$$

$$3 [4] 2 = 3 [3] 3 = 27; \quad (1.27.2)$$

$$2 [4] 4 = 2 [3] 2 [3] 2 [3] 2 = X; \quad (1.27.3)$$

In the last formula there are two options – the calculation in the right-hand side of the equation (1.27.3) to make from left to right (get 256), or from right to left (get 65 536). How to choose the right option? And both options are correct. Both options should be considered. But in this work, we will limit ourselves to one option – from left to right. Therefore, the equation (1.27.3) takes the form:

$$2 [4] 4 = ((2 [3] 2) [3] 2) [3] 2 = 256; \quad (1.27.3)$$

Now the fourth direct operation [4] is determined. And we'll be filling in the Table I.39:

[4]	1	2	3	b
1				
2				
3				
a				c

Table I.39

In this Table we will put the formula:

$$a [4] b = c; \quad (1.27.5)$$

$$1 [4] 1 = 1; \quad (1.27.6)$$

To the right of [4] is 1, which means, that after the equal sign there is only one operand 1 and no operation [3].

$$2 [4] 1 = 2; \quad (1.27.7)$$

$$3 [4] 1 = 3; \quad (1.27.8)$$

$$1 [4] 2 = 1 [3] 1 = 1; \quad (1.27.9)$$

$$1 [4] 3 = (1 [3] 1) [3] 1 = 1 [3] 1 = 1; \quad (1.27.10)$$

$$2 [4] 3 = (2 [3] 2) [3] 2 = 4 [3] 2 = 16; \quad (I.27.11)$$

$$3 [4] 2 = 3 [3] 3 = 27; \quad (I.27.12)$$

$$3 [4] 3 = (3 [3] 3) [3] 3 = 27 [3] 3 = 19\ 683; \quad (I.27.13)$$

[4]	1	2	3	b
1	1	1	1	
2	2	4	16	
3	3	27	19 683	
a				c

Table I.40

So we filled in the Table I.39. And now it is called the Table I.40.

And now we show a shortened method of recording the fourth direct operation [4] through the third direct operation [3]. Take the formula (I.27.5):

$$a [4] b = c;$$

And let $b = 5$. By definition of the fourth operation, we write this formula as:

$$c = a [4] 5 = (((a [3] a) [3] a) [3] a) [3] a; \quad (I.27.14)$$

Since the properties [3] and [2] we have:

$$(a [3] a) [3] a = a [3] (a [2] a); \quad (I.27.15)$$

Then (I.27.14) will look like this:

$$\begin{aligned} c &= a [4] 5 = ((a [3] (a [2] a)) [3] a) [3] a = \\ &= (a [3] (a [2] a [2] a)) [3] a = \\ &= a [3] (a [2] a [2] a [2] a) = a [3] (a [3] 4); \quad (I.27.16) \end{aligned}$$

This example shows the following formula:

$$a [4] b = a [3] (a [3] (b - 1)); \quad (I.27.17)$$

This conclusion is suitable for natural b . But perhaps the final formula (I.27.17) is also true for a wider class of numbers.

I.28 The first inverse operation [4]1 for the fourth direct operation [4].

Let's look at item I.8. The formula (I.8.1) for $n = 4$ has the form:

$$a [4] b = c; \quad (I.28.1)$$

And the formula (I.8.2) for $n = 4$ has the form:

$$c [4]1 b = a; \quad (I.28.2)$$

Let us rearrange “ a ” and “ c ” in Table I.40, and we get Table I.41:

[4]1	1	2	3	b
1	1	1	1	
2	2			
3	3			
4		2		
16			2	
27		3		
19 683			3	
c				a

Table I.41

Using (I.28.1) and (I.28.2) we will try to fill in the empty cells in Table I.41:

$$c = 4; b = 1; \text{ from (I.28.1) we obtain: } a = 4; \quad (I.28.3)$$

$$c = 16; b = 1; \text{ from (I.28.1) we obtain: } a = 16; \quad (I.28.4)$$

$$c = 27; b = 1; \text{ from (I.28.1) we obtain: } a = 27; \quad (I.28.5)$$

$$c = 19\ 683; b = 1; \text{ from (I.28.1) we obtain: } a = 19\ 683; \quad (\text{I.28.6})$$

The equations for **a** are not fit to the cells of the Table, and we will indicate these equations by a number with a plus (we will use (I.27.17)):

$$1+ = a; c = 2; b = 2; a [4] 2 = a [3] a = c = 2; \quad (\text{I.28.7})$$

$$2+ = a; c = 2; b = 3; a [4] 3 = a [3] (a [3] 2) = c = 2; \quad (\text{I.28.8})$$

$$3+ = a; c = 3; b = 2; a [4] 2 = a [3] a = c = 3; \quad (\text{I.28.9})$$

$$4+ = a; c = 3; b = 3; a [4] 3 = a [3] (a [3] 2) = c = 3; \quad (\text{I.28.10})$$

$$5+ = a; c = 4; b = 3; a [4] 3 = a [3] (a [3] 2) = c = 4; \quad (\text{I.28.11})$$

$$6+ = a; c = 16; b = 2; a [4] 2 = a [3] a = c = 16; \quad (\text{I.28.12})$$

$$7+ = a; c = 27; b = 3; a [4] 3 = a [3] (a [3] 2) = c = 27; \quad (\text{I.28.13})$$

$$8+ = a; c = 19\ 683; b = 2; a [4] 2 = a [3] a = c = 19\ 683; \quad (\text{I.28.14})$$

The completed Table I.41 is now called Table I.42:

[4]1	1	2	3	b
1	1	1	1	
2	2	1+	2+	
3	3	3+	4+	
4	4	2	5+	
16	16	6+	2	
27	27	3	7+	
19 683	19 683	8+	3	
c				a

Table I.42

The numbers from 1+ to 8+ are new, because they are derived from the new equations – from (I.28.7) to (I.28.14). And, since

these equations are constructed using the fourth direct operation [4], we will call the new numbers by "fourth numbers".

I.29 The second inverse operation [4]2] for the fourth direct operation [4].

Formula (I.8.1) at $n = 4$ gives:

$$a [4] b = c; \tag{I.29.1}$$

We take the formula (I.8.3) at $n = 4$:

$$a [4]2] c = b; \tag{I.29.2}$$

Let's change "b" and "c" in the Table I.40, and get the Table I.43:

[4]2]	1	2	3	4	16	27	19 683	c
1	1, 2, 3							
2		1		2	3			
3			1			2	3	
a								b

Table I.43

Here we are again faced with a new type of numbers: numbers – strings. In cell **b**, where **a** = 1, **c** = 1 of Table I.43 are just 3 numbers: 1, 2, 3. And, if some formula will meet this number b, it is unclear which of its three values (1, 2, 3) to take for calculations. We have seen this in Table I.19 and in Table I.20. The only thing we can do now is to develop a special designation for such numbers-strings, so that it is entirely placed in one cell. If the first value of the number-line we denote - "m", the number of values in the number-line denote "N", additive to move to the next value denote "n", and as a separator of these symbols we take a vertical line "|", we get such a record:

$$m|N|n \tag{I.29.3}$$

In our example (1, 2, 3) it will be written as:

$$1|3|1 \quad (I.29.4)$$

Now we fill in the empty cells in Table I.43 and get Table I.44:

[4]2	1	2	3	4	16	27	19 683	c
1	1 3 1	3+	4+	5+	6+	7+	8+	
2	1+	1	9+	2	3	10+	11+	
3	2+	12+	1	13+	14+	2	3	
a								b

Table I.44

$$a = 2; c = 1; 2 [4] b = 1; 2 [4]2] 1 = b = 1+; \quad (I.29.5)$$

$$a = 3; c = 1; 3 [4] b = 1; 3 [4]2] 1 = b = 2+; \quad (I.29.6)$$

$$a = 1; c = 2; 1 [4] b = 2; 1 [4]2] 2 = b = 3+; \quad (I.29.7)$$

$$a = 1; c = 3; 1 [4] b = 3; 1 [4]2] 3 = b = 4+; \quad (I.29.8)$$

$$a = 1; c = 4; 1 [4] b = 4; 1 [4]2] 4 = b = 5+; \quad (I.29.9)$$

$$a = 1; c = 16; 1 [4] b = 16; 1 [4]2] 16 = b = 6+; \quad (I.29.10)$$

$$a = 1; c = 27; 1 [4] b = 27; 1 [4]2] 27 = b = 7+; \quad (I.29.11)$$

$$a = 1; c = 19\ 683; 1 [4] b = 19\ 683; 1 [4]2] 19\ 683 = b = 8+; \quad (I.29.12)$$

$$a = 2; c = 3; 2 [4] b = 3; 2 [4]2] 3 = b = 9+; \quad (I.29.13)$$

$$a = 2; c = 27; 2 [4] b = 27; 2 [4]2] 27 = b = 10+; \quad (I.29.14)$$

$$a = 2; c = 19\ 683; 2 [4] b = 19\ 683; 2 [4]2] 19\ 683 = b = 11+; \quad (I.29.15)$$

$$a = 3; c = 2; 3 [4] b = 2; 3 [4]2] 2 = b = 12+; \quad (I.29.16)$$

$$a = 3; c = 4; 3 [4] b = 4; 3 [4]2] 4 = b = 13+; \quad (I.29.17)$$

$$a = 3; c = 16; 3 [4] b = 16; 3 [4]2] 16 = b = 14+; \quad (I.29.18)$$

The numbers from 1+ to 14+ are derived from the new equations – from (I.29.5) to (I.29.18) – which (equations) are constructed

through a new direct operation [4]. Therefore, we will classify them as the "fourth numbers".

I.30 Question numbers. Extension of the concept to new direct operations.

Remember the definition of question numbers in paragraph I.17:

$$1 [3] ?a = 1 \wedge ?a = a; \tag{I.17.4}$$

Here a is any complex number. And the number ?a will be called a "question number".

- - - - -

Now that the fourth direct operation has appeared, this definition can be generalized to any direct operation with number n:

$$1 [n] n?a = a; \tag{I.30.1}$$

Here a is any complex or question number. And number n?a will be called a "question number". The connection of the old "question numbers" ?a and new "question numbers" n?a is this:

$$3?a = ?a; \tag{I.30.2}$$

$$1 [3] 3?a = a; \tag{I.30.2}$$

And for the fourth direct operation [4] definition (I.30.1) gives:

$$1 [4] 4?a = a; \tag{I.30.3}$$

For a = 2 we obtain the equation:

$$1 [4] 4?2 = 2; \tag{I.30.4}$$

Here the number 4^2 increases 1 to 2. So this is an infinitely large number. Number 4^3 it turns out, too, endlessly large number. Also infinitely large numbers are 4^4 , 4^{16} , 4^{27} , 4^{19683} , and so on.

And number 4^1 gives the equation:

$$1 [4] 4^1 = 1; \quad (I.30.6)$$

It stores 1 and thus is itself equal to 1:

$$4^1 = 1; \quad (I.30.7)$$

I.31 Operation [4] for question numbers ?a (3?a).

From point I.27 we take the formula:

$$(I.27.17) \quad a [4] b = a [3] (a [3] (b-1));$$

For the brevity of the formulas, we will use the old notation. Recall that $?1 = 1$.

$$\begin{aligned} ?1 [4] ?1 &= ?1 [3] (?1 [3] (?1 - 1)) = 1 [3] (1 [3] (1 - 1)) = \\ &= 1 [3] (1 [3] 0) = 1 [3] 1 = 1; \end{aligned} \quad (I.31.1)$$

$$\begin{aligned} ?1 [4] ?2 &= ?1 [3] (?1 [3] (?2 - 1)) = 1 [3] (1 [3] (?2 - 1)) = \\ &= 1 [3] ((1 [3] ?2) [2] (1 [3] (-1))) = 1 [3] (2 [2] 1) = \\ &= 1 [3] 2 = 1; \end{aligned} \quad (I.31.2)$$

$$\begin{aligned} ?1 [4] ?3 &= 1 [3] (1 [3] (?3 - 1)) = 1 [3] ((1 [3] ?3) [2] (1 [3] (-1))) = \\ &= 1 [3] (3 [2] 1) = 1 [3] 3 = 1; \end{aligned} \quad (I.31.3)$$

$$\begin{aligned} ?2 [4] ?1 &= ?2 [3] (?2 [3] (?1 - 1)) = ?2 [3] (?2 [3] (1 - 1)) = \\ &= ?2 [3] ((?2 [3] 1) / (?2 [3] 1)) = ?2 [3] (?2 / ?2) = ?2 [3] 1 = \\ &= ?2; \end{aligned} \quad (I.31.4)$$

$$\begin{aligned}
?3 [4] ?1 &= ?3 [3] (?3 [3] (?1 - 1)) = ?3 [3] (?3 [3] (1 - 1)) = \\
&= ?3 [3] ((?3 [3] 1) / (?3 [3] 1)) = ?3 [3] (?3 / ?3) = \\
&= ?3 [3] 1 = ?3; \tag{I.31.5}
\end{aligned}$$

$$\begin{aligned}
?2 [4] ?2 &= ?2 [3] (?2 [3] (?2 - 1)) = \\
&\text{(From Table I.24: } ?2 - 1 = ?2 [1]1) ?1 = ?2) \\
&= ?2 [3] (?2 [3] ?2); \tag{I.31.6}
\end{aligned}$$

$$\begin{aligned}
?2 [4] ?3 &= ?2 [3] (?2 [3] (?3 - 1)) = \\
&= ?2 [3] (?2 [3] ?3); \tag{I.31.7}
\end{aligned}$$

$$\begin{aligned}
?3 [4] ?2 &= ?3 [3] (?3 [3] (?2 - 1)) = \\
&= ?3 [3] (?3 [3] ?2); \tag{I.31.8}
\end{aligned}$$

$$\begin{aligned}
?3 [4] ?3 &= ?3 [3] (?3 [3] (?3 - 1)) = \\
&= ?3 [3] (?3 [3] ?3); \tag{I.31.9}
\end{aligned}$$

These results are summarized in Table I.45:

[4]	?1	?2	?3	b
?1	1	1	1	
?2	?2	?2 [3] (?2 [3] ?2)	?2 [3] (?2 [3] ?3)	
?3	?3	?3 [3] (?3 [3] ?2)	?3 [3] (?3 [3] ?3)	
a				c

Table I.45

Table I.45 contains the formula: $a [4] b = c;$ (I.31.10)

A little note. From (I.30.4) we have:

$$1 [4] 4?2 = 2;$$

And from (I.31.2) should (given $3?2 = ?2$):

$$1 [4] 3?2 = 1;$$

That means $4?2 > 3?2;$ (I.31.11)

I.32 The first inverse operation [4]1 for the fourth direct operation [4] for question numbers ?a (3?a).

Let's look at item I.8. The formula (I.8.1) for $n = 4$ has the form:

$$a [4] b = c; \quad (I.32.1)$$

And the formula (I.8.2) for $n = 4$ has the form:

$$c [4]1 b = a; \quad (I.32.2)$$

We introduce the following notations:

$$?2 [3] (?2 [3] ?2) = "222"; ?2 [3] (?2 [3] ?3) = "223"; \quad (I.32.3)$$

$$?3 [3] (?3 [3] ?2) = "332"; ?3 [3] (?3 [3] ?3) = "333"; \quad (I.32.4)$$

If we rearrange "a" and "c" in Table I.45, we get Table I.46 for operation [4]1 applied to the numbers ?a (3?a):

[4]1	?1	?2	?3	b
1	1	1	1	
?2	?2			
?3	?3			
222		?2		
223			?2	
332		?3		
333			?3	
c				a

Table I.46

Using the formula (I.32.1) set $b = 1$ and obtain $a = c$. Table I.46 takes the form of a Table I.47:

[4]1	?1	?2	?3	b
1	1	1	1	
?2	?2			
?3	?3			
222	222	?2		
223	223		?2	
332	332	?3		
333	333		?3	
c				a

Table I.47

We fill in the empty cells in the Table I.47 with signs – 1+, 2+, ..., 7+, 8+ - and get Table I.48:

[4]1	?1	?2	?3	b
1	1	1	1	
?2	?2	1+	2+	
?3	?3	3+	4+	
222	222	?2	5+	
223	223	6+	?2	
332	332	?3	7+	
333	333	8+	?3	
c				a

Table I.48

$$a [4] b = c;$$

Let's give for signs their corresponding values:

$$c = ?2; b = ?2; a [4] ?2 = ?2; 1+ = a; \tag{I.32.5}$$

$$c = ?2; b = ?3; a [4] ?3 = ?2; 2+ = a; \tag{I.32.6}$$

$$c = ?3; b = ?2; a [4] ?2 = ?3; 3+ = a; \tag{I.32.7}$$

$$c = ?3; b = ?3; a [4] ?3 = ?3; 4+ = a; \tag{I.32.8}$$

$$c = 222; b = ?3; a [4] ?3 = 222; 5+ = a; \tag{I.32.9}$$

$$c = 223; b = ?2; a [4] ?2 = 223; 6+ = a; \tag{I.32.10}$$

$$c = 332; b = ?3; a [4] ?3 = 332; 7+ = a; \tag{I.32.11}$$

$$c = 333; b = ?2; a [4] ?2 = 333; 8+ = a; \tag{I.32.12}$$

I.33 The second inverse operation [4]2 for the fourth direct operation [4] for question numbers ?a (3?a).

The formula (I.8.1) for n = 4 gives:

$$a [4] b = c; \tag{I.33.1}$$

Take the formula (I.8.3) for n = 4:

$$a [4]2 c = b; \tag{I.33.2}$$

We introduce the following notations:

$$?2 [3] (?2 [3] ?2) = "222"; ?2 [3] (?2 [3] ?3) = "223"; \tag{I.33.3}$$

$$?3 [3] (?3 [3] ?2) = "332"; ?3 [3] (?3 [3] ?3) = "333"; \tag{I.33.4}$$

Let's rearrange b and c in Table I.45 and get Table I.49:

[4]2	1	?2	?3	222	223	332	333	c
?1	1+							
?2		1		?2	?3			
?3			1			?2	?3	
a								b

Table I.49

Here 1+ is a number-string: 1+ = 1,?2, ?3; (I.33.5)

Fill the remaining empty cells of the Table I.49 with signs 2+, 3+, ..., 14+, 15+ and get the Table I.50:

[4]2	1	?2	?3	222	223	332	333	c
1	1+	4+	5+	6+	7+	8+	9+	
?2	2+	1	11+	?2	?3	14+	15+	
?3	3+	10+	1	12+	13+	?2	?3	
a								b

Table I.50

Give the values for the new signs in the Table I.50:

$$(a [4] b = c) (a [4]2] c = b)$$

$$a = ?2; c = 1; ?2 [4] 2+ = 1; \quad (I.33.6)$$

$$a = ?3; c = 1; ?3 [4] 3+ = 1; \quad (I.33.7)$$

$$a = 1; c = ?2; 1 [4] 4+ = ?2; \quad (I.33.8)$$

$$a = 1; c = ?3; 1 [4] 5+ = ?3; \quad (I.33.9)$$

$$a = 1; c = 222; 1 [4] 6+ = 222; \quad (I.33.10)$$

$$a = 1; c = 223; 1 [4] 7+ = 223; \quad (I.33.11)$$

$$a = 1; c = 332; 1 [4] 8+ = 332; \quad (I.33.12)$$

$$a = 1; c = 333; 1 [4] 9+ = 333; \quad (I.33.13)$$

$$a = ?3; c = ?2; ?3 [4] 10+ = ?2; \quad (I.33.14)$$

$$a = ?2; c = ?3; ?2 [4] 11+ = ?3; \quad (I.33.15)$$

$$a = ?3; c = 222; ?3 [4] 12+ = 222; \quad (I.33.16)$$

$$a = ?3; c = 223; ?3 [4] 13+ = 223; \quad (I.33.17)$$

$$a = ?2; c = 332; ?2 [4] 14+ = 332; \quad (I.33.18)$$

$$a = ?2; c = 333; ?2 [4] 15+ = 333; \quad (I.33.19)$$

I.34 Operation [4] for question numbers 4?a.

$$a [4] b = c; \quad (I.34.1)$$

From (I.30.7) take the equation: $4?1 = 1$;

Then we will get:

$$4?1 [4] 4?1 = 1; \quad (I.34.2)$$

$$4?1 [4] 4?2 = 2; \quad (I.34.3)$$

$$4?1 [4] 4?3 = 3; \quad (I.34.4)$$

$$4?2 [4] 4?1 = 4?2; \quad (I.34.5)$$

$$4?3 [4] 4?1 = 4?3; \quad (I.34.6)$$

For the formula (I.34.1) it will be the Table I.51:

[4]	4?1	4?2	4?3	b
4?1	1	2	3	
4?2	4?2	4?2 [4] 4?2	4?2 [4] 4?3	
4?3	4?3	4?3 [4] 4?2	4?3 [4] 4?3	
a				c

Table I.51

I.35 The first inverse operation [4]1 to the fourth direct operation [4] on the set of question numbers 4?a.

To obtain the first inverse operation [4]1 to the fourth direct operation [4] it is necessary to rearrange "a" and "c" in Table I.51:

$$c [4]1 b = a; \quad (I.35.1)$$

And then we get a Table I.52:

[4]1	4?1	4?2	4?3	b
1	1			
2		1		
3			1	
4?2	4?2			
4?3	4?3			
4?2 [4] 4?2		4?2		
4?2 [4] 4?3			4?2	
4?3 [4] 4?2		4?3		
4?3 [4] 4?3			4?3	
c				a

Table I.52

Using the formulas (I.32.1), (I.30.7) and the Table I.52 (a [4] 1 = c), we fill in a few more cells, and get Table I.53:

[4]1	4?1	4?2	4?3	b
1	1			
2	2	1		
3	3		1	
4?2	4?2			
4?3	4?3			
4?2 [4] 4?2	4?2 [4] 4?2	4?2		
4?2 [4] 4?3	4?2 [4] 4?3		4?2	
4?3 [4] 4?2	4?3 [4] 4?2	4?3		
4?3 [4] 4?3	4?3 [4] 4?3		4?3	
c				a

Table 53

The full expressions will not fit in the remaining empty cells. So we'll fill them with short signs and then match the signs to the expressions. Table with signs will be called Table I.54:

[4]1	4?1	4?2	4?3	b
1	1	1+	2+	
2	2	1	3+	
3	3	4+	1	
4?2	4?2	5+	6+	
4?3	4?3	7+	8+	
4?2 [4] 4?2	4?2 [4] 4?2	4?2	9+	
4?2 [4] 4?3	4?2 [4] 4?3	10+	4?2	
4?3 [4] 4?2	4?3 [4] 4?2	4?3	11+	
4?3 [4] 4?3	4?3 [4] 4?3	12+	4?3	
c				a

Table I.54

$$1+ [4] 4?2 = 1; \quad (I.35.2)$$

$$2+ [4] 4?3 = 1; \quad (I.35.3)$$

$$3+ [4] 4?3 = 2; \quad (I.35.4)$$

$$4+ [4] 4?2 = 3; \quad (I.35.5)$$

$$5+ [4] 4?2 = 4?2; \quad (I.35.6)$$

$$6+ [4] 4?3 = 4?2; \quad (I.35.7)$$

$$7+ [4] 4?2 = 4?3; \quad (I.35.8)$$

$$8+ [4] 4?3 = 4?3; \quad (I.35.9)$$

$$9+ [4] 4?3 = 4?2 [4] 4?2; \quad (I.35.10)$$

$$10+ [4] 4?2 = 4?2 [4] 4?3; \quad (I.35.11)$$

$$11+ [4] 4?3 = 4?3 [4] 4?2; \quad (I.35.12)$$

$$12+ [4] 4?2 = 4?3 [4] 4?3; \quad (I.35.13)$$

I.36 The second inverse operation [4]2] to the fourth direct operation [4] on the set of question numbers 4?a.

To obtain the second inverse operation [4]2] to the fourth direct operation [4] it is necessary to rearrange "b" and "c" in Table I.51, and we get the Table I.55:

[4]2]	1	2	3	4?2	4?3	22?	23?	32?	33?	c
4?1	1	4?2	4?3							
4?2				4?1		4?2	4?3			
4?3					4?1			4?2	4?3	
a										b

Table I.55

$$a [4] b = c; \quad (I.36.1)$$

$$a [4]2] c = b; \quad (I.36.2)$$

In Table 55 there are applied the reductions:

$$"22?" = "4?2 [4] 4?2"; "23?" = "4?2 [4] 4?3"; \quad (I.36.3)$$

$$"32?" = "4?3 [4] 4?2"; "33?" = "4?3 [4] 4?3"; \quad (I.36.4)$$

From (I.30.7) it is taken the equation: $4?1 = 1$;

Let's set "a" = $4?1 = 1$ and we will substitute in (I.36.1) different "c". We will get different equations for "b":

$$c = 4?2; 1 [4] b = 4?2; b = 4?(4?2); b1 = b; \quad (I.36.5)$$

$$c = 4?3; 1 [4] b = 4?3; b = 4?(4?3); b2 = b; \quad (I.36.6)$$

$$c = 22?; 1 [4] b = 22?; b = 4?(22?); b3 = b; \quad (I.36.7)$$

$$c = 23?; 1 [4] b = 23?; b = 4?(23?); b4 = b; \quad (I.36.8)$$

$$c = 32?; 1 [4] b = 32?; b = 4?(32?); b5 = b; \quad (I.36.9)$$

$$c = 33?; 1 [4] b = 33?; b = 4?(33?); b6 = b; \quad (I.36.10)$$

Let's put into a Table I.55 the values: b1, b2, b3, b4, b5, b6 in appropriate places, and get a Table I.56:

[4]2]	1	2	3	4?2	4?3	22?	23?	32?	33?	c
4?1	1	4?2	4?3	b1	b2	b3	b4	b5	b6	
4?2				1		4?2	4?3			
4?3					1			4?2	4?3	
a										b

Table I.56

Let's fill the remaining empty cells by indexes: 1+, 2+, ..., 11+, 12+ and get a Table I.57:

[4]2]	1	2	3	4?2	4?3	22?	23?	32?	33?	c
4?1	1	4?2	4?3	b1	b2	b3	b4	b5	b6	
4?2	1+	3+	5+	1	8+	4?2	4?3	11+	12+	
4?3	2+	4+	6+	7+	1	9+	10+	4?2	4?3	
a										b

Table I.57

Now let's compare to each index its value:

$$(a [4] b = c) (a [4]2] c = b)$$

$$a = 4?2; c = 1; 4?2 [4] b = 1; 1+ = b; \quad (I.36.11)$$

$$a = 4?3; c = 1; 4?3 [4] b = 1; 2+ = b; \quad (I.36.12)$$

$$a = 4?2; c = 2; 4?2 [4] b = 2; 3+ = b; \quad (I.36.13)$$

$$a = 4?3; c = 2; 4?3 [4] b = 2; 4+ = b; \quad (I.36.14)$$

$$a = 4?2; c = 3; 4?2 [4] b = 3; 5+ = b; \quad (I.36.15)$$

$$a = 4?3; c = 3; 4?3 [4] b = 3; 6+ = b; \quad (I.36.16)$$

$$a = 4?3; c = 4?2; 4?3 [4] b = 4?2; 7+ = b; \quad (I.36.17)$$

$$a = 4?2; c = 4?3; 4?2 [4] b = 4?3; 8+ = b; \tag{I.36.18}$$

$$a = 4?3; c = 22?; 4?3 [4] b = 22?; 9+ = b; \tag{I.36.19}$$

$$a = 4?3; c = 23?; 4?3 [4] b = 23?; 10+ = b; \tag{I.36.20}$$

$$a = 4?2; c = 32?; 4?2 [4] b = 32?; 11+ = b; \tag{I.36.21}$$

$$a = 4?2; c = 33?; 4?2 [4] b = 33?; 12+ = b; \tag{I.36.22}$$

In the last four formulas the indexes 9+, 10+, 11+, 12+ are infinitely large.

I.37 The construction of the fifth direct operation - [5].

Paragraph I.26 describes the general rule of constructing [n + 1] direct operation by [n] direct operation. Here it is:

- - - - -

Operation [n+1] is formed by operation [n] as follows: the operand for operation [n] is written to the left of [n+1], and the number of these operands is written to the right of [n+1].

- - - - -

You can look at paragraph I.27 - it describes the construction of the fourth direct operation [4]. It obtains a general rule of parenthesis arrangements – so, that the calculations are performed from left to right. In our case n = 4. Let's look at some examples:

$$a [5] b = c; \tag{I.37.1}$$

$$2 [5] 2 = 2 [4] 2 = 4; \tag{I.37.2}$$

$$2 [5] 3 = (2 [4] 2) [4] 2 = 4 [4] 2 = 4 [3] 4 = 256; \tag{I.37.3}$$

$$3 [5] 2 = 3 [4] 3 = 19\ 683; \tag{I.37.4}$$

$$3 [5] 3 = (3 [4] 3) [4] 3 = 19\ 683 [4] 3; \quad (I.37.5)$$

$$1 [5] 1 = 1; 1 [5] 2 = 1; 1 [5] 3 = 1; \quad (I.37.6)$$

$$2 [5] 1 = 2; 3 [5] 1 = 3; \quad (I.37.7)$$

Let us summarize them in Table I.58:

[5]	1	2	3	b
1	1	1	1	
2	2	4	256	
3	3	19 683	19 683 [4] 3	
a				c

Table I.58

I.38 The first inverse operation [5]1 for the fifth direct operation [5].

From the definition of the fifth operation

$$a [5] b = c; \quad (I.38.1)$$

the definition of the first inverse to the fifth direct follows:

$$c [5]1 b = a; \quad (I.38.2)$$

Rearrange **a** and **c** in the Table I.58 and get the Table I.59:

[5]1	1	2	3	b
1	1	1	1	
2	2			
3	3			
4		2		
256			2	
19683		3		
19683 [4] 3			3	
c				a

Table I.59

Take $b = 1$. Then from (I.38.1) should:

$$a [5] 1 = c; \quad (I.38.3)$$

From here we have:

$$a = c; \quad (I.38.4)$$

And the Table I.59 becomes by Table I.60:

[5]1	1	2	3	b
1	1	1	1	
2	2			
3	3			
4	4	2		
256	256		2	
19683	19683	3		
19683 [4] 3	19683 [4] 3		3	
c				a

Table I.60

The remaining empty cells we fill in by the digits with + and get the Table I.61:

[5]1	1	2	3	b
1	1	1	1	
2	2	1+	2+	
3	3	3+	4+	
4	4	2	5+	
256	256	6+	2	
19683	19683	3	7+	
19683 [4] 3	19683 [4] 3	8+	3	
c				a

Table I.61

Now we fill by meaning these pairs of characters:

$$1+ [5] 2 = 2; 1+ [4] 1+ = 2; \quad (I.38.5)$$

$$2+ [5] 3 = 2; (2+ [4] 2+) [4] 2+ = 2; \quad (I.38.6)$$

$$3+ [5] 2 = 3; 3+ [4] 3+ = 3; \quad (I.38.7)$$

$$4+ [5] 3 = 3; (4+ [4] 4+) [4] 4+ = 3; \quad (I.38.8)$$

$$5+ [5] 3 = 4; (5+ [4] 5+) [4] 5+ = 4; \quad (I.38.9)$$

$$6+ [5] 2 = 256; 6+ [4] 6+ = 256; \quad (I.38.10)$$

$$7+ [5] 3 = (7+ [4] 7+) [4] 7+ = 19683; \quad (I.38.11)$$

$$8+ [5] 2 = 8+ [4] 8+ = 19683 [4] 3; \quad (I.38.12)$$

The numbers from 1+ to 8+ are derived from the new equations – from (I.38.5) to (I. 38.12) – which (numbers) are constructed through the first inverse operation to the new direct operation [5]. So we'll call them the " fifth numbers."

I.39 The second inverse operation [5]2] for the fifth direct operation [5].

$$\text{From} \quad a [5] b = c; \quad (I.39.1)$$

it should follow the definition of the second reverse for the fifth direct:

$$a [5]2] c = b; \quad (I.39.2)$$

$$\text{We introduce the notation: } d1 = 19683; d2 = d1 [4] 3; \quad (I.39.3)$$

Let's rearrange **b** and **c** in Table I.58 and get Table I.62:

[5]2]	1	2	3	4	256	d1	d2	c
1	1, 2, 3							
2		1		2	3			
3			1			2	3	
a								b

Table I.62

Fill the remaining empty cells by numbers with “+” and get the Table I.63:

[5]2]	1	2	3	4	256	d1	d2	c
1	1, 2, 3	3+	5+	7+	9+	11+	13+	
2	1+	1	6+	2	3	12+	14+	
3	2+	4+	1	8+	10+	2	3	
a								b

Table I.63

$$2 [5] 1+ = 1; \quad (I.39.4)$$

$$3 [5] 2+ = 1; \quad (I.39.5)$$

$$1 [5] 3+ = 2; \quad (I.39.6)$$

$$3 [5] 4+ = 2; \quad (I.39.7)$$

$$1 [5] 5+ = 3; \quad (I.39.8)$$

$$2 [5] 6+ = 3; \quad (I.39.9)$$

$$1 [5] 7+ = 4; \quad (I.39.10)$$

$$3 [5] 8+ = 4; \quad (I.39.11)$$

$$1 [5] 9+ = 256; \quad (I.39.12)$$

$$3 [5] 10+ = 256; \quad (I.39.13)$$

$$1 [5] 11+ = d1; \quad (I.39.14)$$

$$2 [5] 12+ = d1; \quad (I.39.15)$$

$$1 [5] 13+ = d2; \quad (I.39.16)$$

$$2 [5] 14+ = d2; \quad (I.39.17)$$

The numbers from 1+ to 14+ are derived from the new equations – from (I.39.4) to (I.39.17) – which (numbers) are constructed

through a second inverse operation to a new direct operation [5].
Therefore, we will classify them as "the fifth numbers".

Then you should build the sixth operation, and so on. This concludes Chapter I.

CHAPTER II

II.2 Determination of direct zero [0] and direct negative [-m] operations.

Chapter I establishes a connection between direct operation [n] and direct operation [n + 1]. Here are a couple of examples of this connection:

$$3 [n + 1] 2 = 3 [n] 3; \quad (\text{II.2.1})$$

$$2 [n + 1] 3 = (2 [n] 2) [n] 2; \quad (\text{II.2.2})$$

If we take $n = 0$, we can determine a new direct operation [0] by the known direct operation [1] (summation). And then we can build a new direct operation [-1] by direct operation [0]. Thus we can construct all direct operations with negative numbers.

So the number of the direct operation can take the value of any integer.

II.3 Zero direct operation [0].

Take $n = 0$. Use also: $2 [1] 2 = 2 + 2 = 4;$ (II.3.1)

In the sample (II.2.1) replace 3 with 2, and then we get the equation for 2 [0] 2:

$$4 = 2 [1] 2 = 2 [0] 2; \quad (\text{II.3.2})$$

Further $2 [1] 3 = 2 + 3 = 5;$ (II.3.3)

Apply the sample (II.2.2) and obtain the equation for 4 [0] 2:

$$5 = 2 [1] 3 = (2 [0] 2) [0] 2 = 4 [0] 2; \quad (\text{II.3.4})$$

Now, considering (II.3.4), we construct the equation for 5 [0] 2:

$$6 = 2 [1] 4 = ((2 [0] 2) [0] 2) [0] 2 = (4 [0] 2) [0] 2 = 5 [0] 2; \quad (\text{II.3.5})$$

And now, considering (II.3.5), the equation for 6 [0] 2:

$$7 = 2 [1] 5 = (5 [0] 2) [0] 2 = 6 [0] 2; \quad (\text{II.3.6})$$

Next, we get a couple of equations:

$$8 = 2 [1] 6 = (6 [0] 2) [0] 2 = 7 [0] 2; \quad (\text{II.3.7})$$

$$9 = 2 [1] 7 = (7 [0] 2) [0] 2 = 8 [0] 2; \quad (\text{II. 3.8})$$

The equations obtained are generalized by the formula:

$$a [0] b = c; \quad (\text{II.3.9})$$

Now let's build five more equations, but with $b = 3$:

$$5 = 3 [1] 2 = 3 [0] 3; \quad (\text{II.3.10})$$

$$6 = 3 [1] 3 = (3 [0] 3) [0] 3 = 5 [0] 3; \quad (\text{II.3.11})$$

$$7 = 3 [1] 4 = 6 [0] 3; \quad (\text{II.3.12})$$

$$8 = 3 [1] 5 = 7 [0] 3; \quad (\text{II.3.13})$$

$$9 = 3 [1] 6 = 8 [0] 3; \quad (\text{II.3.14})$$

And now four more equations with $b = 4$:

$$6 = 4 [1] 2 = 4 [0] 4; \quad (\text{II.3.15})$$

$$7 = 4 [1] 3 = (4 [0] 4) [0] 4 = 6 [0] 4; \quad (\text{II.3.16})$$

$$8 = 4 [1] 4 = 7 [0] 4; \quad (\text{II.3.17})$$

$$9 = 4 [1] 5 = 8 [0] 4; \quad (\text{II.3.18})$$

We construct three equations with $b = 5$:

$$7 = 5 [1] 2 = 5 [0] 5; \quad (\text{II.3.19})$$

$$8 = 5 [1] 3 = (5 [0] 5) [0] 5 = 7 [0] 5; \quad (\text{II.3.20})$$

$$9 = 5 [1] 4 = 8 [0] 5; \quad (\text{II.3.21})$$

Now let's build two equations with $b = 6$:

$$8 = 6 [1] \quad 2 = 6 [0] \quad 6; \tag{II.3.22}$$

$$9 = 6 [1] \quad 3 = (6 [0] \quad 6) [0] \quad 6 = 8 [0] \quad 6; \tag{II.3.23}$$

And finally, the last equation (with $b = 7$):

$$9 = 7 [1] \quad 2 = 7 [0] \quad 7; \tag{II.3.24}$$

$$a [0] \quad b = c; \tag{II.3.9}$$

All these equations are presented in the form of Table II.1:

[0]	2	3	4	5	6	7	b
2	4						
3		5					
4	5		6				
5	6	6		7			
6	7	7	7		8		
7	8	8	8	8		9	
a							c

Table II.1

It is impossible to make formulas for empty cells. It remains only to speculate. If you try to make it logical, balanced and beautiful, you will get the Table II.2:

[0]	2	3	4	5	6	7	b
2	4	4.5	5	5	5	5	
3	4.5	5	5.5	6	6	6	
4	5	5.5	6	6.5	7	7	
5	6	6	6.5	7	7.5	8	
6	7	7	7	7.5	8	8.5	
7	8	8	8	8	8.5	9	
a							c

Table II.2

Here is an extended table for [0]:

[0]	-2	-1	0	1	2	3	4	5	6	7	b
-2	0	0.5	1	1	1	1	1	1	1	1	
-1	0.5	1	1.5	2	2	2	2	2	2	2	
0	1	1.5	2	2.5	3	3	3	3	3	3	
1	2	2	2.5	3	3.5	4	4	4	4	4	
2	3	3	3	3.5	4	4.5	5	5	5	5	
3	4	4	4	4	4.5	5	5.5	6	6	6	
4	5	5	5	5	5	5.5	6	6.5	7	7	
5	6	6	6	6	6	6	6.5	7	7.5	8	
6	7	7	7	7	7	7	7	7.5	8	8.5	
7	8	8	8	8	8	8	8	8	8.5	9	
a											c

Table II.2.1

II.4 The first inverse operation [0]1] to the zero direct operation [0].

Permute "a" and "c" in Table II.2 and get Table II.3 for the first inverse to zero:

$$c [0]1] b = a; \quad (II.4.1)$$

[0]1]	2	3	4	5	6	7	b
4	2						
4.5	3	2					
5	4	3	2	2	2	2	
5.5		4	3				
6	5	5	4	3	3	3	
6.5			5	4			
7	6	6	6	5	4	4	
7.5				6	5		
8	7	7	7	7	6	5	
8.5					7	6	
9	8	8	8	8	8	7	
c							a

Table II.3

The contents of the empty cells conjectures, logical and simple.
The result is in the Table II.3.1:

[0]1	2	3	4	5	6	7	b
4	2	1	1	1	1	1	
4.5	3	2	1.5	1.5	1.5	1.5	
5	4	3	2	2	2	2	
5.5	4.5	4	3	2.5	2.5	2.5	
6	5	5	4	3	3	3	
6.5	5.5	5.5	5	4	3.5	3.5	
7	6	6	6	5	4	4	
7.5	6.5	6.5	6.5	6	5	4.5	
8	7	7	7	7	6	5	
8.5	7.5	7.5	7.5	7.5	7	6	
9	8	8	8	8	8	7	
c							a

Table II.3.1

II.5 The second inverse operation [0]2] to the zero direct operation [0]. Part 1.

Swap the “c” and “b” in Table II.2 and obtain the Table II.4:

$$a [0]2] c = b; \quad (II.5.1)$$

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	1+									
3		2	3	4	2+							
4			2	3	4	5	4+					
5					3+	4	5	6	7			
6							5+	5	6	7		
7									6+	6	7	
a												b

Table II.4 [0]2]

$$1+ = 4, 5, 6, 7; 2+ = 5, 6, 7; 4+ = 6, 7; \quad (II.5.2)$$

$$3+ = 2, 3; 5+ = 2, 3, 4; 6+ = 2, 3, 4, 5; \quad (II.5.3)$$

In this table we again met the number-lines: 1+, ..., 6+. We will write them down in the designations defined in paragraph I.29:

If the first value of the number-line denote "m", the number of values in the number-line denote "N", additive to move to the next value denote "n", and as a separator of these symbols take a vertical line "|", we got such a record:

$$(I.29.3): \quad m|N|n$$

First, do this with the formulas (II.5.2):

$$1+ = 4|4|1; 2+ = 5|3|1; 4+ = 6|2|1; \quad (II.5.4)$$

Let's look at Table II.2 again:

[0]	2	3	4	5	6	7	b
2	4	4.5	5	5	5	5	
3	4.5	5	5.5	6	6	6	
4	5	5.5	6	6.5	7	7	
5	6	6	6.5	7	7.5	8	
6	7	7	7	7.5	8	8.5	
7	8	8	8	8	8.5	9	
a							c

Table II.2

There is a continuous row of fives at a = 2 after the number of 4.5 and up to the border of the Table II.2. The first of these is bordered by a number of 4.5 and has b = 4. So in 1+ for "m" we get b = 4. The number of these fives from 4.5 to the right border of the table = 4. Therefore, for N we took 4. The first five has b = 4. The second five has b = 5. Additive = 1. The third five has b = 6. Supplement,

too, = 1. The fourth five has b = 7. Supplement, too, = 1. Therefore, n = 1. And we have for a series of four fives:

$$1+ = m | N | n = 4 | 4 | 1; \quad (\text{II.5.5})$$

For a series of three sixes at a = 3 we have after the number 5.5 the first six with b = 5. So in 2+ for “m” we get b = 5. The number of these sixes from the number 5.5 to the right border of the Table = 3. Therefore, for N we took 3. The first six has b = 5. The second six has b = 6. Addition to the first six = 1. The third six has b = 7. Additive to the second six = 1. So n = 1. And we have for a series of three sixes:

$$2+ = m | N | n = 5 | 3 | 1; \quad (\text{II.5.6})$$

For a series of two sevens at a = 4 we have the number 6.5 on the left as the boundary. The first seven to the right of 6.5 has b = 6. Therefore, in 4+ we took m = 6. The number of these sevens from 6.5 on the left, and to the right border of Table II.2 equals to 2. Therefore, N = 2. The first seven has b = 6. The second seven has b = 7. Addition to the first seven = 1. Therefore, n = 1. And for a series of two sevens we have:

$$4+ = m | N | n = 6 | 2 | 1; \quad (\text{II.5.7})$$

We have taken for right boundary of the Table II.2 b = 7. If it was not for the size limit of Table II.2 for the zero direct operation [0] by the page frames in this book, then from the general view of Table II.2 it is clear that the row of fives at a = 2 continues to the right endlessly. That is N = N1. (Number N1 coincides with the number of natural numbers.) And then for 1+ we have:

$$1+ = 4 | N1 | 1; \quad (\text{II.5.8})$$

The same reasoning is true for the row of three sixes. And we have for 2+:

$$2+ = 5 | N1 | 1; \quad (II.5.9)$$

The same thing happens with a row of two sevens at a = 4:

$$4+ = 6 | N1 | 1; \quad (II.5.10)$$

Now let's move on to the formulas (II.5.3). Let's start with 3+ in Table II.4. There are two sixes at a = 5 in Table II.2 to the left of the number 6.5. The left border of the Table II.2 is located to the left of them. And to the left of it hides an endless series of sixes. Therefore, it is possible to start the numbering of sixes in the number-line 3+ only from the right – with the number b = 3: m = 3. The number of sixes is infinite: N = N1 (number of positive integers). The six to the left of the first has b = 2. So the addition is -1, n = -1. And then we have:

$$3+ = m | N | n = 3 | N1 | -1; \quad (II.5.11)$$

Continue 5+ in Table II.4. There are three sevens at a = 6 in Table II.2 to the left of 7.5. The left border of the Table II.2 is located to the left of them. And to the left of this border hides an infinite number of sevens. Therefore, it is possible to start the numbering of sevens only from the right – with a seven having b = 4: m = 4. N = N1. The second seven in the number-line 5+ has b = 3. So the addition is -1, n = -1. And then we have:

$$5+ = m | N | n = 4 | N1 | -1; \quad (II.5.12)$$

Now is the turn to the 6+ in Table II.4. There are four eights at a = 7 in Table II.2 to the left of 8.5. To the left of them out of a border of the Table II.2 is an infinite number of eights. Then N = N1. And the rightmost eight has b = 5: m = 5. The next eight to the left of it has b = 4. Then increment = -1, n = -1. And we now have:

$$6+ = m | N | n = 5 | N1 | -1; \quad (II.5.13)$$

Let us introduce four more numbers-strings in Table II.4:

$$7+ = 2 | N1 | -1; 8+ = 1 | N1 | -1; 9+ = 7 | N1 | 1; 0+ = 8 | N1 | 1; \text{ (II.5.14)}$$

And we got the Table II.5:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	1+									
3	8+	2	3	4	2+							
4			7+	3	4	5	4+					
5					3+	4	5	6	9+			
6							5+	5	6	7	0+	
7									6+	6	7	
a												b

Table II.5 **[0]2]**

Now let's explain - where from did these four numbers-lines appeared. The single five is placed in the Table II.2 with a = 4, at the left border. And on the other side of the border there is an endless row of fives. And the beginning of all these fives is the right five with b = 2: m = 2. N = N1. To the left of it (just abroad) is a five with b = 1. Then increment = -1, n = -1. Here is the result:

$$7+ = m | N | n = 2 | N1 | -1; \text{ (II.5.15)}$$

In Table II.2 at a = 3 and b = 1 (just beyond the left border) is 4. To the right of it there is the number 4.5, and to the left of it there is an infinite number of fours. That is, m = b = 1 and N = N1. The fours extend to the left in descending order of b. Additive = -1: n = -1. And we got:

$$8+ = m | N | n = 1 | N1 | -1; \text{ (II.5.16)}$$

In Table II.2, with a = 5 and b = 7 is 8. To its left is the number 7.5. To its right is the right border of Table II.2, and just over the border - an endless series of figures 8. So m = b = 7. N = N1. Eights are

numbered in ascending order of b. It means, that supplement = 1. And we have:

$$9+ = m | N | n = 7 | N1 | 1; \quad (II.5.17)$$

In Table II.2 for a = 6 and b = 8 is located 9 (to the right of 8.5 – just beyond the right border of Table II.2). It is followed to the right by an infinite number of nines with an increment of 1: n = 1. Also we have m = b = 8. N = N1. And as a result:

$$0+ = m | N | n = 8 | N1 | 1; \quad (II.5.18)$$

II.6 The second inverse operation [0]2] to the zero direct operation [0]. Part 2. Floor 1st.

Let us reproduce the final results obtained in Part 1:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	1+									
3	8+	2	3	4	2+							
4			7+	3	4	5	4+					
5					3+	4	5	6	9+			
6							5+	5	6	7	0+	
7									6+	6	7	
a												b

Table II.5 [0]2]

$$1+ = 4 | N1 | 1; \quad (II.5.8)$$

$$2+ = 5 | N1 | 1; \quad (II.5.9)$$

$$4+ = 6 | N1 | 1; \quad (II.5.10)$$

$$9+ = m | N | n = 7 | N1 | 1; \quad (II.5.17)$$

$$0+ = m | N | n = 8 | N1 | 1; \quad (II.5.18)$$

$$8+ = m |N| n = 1 |N1| -1; \quad (II.5.16)$$

$$7+ = m |N| n = 2 |N1| -1; \quad (II.5.15)$$

$$3+ = m |N| n = 3 |N1| -1; \quad (II.5.11)$$

$$5+ = m |N| n = 4 |N1| -1; \quad (II.5.12)$$

$$6+ = m |N| n = 5 |N1| -1; \quad (II.5.13)$$

The number-lines. What to do with them? In what order to extract numbers from them?

Let's assume that all initial values of the numbers-lines are in the plane of Table II.5. And all the others are above them. Each consecutive number from the number-line is on its floor. All the second numbers-rows from 1+ to 6+ - are on the second floor of the three-dimensional Table II.5. All third numbers of all numbers-lines are on the third floor of Table II.5. And so on.

Let's look at the first floor of the Table II.5:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	4									
3	1	2	3	4	5							
4			2	3	4	5	6					
5					3	4	5	6	7			
6							4	5	6	7	8	
7									5	6	7	
a												b

Table II.5 **[0]2** 1-st floor

This turned out to be the "skeleton" of the table. This is what is obtained from formulas linking operations [0] and [1]. We must to dream (most likely) and build on this "skeleton" everything else.

If you look closely at the "skeleton", you can see this feature of the "skeleton": horizontal numbers are parts of integer series –

they can be extended to the left and to the right from the "skeletal" middle.

If you look at this "skeleton" again, you can see its second feature: vertical numbers are parts of integer series – they can be extended up and down from the "skeletal" middle.

There is also a third feature: the horizontal addition of integers does not interfere with the vertical addition of integers. And vertical additions of integers also do not interfere with horizontal additions of integers.

After these observations let us grow "meat" on the "skeleton" and get Table II.6:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	4	5	6	7	8	9	10	11	12	
3	1	2	3	4	5	6	7	8	9	10	11	
4	0	1	2	3	4	5	6	7	8	9	10	
5	-1	0	1	2	3	4	5	6	7	8	9	
6	-2	-1	0	1	2	3	4	5	6	7	8	
7	-3	-2	-1	0	1	2	3	4	5	6	7	
a												b

Table II.6 **[0]2** 1-st floor

II.7 The second inverse operation **[0]2** to the zero direct operation **[0]**. Part 2. Floor 2nd.

Now let's go to the second floor. Insert in the cells of Table II.5 the second numbers of the corresponding numbers-strings and get the "skeleton" of Table II.7:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	5									
3	0	2	3	4	6							
4			1	3	4	5	7					
5					2	4	5	6	8			
6							3	5	6	7	9	
7									4	6	7	
a												b

Table II.7 [0]2 Floor 2nd

If you look at this "skeleton", you can see that it has no horizontal ordering. At least there is no horizontal integer order.

A vertical ordering may be present. Where "c" is semi-integer, integer ordering is possible. And where "c" is an integer, an even or odd ordering is possible.

Nothing else comes to mind, so let's increase our vertical "meat" on this "skeleton" and get the Table II.8:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	5	5	8	7	11	9	14	11	17	
3	0	2	3	4	6	6	9	8	12	10	15	
4	-2	1	1	3	4	5	7	7	10	9	13	
5	-4	0	-1	2	2	4	5	6	8	8	11	
6	-6	-1	-3	1	0	3	3	5	6	7	9	
7	-8	-2	-5	0	-2	2	1	4	4	6	7	
a												b

Table II.8 [0]2 Floor 2nd

II.8 The second inverse operation [0]2 to the zero direct operation [0]. Part 2. The 3rd floor.

Now let's go to the third floor. Insert in the cells of Table II.5 the third numbers of the corresponding numbers-rows and get the Table II.9:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	6									
3	-1	2	3	4	7							
4			0	3	4	5	8					
5					1	4	5	6	9			
6							2	5	6	7	10	
7									3	6	7	
a												b

Table II.9 [0]2 The 3rd floor

There is no horizontal ordering in this "skeleton". While the vertical ordering is present. Where "c" is semi-integer, integer ordering is possible. And where "c" is an integer, the numbers are located with step 3.

Let's build "meat" on this "skeleton" and get Table II.10:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9
2	2	3	6	5	10	7	14	9	18	11	22
3	-1	2	3	4	7	6	11	8	15	10	19
4	-4	1	0	3	4	5	8	7	12	9	16
5	-7	0	-3	2	1	4	5	6	9	8	13
6	-11	-1	-6	1	-2	3	2	5	6	7	10
7	-14	-2	-9	0	-5	2	-1	4	3	6	7
a											

Table II.10 [0]2 The 3-rd floor

II.9 The second inverse operation [0]2] to the zero direct operation [0]. Part 2. Floor 4th and above.

Now it is clear what the ordering is on the 4th floor and above. Where "c" is half-integer, there the integer vertical alignment is possible. And where "c" is an integer, the numbers vertically go in increments equal to the floor number.

II.10 Zero direct operation [0] on the set of question numbers ?a.

Now we will fill in Table II.11:

$$?a [0] ?b = c; \tag{II.10.1}$$

[0]	?1	?2	?3	?4	?b
?1					
?2					
?3					
?4					
?a					c

Table II.11

$$?1 [0] ?1 = 1 [0] 1 = 1 [1] 2 = 1 + 2 = 3; \tag{II.10.2}$$

$$\begin{aligned} ?2 [0] ?2 = ?2 [1] 2 = ?2 [1] (1 [1] 1) = (?2 [1] 1) [1] 1 = \\ = ?2 [1] 1 = ?2; \end{aligned} \tag{II.10.3}$$

$$\begin{aligned} ?3 [0] ?3 = ?3 [1] 2 = ?3 [1] (1 [1] 1) = (?3 [1] 1) [1] 1 = \\ = ?3 [1] 1 = ?3; \end{aligned} \tag{II.10.4}$$

$$\begin{aligned} ?4 [0] ?4 = ?4 [1] 2 = ?4 [1] (1 [1] 1) = (?4 [1] 1) [1] 1 = \\ = ?4 [1] 1 = ?4; \end{aligned} \tag{II.10.5}$$

$$?2 [0] ?1 = ?2 [0] 1 = ?2; \tag{II.10.6}$$

$$?3 [0] ?1 = ?3 [0] 1 = ?3; \tag{II.10.7}$$

$$?4 [0] ?1 = ?4 [0] 1 = ?4; \quad (\text{II.10.8})$$

$$\begin{aligned} ?1 [0] ?2 &= 1 [0] ?2 = (((1 \text{ } [-1] \text{ } 1) \text{ } [-1] \text{ } 1) \text{ } [-1] \text{ } 1) \text{ } [-1] \text{ } 1 \dots = \\ &= (((1 \text{ } [0] \text{ } 2) \text{ } [-1] \text{ } 1) \text{ } [-1] \text{ } 1) \text{ } [-1] \text{ } 1 \dots = \end{aligned}$$

Use Table II.2.1 (expansion of Table II.2)

$$= ((3.5 \text{ } [-1] \text{ } 1) \text{ } [-1] \text{ } 1) \text{ } [-1] \text{ } 1 \dots =$$

Use Table II.26:

$$= (4 \text{ } [-1] \text{ } 1) \text{ } [-1] \text{ } 1 \dots =$$

Use Table II.26 and (II.13.76) ?2 times:

$$= 4 \text{ } [-1] \text{ } 1 \dots = 4 \text{ } [-1] \text{ } 1 \dots = 4; \quad (\text{II.10.9})$$

$$\text{That is: } ?1 [0] ?2 = 4; \quad (\text{II.10.9})$$

Similarly (?3 times):

$$\begin{aligned} ?1 [0] ?3 &= 1 [0] ?3 = (((1 \text{ } [-1] \text{ } 1) \text{ } [-1] \text{ } 1) \text{ } [-1] \text{ } 1) \text{ } [-1] \text{ } 1 \dots = \\ &= (4 \text{ } [0] \text{ } 1) [0] 1 \dots = 4 [0] 1 \dots = 4; \end{aligned} \quad (\text{II.10.10})$$

Also (?4 times):

$$?1 [0] ?4 = 4; \quad (\text{II.10.11})$$

We did the Table II.12:

[0]	?1	?2	?3	?4	?b
?1	3	?2	?3	?4	
?2	?2	?2			
?3	?3		?3		
?4	?4			?4	
?a					c

Table II.12

We made a "skeleton". The remaining empty cells are filled with imagination. And the result is Table II.13:

[0]	?1	?2	?3	?b
?1	3	4	4	
?2	?2	?2	?2	
?3	?3	?3	?3	
?a				c

Table II.13

$$?a [0] ?b = ?a; (a = 2, 3, 4, \dots; b = 1, 2, 3, 4, \dots); \quad (\text{II.10.16})$$

Looking at the expanded Table II.2.1 for [0] we will see (?1 = 1):

$$1 [0] 0 = 2.5; \quad (\text{II.10.17})$$

$$0 [0] 0 = 2; \quad (\text{II.10.18})$$

$$0 [0] 1 = 2.5; \quad (\text{II.10.19})$$

$$0 [0] 2 = 3; \quad (\text{II.10.20})$$

$$0 [0] 3 = 3; \quad (\text{II.10.21})$$

$$0 [0] 3 = (0 [-1] 0) [-1] 0 = (0 [0] 2) [-1] 0 = 3 [-1] 0; \quad (\text{II.10.22})$$

$$(\text{II.10.21}) + (\text{II.10.22}) = (\text{II.10.23}):$$

$$3 [-1] 0 = 3; \quad (\text{II.10.23})$$

$$\begin{aligned} 0 [0] ?2 &= ((\mathbf{0} [-1] \mathbf{0}) [-1] 0) [-1] 0 \dots = ((\mathbf{0} [\mathbf{0}] \mathbf{2}) [-1] 0) [-1] 0 \dots = \\ &= (\mathbf{3} [-1] 0) [-1] 0 \dots = \mathbf{3} [-1] 0 \dots = \dots ?2 \text{ times and } = \mathbf{3}; \end{aligned} \quad (\text{II.10.24})$$

$$\text{That is } 0 [0]?2 = 3; \quad (\text{II.10.24})$$

$$0 [0]?3 = \dots ?3 \text{ times and } = 3; \quad (\text{II.10.25})$$

Insert these values into Table II.13 and get Table II.13.1:

[0]	0	?1	?2	?3	?b
0	2	2.5	3	3	
?1	2.5	3	4	4	
?2		?2	?2	?2	
?3		?3	?3	?3	
?a					c

Table II.13.1

Two cells are not clear how to fill, so let's limit ourselves to Table II.13.

II.11 The first inverse operation [0]1 to the zero direct operation [0] on the set of question numbers ?a.

Rearrange "?a" and "c" in the Table II.13 and get the Table II.14:

[0]1	?1	?2	?3	?b
3	?1			
4		?1	?1	
?2	?2	?2	?2	
?3	?3	?3	?3	
c				?a

Table II.14

?a [0] ?1 = ?a [0] 1 = 4; (II.11.1)

If you look at Table II.2.1 (expanded to Table II.2) then we'll see that ?a = 3. That is 4 [0]1]?1 = 3; (II.11.2)

?a [0]?2 = 3; according to Table II.13.1 see: ?a = 0; (II.11.3)

3 [0]1]?2 = 0; (II.11.4)

?a [0]?3 = 3; according to Table II.13.1 see: ?a = 0; (II.11.5)

3 [0]1]?3 = 0; (II.11.6)

Table II.14 will become Table II.15:

[0]1	?1	?2	?3	?b
3	1	0	0	
4	3	1	1	
?2	?2	?2	?2	
?3	?3	?3	?3	
c				?a

Table II.15

And as a result - for the first inverse operation [0]1] to the zero direct operation [0] for numbers ?a – we have Table II.15.

II.12 The second inverse operation [0]2] to the zero direct operation [0] on the set of question numbers ?a.

Let's see in details how numbers-strings are born. Consider the first row in Table II.13 and leave it in Table II.16:

[0]	?1	?2	?3	?b
?1	3	4	4	c
?2				
?3				
?a				

Table II.16

Rearrange the "c" and "?b". Get Table II.16.1:

[0]2]	3	4	4	c
?1	?1	?2	?3	?b
?2				
?3				
?a				

Table II.16.1

The column headings should be different, so let's combine the columns with the same headings in Table II.16.2:

[0]2	3	4		c
?1	?1	?2, ?3		?b
?2				
?3				
?a				

Table II.16.2

Consider the second row in Table II.13 and leave it in Table II.16.3:

[0]	?1	?2	?3	?b
?1				
?2	?2	?2	?2	c
?3				
?a				

Table II.16.3

Rearrange the "c" and "?b". Get Table II.16.4:

[0]2	?2	?2	?2	c
?1				
?2	?1	?2	?3	?b
?3				
?a				

Table II.16.4

Again, the same column headings. Again, combine the contents of these columns under a common name and get Table II.16.5:

[0]2		?2		c
?1				
?2		?1, ?2, ?3		?b
?3				
?a				

Table II.16.5

Consider the third row in Table II.13 and leave it in Table II.16.6:

[0]	?1	?2	?3	?b
?1				
?2				
?3	?3	?3	?3	c
?a				

Table II.16.6

Rearrange the "c" and "?b". Get Table II.16.7:

[0]2]	?3	?3	?3	c
?1				
?2				
?3	?1	?2	?3	?b
?a				

Table II.16.7

Again, the same column headings. Again, combine the contents of these columns under a common name and get Table II.16.8:

[0]2]			?3	c
?1				
?2				
?3			?1, ?2, ?3	?b
?a				

Table II.16.8

Combine the Tables: II.16.2, II.16.5, II.16.8 and get Table II.20:

[0]2]	3	4	?2	?3	c
?1	?1	?2, ?3			
?2			?1, ?2, ?3		
?3				?1, ?2, ?3	
?a					?b

Table II.20

Numbers-lines denote as follows:

$$1+ = ?2, ?3; \quad 2+ = ?1, ?2, ?3; \quad 3+ = ?1, ?2, ?3; \quad (\text{II.12.1})$$

Table II.20 will take the form of Table II.20.1:

[0]2]	3	4	?2	?3	c
?1	?1	1+			
?2			2+		
?3				3+	
?a					?b

Table II.20.1

For numbers-lines, also there is a variant of record:

$$1+ = ?2 | 2 | ?+1; \quad (\text{II.12.2})$$

$$2+ = ?1 | 3 | ?+1; \quad (\text{II.12.3})$$

$$3+ = ?1 | 3 | ?+1; \quad (\text{II.12.4})$$

Before the first vertical line - "|" – the first number in the number-line is specified. It is followed by the number of numbers in the number-line. Then comes the second vertical line. It is followed by the step between the numbers in the number-line. In this case, for quantities beginning with the sign "?", specified index step following sign "?".

We introduce the floors above the table.

The very first value in all numbers-lines is located on the first floor in Table II.21:

[0]2]	3	4	?2	?3	c
?1	?1	?2			
?2			?1		
?3				?1	
?a					?b

Table II.21 1st floor

On the second floor of operation [0]2] for ?a each number-line contains its second value. This is shown in Table II.22:

[0]2]	3	4	?2	?3	c
?1	?1	?3			
?2			?2		
?3				?2	
?a					?b

Table II.22 2nd floor

The third floor contains the third value of numbers-lines. Once in a number-line 1+ there are only two values, then its last value - ?3 – is taken. The result is presented in Table II.23:

[0]2]	3	4	?2	?3	c
?1	?1	?3			
?2			?3		
?3				?3	
?a					?b

Table II.23 3rd floor and above

The fourth floor contains in each cell the fourth value of numbers-lines. In those cells, where the number of values in the number-line is less than four, the most recent values of these numbers-lines are located. We again get the Table II.23. It is true also for higher floors.

We got three "skeletal" Tables: II.21, II.22, II.23. Let's try to ramp up on them "meat". It is not clear how to fill in the first two Tables – II.21 and II.22. But here is the third Table - II.23, which forms itself in the Table II.24:

[0]2]	3	4	?2	?3	c
?1	?1	?3	?5	?7	
?2	?(-1)	?1	?3	?5	
?3	?(-3)	?(-1)	?1	?3	
?a					?b

Table II.24 3rd floor and above

II.13 Direct operation [-1].

In Chapter II, paragraph 2, it is written, how to define an unknown operation [-1] by a known operation [0]:

$$(II.2.1): \quad 3 [n + 1] 2 = 3 [n] 3;$$

$$(II.2.2): \quad 2 [n + 1] 3 = (2 [n] 2) [n] 2;$$

At $n = -1$ we obtain:

$$3 [0] 2 = 3 [-1] 3; \quad (II.13.1)$$

$$2 [0] 3 = (2 [-1] 2) [-1] 2; \quad (II.13.2)$$

Values for operation [0] will be taken from Table II.2. If the desired value lies outside the boundary of Table II.2, then you can push the boundaries of Table II.1 as far as you need, by entering new rows and new columns, and filling them with new values, and then on the principle of symmetry fill the remaining empty cells. In further calculations, the values for operation [0] are taken from such an extended Table II.2.1.

$$1 = -2 [0] 2 = -2 [-1] -2; \quad (II.13.3)$$

$$1 = -2 [0] 3 = (-2 [-1] -2) [-1] -2 = 1 [-1] -2; \quad (II.13.4)$$

$$1 = -2 [0] 4 = 1 [-1] -2; \quad (II.13.5)$$

...

The repeats are endless, so

$$1+ = 1 | N1 | 0; \text{ in cell } 1[-1] -2; \quad (II.13.6)$$

$$2 = -1 [0] 2 = -1 [-1] -1; \quad (II.13.7)$$

$$2 = -1 [0] 3 = (-1 [-1] -1) [-1] -1 = 2 [-1] -1; \quad (II.13.8)$$

$$2 = -1 [0] 4 = 2 [-1] -1; \quad (\text{II.13.9})$$

...

The repeats are endless, so

$$2+ = 2 |N1| 0; \text{ in cell } 2 [-1] -1; \quad (\text{II.13.10})$$

$$3 = 0 [0] 2 = 0 [-1] 0; \quad (\text{II.13.11})$$

$$3 = 0 [0] 3 = (0 [-1] 0) [-1] 0 = 3 [-1] 0; \quad (\text{II.13.12})$$

$$3 = 0 [0] 4 = 3 [-1] 0; \quad (\text{II.13.13})$$

...

The repeats are endless, so

$$3+ = 3 |N1| 0; \text{ in cell } 3 [-1] 0; \quad (\text{II.13.14})$$

$$3.5 = 1 [0] 2 = 1 [-1] 1; \quad (\text{II.13.15})$$

$$4 = 1 [0] 3 = (1 [-1] 1) [-1] 1 = 3.5 [-1] 1; \quad (\text{II.13.16})$$

$$4 = 1 [0] 4 = 4 [-1] 1; \quad (\text{II.13.17})$$

$$4 = 1 [0] 5 = 4 [-1] 1; \quad (\text{II.13.18})$$

$$4 = 1 [0] 6 = 4 [-1] 1; \quad (\text{II.13.19})$$

...

The repeats are endless, so

$$4+ = 4 |N1| 0; \text{ in cell } 4 [-1] 1; \quad (\text{II.13.20})$$

$$4 = 2 [0] 2 = 2 [-1] 2; \quad (\text{II.13.21})$$

$$4.5 = 2 [0] 3 = (2 [-1] 2) [-1] 2 = 4 [-1] 2; \quad (\text{II.13.22})$$

$$5 = 2 [0] 4 = ((2 [-1] 2) [-1] 2) [-1] 2 = 4.5 [-1] 2; \quad (\text{II.13.23})$$

$$5 = 2 [0] 5 = 5 [-1] 2; \quad (\text{II.13.24})$$

$$5 = 2 [0] 6 = 5 [-1] 2; \quad (\text{II.13.25})$$

$$5 = 2 [0] 7 = 5 [-1] 2; \quad (\text{II.13.26})$$

...

The repeats are endless, so

$$5+ = 5 | N1 | 0; \text{ in cell } 5 [-1] 2; \quad (\text{II.13.27})$$

$$4.5 = 3 [0] 2 = 3 [-1] 3; \quad (\text{II.13.28})$$

$$5 = 3 [0] 3 = (3 [-1] 3) [-1] 3 = 4.5 [-1] 3; \quad (\text{II.13.29})$$

$$5.5 = 3 [0] 4 = 5 [-1] 3; \quad (\text{II.13.30})$$

$$6 = 3 [0] 5 = 5.5 [-1] 3; \quad (\text{II.13.31})$$

$$6 = 3 [0] 6 = 6 [-1] 3; \quad (\text{II.13.32})$$

$$6 = 3 [0] 7 = 6 [-1] 3; \quad (\text{II.13.33})$$

...

The repeats are endless, so

$$6+ = 6 | N1 | 0; \text{ in cell } 6 [-1] 3; \quad (\text{II.13.34})$$

$$5 = 4 [0] 2 = 4 [-1] 4; \quad (\text{II.13.35})$$

$$5.5 = 4 [0] 3 = 5 [-1] 4; \quad (\text{II.13.36})$$

$$6 = 4 [0] 4 = 5.5 [-1] 4; \quad (\text{II.13.37})$$

$$6.5 = 4 [0] 5 = 6 [-1] 4; \quad (\text{II.13.38})$$

$$7 = 4 [0] 6 = 6.5 [-1] 4; \quad (\text{II.13.39})$$

$$7 = 4 [0] 7 = 7 [-1] 4; \quad (\text{II.13.40})$$

$$7 = 4 [0] \quad 8 = 7 [-1] \quad 4; \quad (\text{II.13.41})$$

...

The repeats are endless, so

$$7+ = 7 | N1 | 0; \text{ in cell } 7 [-1] \quad 4; \quad (\text{II.13.42})$$

$$6 = 5 [0] \quad 2 = 5 [-1] \quad 5; \quad (\text{II.13.43})$$

$$6 = 5 [0] \quad 3 = 6 [-1] \quad 5; \quad (\text{II.13.44})$$

$$6.5 = 5 [0] \quad 4 = 6 [-1] \quad 5; \quad (\text{II.13.45})$$

$$11+ = 6, 6.5; \text{ in cell } 6 [-1] \quad 5; \quad (\text{II.13.46})$$

$$7 = 5 [0] \quad 5 = 6.5 [-1] \quad 5; \quad (\text{II.13.47})$$

$$7.5 = 5 [0] \quad 6 = 7 [-1] \quad 5; \quad (\text{II.13.48})$$

$$8 = 5 [0] \quad 7 = 7.5 [-1] \quad 5; \quad (\text{II.13.49})$$

$$8 = 5 [0] \quad 8 = 8 [-1] \quad 5; \quad (\text{II.13.50})$$

$$8 = 5 [0] \quad 9 = 8 [-1] \quad 5; \quad (\text{II.13.51})$$

...

The repeats are endless, so

$$8+ = 8 | N1 | 0; \text{ in cell } 8 [-1] \quad 5; \quad (\text{II.13.52})$$

$$7 = 6 [0] \quad 2 = 6 [-1] \quad 6; \quad (\text{II.13.53})$$

$$7 = 6 [0] \quad 3 = 7 [-1] \quad 6; \quad (\text{II.13.54})$$

$$7 = 6 [0] \quad 4 = 7 [-1] \quad 6; \quad (\text{II.13.55})$$

$$7.5 = 6 [0] \quad 5 = 7 [-1] \quad 6; \quad (\text{II.13.56})$$

$$10+ = 7, 7, 7.5; \text{ in cell } 7 [-1] \quad 6; \quad (\text{II.13.57})$$

$$8 = 6 [0] \quad 6 = 7.5 [-1] \quad 6; \quad (\text{II.13.58})$$

$$8.5 = 6 [0] \quad 7 = 8 [-1] \quad 6; \quad (\text{II.13.59})$$

$$8 = 7 [0] 2 = 7 [-1] 7; \quad (\text{II.13.60})$$

$$8 = 7 [0] 3 = 8 [-1] 7; \quad (\text{II.13.61})$$

$$8 = 7 [0] 4 = 8 [-1] 7; \quad (\text{II.13.62})$$

$$8 = 7 [0] 5 = 8 [-1] 7; \quad (\text{II.13.63})$$

$$8.5 = 7 [0] 6 = 8 [-1] 7; \quad (\text{II.13.64})$$

$$9+ = 8, 8, 8, 8.5; \text{ in cell } 8 [-1] 7; \quad (\text{II.13.65})$$

All the above 63 equations are considered and the results are summarized in Table II.26 (a [-1] b = c):

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1										
-1		2									
0			3								
1	1+			3.5							
2		2+			4						
3			3+			4.5					
3.5				4							
4				4+	4.5		5				
4.5					5	5					
5					5+	5.5	5.5	6			
5.5						6	6				
6						6+	6.5	11+	7		
6.5							7	7			
7							7+	7.5	10+	8	
7.5								8	8		
8								8+	8.5	9+	
a											c

Table II.26

In the Table II.26 there are the following number-strings:

$$1+ = 1 | N1 | 0; 2+ = 2 | N1 | 0; \quad (II.13.75)$$

$$3+ = 3 | N1 | 0; 4+ = 4 | N1 | 0; \quad (II.13.76)$$

$$5+ = 5 | N1 | 0; 6+ = 6 | N1 | 0; 7+ = 7 | N1 | 0; 8+ = 8 | N1 | 0; \quad (II.13.77)$$

$$9+ = 8, 8, 8, 8.5; 10+ = 7, 7, 7.5; 11+ = 6, 6.5; \quad (II.13.78)$$

For simplicity, let us leave in the Table II.26 only rows with integer headings. We get the Table II.27:

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1										
-1		2									
0			3								
1	1+			3.5							
2		2+			4						
3			3+			4.5					
4				4+	4.5		5				
5					5+	5.5	5.5	6			
6						6+	6.5	11+	7		
7							7+	7.5	10+	8	
8								8+	8.5	9+	
a											c

Table II.27

Now consider the "skeleton" for the first floor. To do this, take the first value from each number-string and place it in the appropriate cell. Instead of 11+ will be the number 6. Instead of 10+ will be 7. Instead of 9+ will stand 8. And for the rest of the numbers-strings it will be enough to remove the sign"+". And we get the Table II.28:

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1										
-1		2									
0			3								
1	1			3.5							
2		2			4						
3			3			4.5					
4				4	4.5		5				
5					5	5.5	5.5	6			
6						6	6.5	6	7		
7							7	7.5	7	8	
8								8	8.5	8	
a											c

Table II.28 1st floor "skeleton"

Now, after thinking, let's ramp up the "meat" on this "skeleton".
And we got the Table II.29:

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1	2	3	3.5	4	4.5	5	6	7	8	
-1	1	2	3	3.5	4	4.5	5	6	7	8	
0	1	2	3	3.5	4	4.5	5	6	7	8	
1	1	2	3	3.5	4	4.5	5	6	7	8	
2	2	2	3	3.75	4	4.5	5	6	7	8	
3	3	3	3	4	4.25	4.5	5	6	7	8	
4	4	4	4	4	4.5	4.75	5	6	7	8	
5	5	5	5	5	5	5.5	5.5	6	7	8	
6	6	6	6	6	6	6	6.5	6	7	8	
7	7	7	7	7	7	7	7	7.5	7	8	
8	8	8	8	8	8	8	8	8	8.5	8	
a											c

Table II.29 1st floor

Now consider the "skeleton" for the second floor. To do this, from each number-line take the second value from the left and put it in the appropriate cell. Instead of 11+ will stand the number 6.5.

Instead of 10+ will be 7. Instead of 9+ will stand 8. And for the rest of the numbers-lines, it will be enough to remove the sign"+". After all, they get the second value by adding 0 to the first value. And we got the Table II.30:

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1										
-1		2									
0			3								
1	1			3.5							
2		2			4						
3			3			4.5					
4				4	4.5		5				
5					5	5.5	5.5	6			
6						6	6.5	6.5	7		
7							7	7.5	7	8	
8								8	8.5	8	
a											c

Table II.30 2nd floor "skeleton"

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1	2	3	3.5	4	4.5	5	6	7	8	
-1	1	2	3	3.5	4	4.5	5	6	7	8	
0	1	2	3	3.5	4	4.5	5	6	7	8	
1	1	2	3	3.5	4	4.5	5	6	7	8	
2	2	2	3	3.75	4	4.5	5	6	7	8	
3	3	3	3	4	4.25	4.5	5	6	7	8	
4	4	4	4	4	4.5	4.75	5	6	7	8	
5	5	5	5	5	5	5.5	5.5	6	7	8	
6	6	6	6	6	6	6	6.5	6.5	7	8	
7	7	7	7	7	7	7	7	7.5	7	8	
8	8	8	8	8	8	8	8	8	8.5	8	
a											c

Table II.31 2nd floor

Now consider the "skeleton" for the third floor. It coincides with the "skeleton" for the second floor, but instead of 10+ is not 7, but 7.5 and we got the Table II.32 and Table II.33:

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1										
-1		2									
0			3								
1	1			3.5							
2		2			4						
3			3			4.5					
4				4	4.5		5				
5					5	5.5	5.5	6			
6						6	6.5	6.5	7		
7							7	7.5	7.5	8	
8								8	8.5	8	
a											c

Table II.32 3rd floor "skeleton"

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1	2	3	3.5	4	4.5	5	6	7	8	
-1	1	2	3	3.5	4	4.5	5	6	7	8	
0	1	2	3	3.5	4	4.5	5	6	7	8	
1	1	2	3	3.5	4	4.5	5	6	7	8	
2	2	2	3	3.75	4	4.5	5	6	7	8	
3	3	3	3	4	4.25	4.5	5	6	7	8	
4	4	4	4	4	4.5	4.75	5	6	7	8	
5	5	5	5	5	5	5.5	5.5	6	7	8	
6	6	6	6	6	6	6	6.5	6.5	7	8	
7	7	7	7	7	7	7	7	7.5	7.5	8	
8	8	8	8	8	8	8	8	8	8.5	8	
a											c

Table II.33 3rd floor

Now consider the "skeleton" for the 4th floor. It coincides with the "skeleton" for the 3rd floor, but instead of 9+ is not 8, but 8.5, and we got the Table II.34 and the Table II.35:

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1										
-1		2									
0			3								
1	1			3.5							
2		2			4						
3			3			4.5					
4				4	4.5		5				
5					5	5.5	5.5	6			
6						6	6.5	6.5	7		
7							7	7.5	7.5	8	
8								8	8.5	8.5	
a											c

Table II.34 4th floor and above "skeleton"

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1	2	3	3.5	4	4.5	5	6	7	8	
-1	1	2	3	3.5	4	4.5	5	6	7	8	
0	1	2	3	3.5	4	4.5	5	6	7	8	
1	1	2	3	3.5	4	4.5	5	6	7	8	
2	2	2	3	3.75	4	4.5	5	6	7	8	
3	3	3	3	4	4.25	4.5	5	6	7	8	
4	4	4	4	4	4.5	4.75	5	6	7	8	
5	5	5	5	5	5	5.5	5.5	6	7	8	
6	6	6	6	6	6	6	6.5	6.5	7	8	
7	7	7	7	7	7	7	7	7.5	7.5	8	
8	8	8	8	8	8	8	8	8	8.5	8.5	
a											c

Table II.35 4th floor and above

Tables II.34 and II.35 are correct also for all overlying floors.

II.14 Direct operation [-1] for question numbers ?a.

Let's fill the Table II.40:

[-1]	?1	?2	?3	?4	?b
?1					
?2					
?3					
?4					
?a					c

Table II.40

In the meantime, let's discuss one rule of calculation with right operand "1". We may do it in the course of filling in the Table II.40. Proceed:

$$?1 [-1] ?1 = 1 [-1] 1 =$$

There are two possible ways:

$$= 1 [0] 2 = 3.5; \quad (\text{II.14.1})$$

$$= 1 [-2] = 1; \quad (\text{II.14.2})$$

It is difficult to justify logically my choice, but - from my practice of calculations - I choose the first way. Resume:

$$?2 [-1] ?1 = ?2 [-1] 1 = ?2 [-2] = ?2; \quad (\text{II.14.7})$$

$$?3 [-1] ?1 = ?3 [-1] 1 = ?3 [-2] = ?3; \quad (\text{II.14.8})$$

$$?4 [-1] ?1 = ?4 [-1] 1 = ?4 [-2] = ?4; \quad (\text{II.14.9})$$

Now we have Table II.42:

[-1]	?1	?2	?3	?4	?b
?1	3.5				
?2	?2				
?3	?3				
?4	?4				
?a					c

Table II.42

Let's continue our calculations:

$$?1 [-1] ?2 = 1 \{-1\} ?2 = (((1 [-1] 1) [-1] 1) [-1] 1) [-1] 1) [-1] 1 \dots =$$

Apply the formula (II.14.1)

$$= (((3.5 [-1] 1) [-1] 1) [-1] 1) [-1] 1 \dots =$$

Apply Table II.26:

$$3.5 [-1] 1 = 4; \tag{II.14.10}$$

$$= ((4 [-1] 1) [-1] 1) [-1] 1 \dots =$$

Apply Table II.26 and (II.13.76):

$$4 [-1] 1 = 4; \tag{II.14.11}$$

$$= (4 [-1] 1) [-1] 1 \dots =$$

Again apply (II.14.11):

$$= 4 [-1] 1 \dots = \dots = \dots =$$

We will apply (II.14.11) ?2 times and got:

$$= 4; \tag{II.14.12}$$

$$\text{That is: } ?1 [-1] ?2 = 4; \tag{II.14.12}$$

Further

$$?1 [-1] ?3 = 1 [-1] ?3 = (((1 [-1] 1) [-1] 1) [-1] 1) [-1] 1) [-1] 1 \dots =$$

Apply (II.14.1):

$$= (((3.5 [-1] 1) [-1] 1) [-1] 1) [-1] 1 \dots =$$

Apply (II.14.10):

$$= ((4 [-1] 1) [-1] 1) [-1] 1 \dots =$$

Apply (II.14.11):

$$= (4 [-1] 1) [-1] 1 \dots =$$

Again apply (II.14.11):

$$= 4 [-1] 1 \dots = \dots = \dots =$$

We will apply (II.14.11) ?3 times and get:

$$= 4; \tag{II.14.13}$$

$$\text{That is: } ?1 [-1] ?3 = 4; \tag{II.14.13}$$

Similarly, we obtain:

$$?1 [-1] ?4 = 4; \tag{II.14.14}$$

And now we have filled three more boxes in the Table II.42, and got the Table II.43:

[-1]	?1	?2	?3	?4	?b
?1	3.5	4	4	4	
?2	?2				
?3	?3				
?4	?4				
?a					c

Table II.43 "skeleton"

We got the “skeleton”. Let's ramp up "meat" on it so, that it was similar to Table II.13.

So we filled in the empty cells in the Table II.43 and obtained the Table II.44:

[-1]	?1	?2	?3	?4	?b
?1	3.5	4	4	4	
?2	?2	?2	?2	?2	
?3	?3	?3	?3	?3	
?4	?4	?4	?4	?4	
?a					c

Table II.44

Recall that Table II.44 represents the direct operation [-1] for the question numbers ?a.

CHAPTER III

III.2 Numbers N1 and n1.

Back in 1984, I tried to extract new numbers from new equations. Also I tried the equation:

$$x + 1 = x; \quad (\text{III.2.1})$$

But then I couldn't find a use for it. And it wasn't until a few years later, that I realized, that this equation describes an infinitely large number. And I designated it as N1:

$$N1 + 1 = N1; \quad (\text{III.2.2})$$

From this definition it follows:

$$N1 - N1 = 1; \quad (\text{III.2.3})$$

$$N1 * (1 - 1) = 1; \quad (\text{III.2.4})$$

We introduce a new value n1 as follows: $N1 * n1 = 1;$ (III.2.5)

It follows: $n1 = 1 / N1;$ (III.2.6)

That is, n1 is an infinitely small number.

$$(\text{III.2.5}) + (\text{III.2.4}) = (\text{III.2.7})$$

$$n1 = 1 - 1; \quad (\text{III.2.7})$$

III.3 Numbers N2 and n2.

Using the number N1 we construct a new infinitely large number N2:

$$N2 = 2 \wedge N1 = 2 [3] N1; \quad (\text{III.3.1})$$

$$N2 > N1; \quad (\text{III.3.2})$$

We construct a new number n2 as follows: $n2 = 1 / N2;$ (III.3.3)

It is an infinitesimal number, and it is less than n1

$$n2 < n1; \quad (\text{III.3.4})$$

This follows from (III.2.6), (III.3.3), (III.3.2)

III.4 Numbers N3 and n3.

Using the number N2 we construct a new infinitely large number N3:

$$N3 = 2 \uparrow N2 = 2 \uparrow [3] N2; \quad (\text{III.4.1})$$

$$N3 > N2; \quad (\text{III.4.2})$$

We construct a new number n3 as follows: $n3 = 1 / N3$; (III.4.3)

It is an infinitesimal number, and it is less than n2

$$n3 < n2; \quad (\text{III.4.4})$$

This follows from (III.4.3), (III.3.3), (III.4.2)

III.5 Numbers NA and nA.

Let's take a series of integers A (... -3, -2, -1, 0, 1, 2, 3, ...) and build a series of infinitely large numbers NA (based on the already constructed numbers N1, N2, N3):

$$N(A + 1) = 2 \uparrow NA = 2 \uparrow [3] NA; \quad (\text{III.5.1})$$

$$N(A + 1) > NA; \quad (\text{III.5.2})$$

When A = 3, we obtain a definition for N4:

$$N4 = 2 \uparrow [3] N3; \quad (\text{III.5.3})$$

For A = 4, we obtain a definition for N5 through N4:

$$N5 = 2 \uparrow [3] N4; \quad (\text{III.5.4})$$

When taking increasing indices A, increasing numbers NA are obtained.

Now let's take A = 0 and got:

$$N1 = 2 \uparrow [3] N0; \quad (\text{III.5.5})$$

This equation defines N_0 by N_1 .

Take $A = -1$ and get:

$$N_0 = 2 [3] N(-1); \quad (III.5.6)$$

Here $N(-1)$ is defined by N_0 .

When $A = -2$, we obtained the definition of $N(-2)$ by $N(-1)$:

$$N(-1) = 2 [3] N(-2); \quad (III.5.7)$$

Taking $A = -3, -4, -5, \dots$ we will produce new positive infinitely large numbers $N(-3), N(-4), N(-5), \dots$ and all of them will fit in the interval from N_1 to 1.

Now let's build a series of infinitesimal numbers n_A :

$$n_A = 1 / N_A; \quad (III.5.8)$$

$$n(A + 1) < n_A; \quad (III.5.9)$$

III.6 Topology of the number line for N-numbers.

The representation of a number line is usually given as follows:

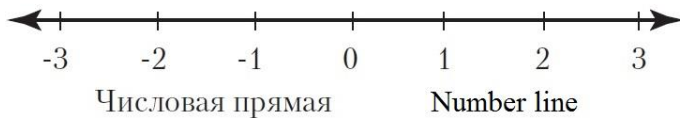


Figure 1

Taking into account the infinitely large N_A and the infinitely small n_A numbers, it gives us the following two figures:

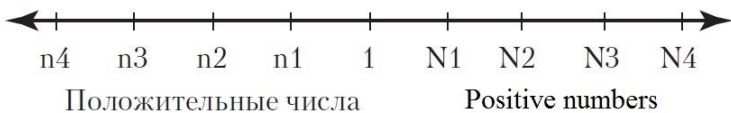


Figure 2

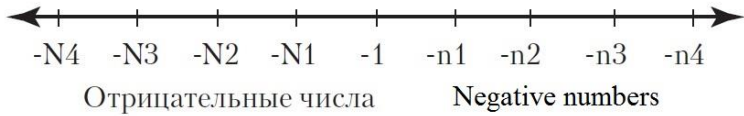
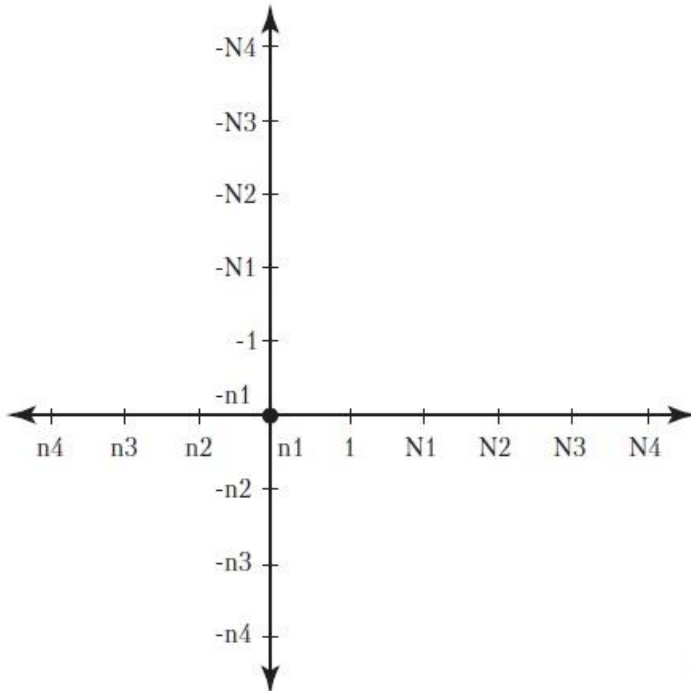


Figure 3

From (III.2.7) should: -- $n1 \text{ .} = - (1 - 1) = -1 + 1 = 1 - 1 = n1$; (III.6.1)

- $n1 = n1$; (III.6.2)

Physically, $n1$ and $-n1$ lie on different number lines, but topologically they coincide. In other words, these two numeric lines intersect at this point. The result is in Figure 4.



IV

Figure 4

(III.2.5): $N1 * n1 = 1;$

(III.6.2): $- n1 = n1;$

$(- N1) * n1 = - (N1 * n1) = N1 * (-n1) = N1 * n1;$ (III.6.3)

It follows: $- N1 = N1;$ (III.6.4)

So these two infinitely large numbers are also topologically identical. In some multidimensional space, these two numerical "lines" bend to intersect at this point. Result is in figure 5:

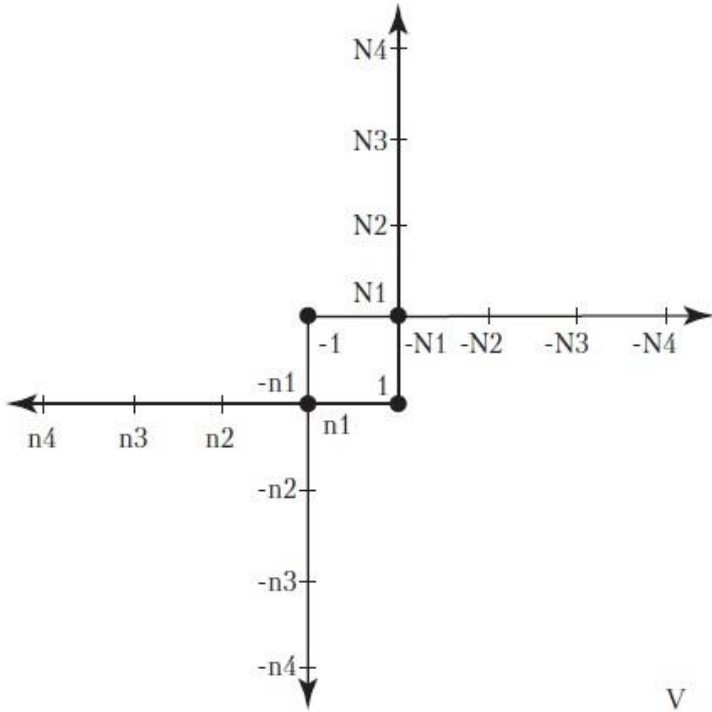


Figure 5

((III.3.1): $N2 = 2^{N1}$; (III.3.3): $n2 = 1 / N2$; (III.6.4): $-N1 = N1$;

$N2 = 2^{N1} = 2^{-N1} = 1 / (2^{N1}) = 1 / N2 = n2$; (III.6.5)

$N2 = n2$; (III.6.6)

So an infinitely large number coincides with an infinitely small number! (opposites coincide - the main law of dialectics). The numbers $N2$ and $n2$ lie on the same straight line of positive numbers, but in different places. And in order to they coincide topologically, this "direct" line need to curve in some multidimensional space so, that they are superimposed on each other. The result is figure 6.

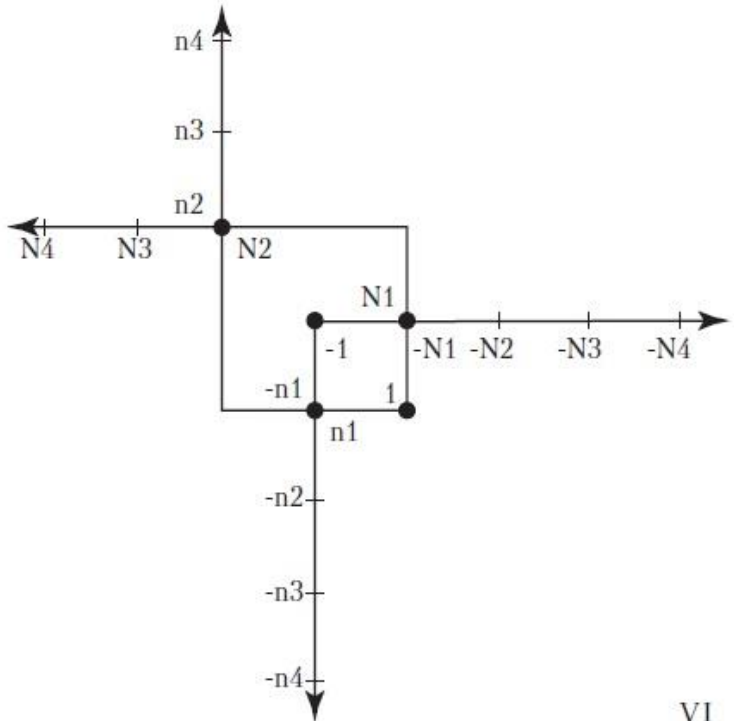


Figure 6

If the equation (III.6.6) is multiplied by -1, we got:

$$- N2 = - n2; \qquad (III.6.7)$$

This pair of numbers (- N2 and -n2) lies on one "direct line" of negative numbers at its different ends. But they are topologically the same. To do this, the "direct line" of negative numbers should be curved in a special multidimensional space so that it has a self-intersection. The result is in figure 7.

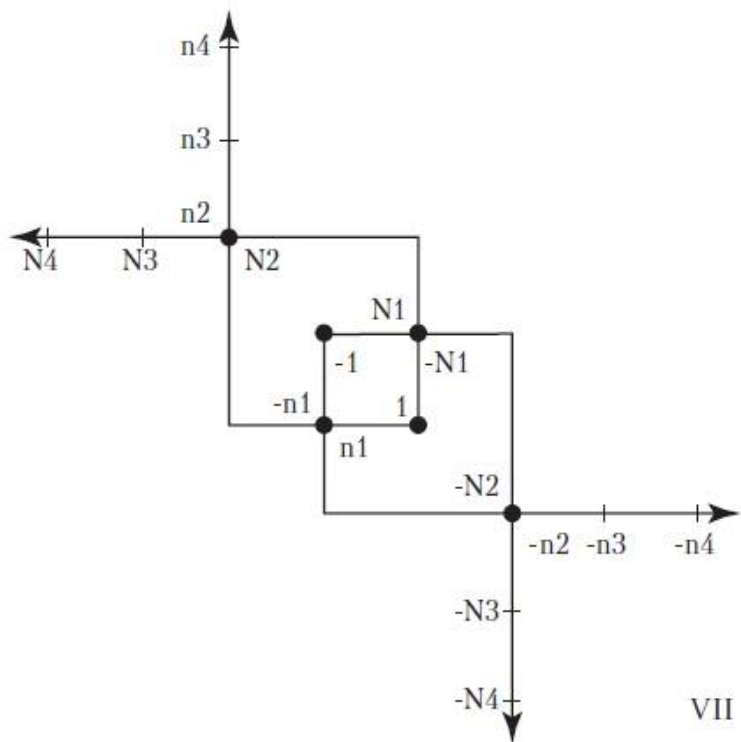


Figure 7

It is possible to obtain new pairs of topologically coinciding different numbers infinitely. Each new pair is based on the

previous pair. But it is increasingly difficult to display these intersections and self-intersections of numerical "direct lines" in flat drawings. Therefore, here are some pairs of topologically matching numbers without their graphical representation in two-dimensional space (three-dimensional space is required here):

$$N3 = 2 \ [3] \ N2 = 2 \ [3] \ n2 = e \ [3] \ (n2 * \ln(2)) =$$

$$= 1 + n2 * \ln(2) + \dots ; \quad (\text{III.6.8})$$

$$n3 = 1 / N3 = 1 - n2 * \ln(2) + \dots ; \quad (\text{III.6.9})$$

$$- N3 = -1 - n2 * \ln(2) - \dots ; \quad (\text{III.6.10})$$

$$- n3 = -1 + n2 * \ln(2) - \dots ; \quad (\text{III.6.11})$$

$$N4 = 2 \ [3] \ N3 = 2 \ [3] \ (1 + n2 * \ln(2) + \dots) =$$

$$= 2 * 2 \ [3] \ (n2 * \ln(2) + \dots) = 2 * e \ [3] \ (\ln(2) * (n2 * \ln(2) + \dots)) =$$

$$= 2 * (1 + n2 * \ln(2) * \ln(2) + \dots) =$$

$$= 2 + n2 * 2 * \ln(2) * \ln(2) + \dots ; \quad (\text{III.6.12})$$

$$N4 = 2 + n2 * 2 * \ln(2) * \ln(2) + \dots ; \quad (\text{III.6.12})$$

$$n4 = 1 / N4 = 1/2 - n2 * 1/2 * \ln(2) * \ln(2) - \dots ; \quad (\text{III.6.13})$$

$$-N4 = -2 - n2 * 2 * \ln(2) * \ln(2) - \dots ; \quad (\text{III.6.14})$$

$$- n4 = -1/2 + n2 * 1/2 * \ln(2) * \ln(2) + \dots ; \quad (\text{III.6.15})$$

We now see that such a simple number line has a very complex topological structure that requires the introduction of a multidimensional space for its complete representation.

III.7 How to travel on a number line using its topological structure.

The simplest journey (figure 7): move from point [1] to point [(N1) (- N1)]. Stop at this point. And turn your face to the left. Then move forward to [-1] and (non-stop) to [(- n1) (n1)]. Here you should stop and turn to the left. Then move straight to the point [1]. So you will make a simple journey and return home.

If at each intersection of the number line you choose the middle path, or just do not stop at all, then you will move along a one-dimensional straight line.

III.8 The connection of ?-numbers with N-numbers. Simplest solution.

We now solve the equation:

$$1 [3] ?a = 1 ^ ?a = a; \tag{III.8.1}$$

At first we try to find the simplest solution.

$$1 [3] ?a = e [3] ((\ln 1) [2] ?a); \tag{III.8.2}$$

$$a = e [3] (\ln a); \tag{III.8.3}$$

$$(\ln 1) [2] ?a = \ln a; \tag{III.8.4}$$

$$e [3] 0 = 1; \ln 1 = 0; \tag{III.8.5}$$

$$0 [2] ?a = \ln a; \tag{III.8.6}$$

Now recall the N-numbers:

$$(III.2.2): \quad N1 + 1 = N1;$$

$$(III.2.7): \quad n1 = 1 - 1 = 0;$$

$$(III.2.5): \quad N1 * n1 = 1;$$

Let's multiply the equation (III.8.6) by N1:

$$N1 * 0 * ?a = N1 * (\ln a); \quad (III.8.7)$$

Applying (III.2.7), and (III.2.5) we get:

$$?a = N1 * (\ln a); \quad (III.8.8)$$

III.9 The connection of ?-numbers with N-numbers. General solution.

Now we solve the equation:

$$1 [3] ?a = 1 ^ ?a = a; \quad (III.9.1)$$

in more general way.

$$1 [3] ?a = e [3] ((\ln 1)^*?a + i*2*PI*m); \quad (III.9.2)$$

$$m = \dots, -2, -1, 0, 1, 2, \dots \quad (III.9.3)$$

$$a = e [3] \ln a = e [3] (\ln a + i*2*PI*q); \quad (III.9.4)$$

$$q = \dots, -2, -1, 0, 1, 2, \dots \quad (III.9.5)$$

$$(\ln 1)^*?a + i*2*PI*m = \ln a + i*2*PI*q; \quad (III.9.6)$$

$$k = q - m = \dots, -2, -1, 0, 1, 2, \dots; \ln 1 = 0; \quad (III.9.7)$$

$$0^*?a = \ln a + i*2*PI*k \quad (III.9.8)$$

Now recall the N-numbers:

$$(III.2.2): \quad N1 + 1 = N1;$$

$$(III.2.7): \quad n1 = 1 - 1 = 0;$$

$$(III.2.5): \quad N1 * n1 = 1;$$

Let's multiply the equation (III.9.8) on N1:

$$N1*0*?a = N1*(\ln a + i*2*PI*k); \quad (III.9.9)$$

$$N1*0*?a = 1*?a = ?a; \quad (III.9.10)$$

$$?a = N1*(\ln a + i*2*PI*k); \quad (III.9.11)$$

$$i*i = -1; PI = 3.14; k = \dots, -2, -1, 0, 1, 2, \dots; \quad (III.9.12)$$

This is more general solution for (III.8.1) then (III.8.8).

III.10 Results.

The birth of numbers from natural numbers to complex numbers was studied in this paper. The law of the construction new numbers on the basis of already known ones is found. On its basis there were built "the fourth numbers" and "the fifth numbers". "Fourth numbers" means that these numbers were constructed by inverse operations to the fourth direct operation [4]. Also "Fifth numbers" means that these numbers were constructed by inverse operations to the fifth direct operation [5].

In these constructions we had to enter numbers-strings (when there are several numbers - separated by commas - in one cell).

Operations on numbers play a significant role. All operations are divided into direct, and reverse to direct. Summation, multiplication, exponentiation ([1], [2], [3]) – are direct operations. Subtraction, division, root, logarithm - inverse operations to these direct operations. There were built several new direct operations: [4], [5], [0], [-1]. Two inverse operations were constructed for each of [4], [5] and [0].

A significant drawback is the lack of equations for the construction of [0] operation by [1] operation (summation), as well as [-1] by [0]. This deficiency needs be compensated by imagination and to

complete Tables of operations by some principles. Not the best way, but there's no other way.

When searching for new numbers, it turned out that the third direct operation (exponentiation) [3] did not exhaust all its possibilities to create new numbers. Its second inverse operation (logarithm) [3]2 gave rise to infinitely large numbers $?2$, $?3$, $?4$, ... These numbers we called $?-numbers$.

N-numbers (infinitely large and infinitely small numbers) are constructed in a completely different way. The complex topological structure of a usual number line is shown with the help of N-numbers.

There was found an explicit connection of $?-numbers$ with N-numbers.

Here I will make a small remark. In the course of work on the text each sign has a mental pronounce. With the appearance of question numbers I was tormented by the mental pronunciation of "question mark". Over time, I replaced this phrase with one word: "question". And then spontaneously became to call "?" by the simple word 'XI'. At first I resisted it because I knew it was the name of one Greek letter. But since I didn't have Greek letters in the text, I accepted it. Maybe someone in reading of my text will come in handy this replacement pronunciation.

09.05.2019