Union of two arithmetic sequences. Formulas for rational and real progressions. (3)

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Abstract. We will derive the formulas for the N-th element of the union of two arithmetic progressions with rational and real common differences.

1 Notation and preliminary findings

In this paper we will keep the entire notation from the last version of the previous papers [1] and [2]. The general principles regarding the construction of the union and the derivation of formulas remain in force.
Other symbols:
$\mathbb{N}^+$, $\mathbb{Q}^+$, $\mathbb{R}^+$ are positive parts of domains.
In this article the unions arguments will be as follows: $a,b\in\mathbb{N}^+$, $s,t\in\mathbb{R}^+$(or $\mathbb{N}^+$, $\mathbb{Q}^+$, if so marked).

2 Formula with commensurable common differences

Definition 2.1 Commensurability
Let $s,t\in\mathbb{R}^+$, $a,b\in\mathbb{N}^+$. If exist such a,b that $\frac{t}{s}=\frac{b}{a}$ then s,t are commensurable numbers. Otherwise s,t are incommensurable.

For any formula with commensurable common divisors s,t:

$$u_N(s,t)=su_N\left(1,\frac{t}{s}\right)=su_N\left(1,\frac{b}{a}\right)$$

We see that formula does not depend on the type s,t, but on their commensurability only. Continuing:

$$u_N(s,t)=\frac{s}{a}u_N(a,b) \tag{1}$$

where $u_N(a,b)$ is formula [2](5) with natural common differences a,b. This is the simplest version of the formula. Complete notation:

$$u_N(s,t)=\frac{s}{a}\max\left(a\left\lceil\frac{b}{a+b}\left(\left\lfloor\frac{N(a+b)}{a+b-\Theta}\right\rfloor-\frac{a}{b}\right)\right\rceil,b\left\lfloor\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}+1\right\rfloor\right\rfloor\right) \tag{2}$$

Special cases:

1. If $s,t\in\mathbb{N}^+$ then $a=s, b=t$ then $u_N(s,t)=u_N(a,b)$.
2. If $s,t\in\mathbb{Q}^+$, $p_s, q_s, p_t, q_t\in\mathbb{N}^+$ and $s=\frac{p_s}{q_s}$, $t=\frac{p_t}{q_t}$:

$$u_N(s,t)=u_N\left(\frac{p_s}{q_s},\frac{p_t}{q_t}\right)=u_N\left(\frac{p_sq_t}{q_sq_t},\frac{p_tq_s}{q_sq_t}\right)=\frac{1}{q_sq_t}u_N\left(p_sq_t,p_tq_s\right)$$

With $a=p_sq_t$, $b=p_tq_s$, $d=q_sq_t$ we get:

$$u_N(s,t)=\frac{1}{d}u_N\left(a,b\right) \tag{3}$$

Example for $s=\frac{8}{11}$, $t=\frac{12}{14}$:
First we must calculate a,b,d: $a=8\cdot14=112$, $b=12\cdot11=132$, $d=11\cdot14=154$
$u_N\left(\frac{8}{11},\frac{12}{14}\right)=\frac{1}{154}u_N\left(112,132\right)$
This can be simplified to $\frac{1}{77}u_N\left(56,66\right)$, although it is not necessary, the result will be the same.

3. If $s,t\in\mathbb{R}^+$ are irrational and commensurable then, after calculating a,b we will use (1).

Example for $s=\frac{8}{11}\pi$, $t=\frac{12}{14}\pi$:

$\frac{t}{s}=\frac{\frac{12}{14}\pi}{\frac{8}{11}\pi}=\frac{12\cdot11}{8\cdot14}=\frac{132}{112} \;\rightarrow\; b=132,\, a=112$

$u_N\left(\frac{8}{11}\pi,\frac{12}{14}\pi\right)=\frac{\frac{8}{11}\pi}{112}u_N\left(112,132\right)=\frac{\pi}{154}u_N\left(112,132\right)$

3 Formula with uncommensurable common differences

Union with uncommensurable common differences with beginnig in the zero has only one common term of union progressions (equal to zero). This means that the whole union is one group G_0. So we'll use relative formulas from [1].

Substituting r in [1](5) with [1](6) and natural a,b with uncommensurable s,t we get:

$$u_n(s,t)=\begin{cases} t\left\lfloor\frac{(n+1)s}{s+t}\right\rfloor \\ s\left(n-\left\lfloor\frac{(n+1)s}{s+t}\right\rfloor\right)\end{cases}=\begin{cases} t\left\lfloor\frac{(n+1)s}{s+t}\right\rfloor & \text{for: } c=0 \\ s\left\lceil\frac{nt-s}{s+t}\right\rceil & \text{for: } c>0\end{cases}$$

Now, we use [2](5):

$$u_n(s,t)=\max\left(s\left\lceil\frac{nt-s}{s+t}\right\rceil,t\left\lfloor\frac{(n+1)s}{s+t}\right\rfloor\right) \tag{4}$$

This is the simplest version of the formula.

4 Unified version of the formula with any real common differences

4.1 Generalized greatest common divisor

From generalized definition of *gcd* for commensurable numbers [3] using Thomae's function $\mathcal{T}$ [4] we get:

$$\gcd(s,t)=s\mathcal{T}\left(\frac{t}{s}\right)=\begin{cases} s\mathcal{T}\left(\frac{b}{a}\right) & \text{for commensurable } s,t \\ 0 & \text{for uncommensurable } s,t \end{cases}$$

Upper formula:

$$s\mathcal{T}\left(\frac{t}{s}\right)=s\mathcal{T}\left(\frac{\frac{b}{\gcd(a,b)}}{\frac{a}{\gcd(a,b)}}\right)=\frac{s}{\frac{a}{\gcd(a,b)}}=\frac{s}{a}\gcd(a,b)$$

finally:

$$\gcd(s,t)=\begin{cases} \frac{s}{a}\gcd(a,b) & \text{for commensurable } s,t \\ 0 & \text{for uncommensurable } s,t \end{cases} \tag{5}$$

4.2 Generalized symbol Θ

Formula [2](5) contains the greatest common divisor Θ. In [1] we set natural Θ for natural union. According to generalized greatest common divisor we generalize Θ definition:

$$\Theta=\gcd(s,t)=\begin{cases} \frac{s}{a}\gcd(a,b) & \text{for commensurable } s,t \\ 0 & \text{for uncommensurable } s,t \end{cases} \tag{6}$$

Conclusions:

1. If $s,t\in\mathbb{N}^+$ then $a=s, b=t$ then $\Theta=\gcd(a,b)$.
2. If $s,t\in\mathbb{Q}^+$, $p_s,q_s,p_t,q_t\in\mathbb{N}^+$ and $s=\frac{p_s}{q_s}$, $t=\frac{p_t}{q_t}$:

$$\gcd(s,t)=\gcd\left(\frac{p_s}{q_s},\frac{p_t}{q_t}\right)=\gcd\left(\frac{p_sq_t}{q_sq_t},\frac{p_tq_s}{q_sq_t}\right)=\frac{1}{q_sq_t}\gcd\left(p_sq_t,p_tq_s\right)$$

   With $a=p_sq_t$, $b=p_tq_s$, $d=q_sq_t$ we get:

$$\Theta=\frac{1}{d}\gcd\left(a,b\right)$$

3. If $s,t\in\mathbb{R}^+$ are irrational and commensurable then $\Theta=\frac{s}{a}\gcd(a,b)$
4. If $s,t\in\mathbb{R}^+$ and at least one of s,t is irrational and s,t are incommensurable then $\Theta=0$.

4.3 Formula with commensurable common divisors

We start with (1) substituting $u(a,b)$ with [2](5):

$$u_N(s,t)=\frac{s}{a}\max\left(a\left\lceil\frac{b}{a+b}\left(\left\lfloor\frac{N(a+b)}{a+b-\Theta}\right\rfloor-\frac{a}{b}\right)\right\rceil,b\left\lfloor\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}+1\right\rfloor\right\rfloor\right)$$

$$=\max\left(\frac{s}{a}a\left\lceil\frac{\frac{s}{a}b}{\frac{s}{a}a+\frac{s}{a}b}\left(\left\lfloor\frac{N(\frac{s}{a}a+\frac{s}{a}b)}{\frac{s}{a}a+\frac{s}{a}b-\frac{s}{a}\gcd(a,b)}\right\rfloor-\frac{\frac{s}{a}a}{\frac{s}{a}b}\right)\right\rceil,\frac{s}{a}b\left\lfloor\frac{\frac{s}{a}a}{\frac{s}{a}a+\frac{s}{a}b}\left\lfloor\frac{N(\frac{s}{a}a+\frac{s}{a}b)}{\frac{s}{a}a+\frac{s}{a}b-\frac{s}{a}\gcd(a,b)}+1\right\rfloor\right\rfloor\right)$$

Substituting (5) we get:

$$u_N(s,t)=\max\left(s\left\lceil\frac{t}{s+t}\left(\left\lfloor\frac{N(s+t)}{s+t-\gcd(s,t)}\right\rfloor-\frac{s}{t}\right)\right\rceil,t\left\lfloor\frac{s}{s+t}\left\lfloor\frac{N(s+t)}{s+t-\gcd(s,t)}+1\right\rfloor\right\rfloor\right)$$

and with generalized Θ:

$$u_N(s,t)=\max\left(s\left\lceil\frac{t}{s+t}\left(\left\lfloor\frac{N(s+t)}{s+t-\Theta}\right\rfloor-\frac{s}{t}\right)\right\rceil,t\left\lfloor\frac{s}{s+t}\left\lfloor\frac{N(s+t)}{s+t-\Theta}+1\right\rfloor\right\rfloor\right) \tag{7}$$

4.4 Formula with uncommensurable common divisors

We transform (4):

$$u_n(s,t)=\max\left(s\left\lceil\frac{nt-s}{s+t}\right\rceil,t\left\lfloor\frac{(n+1)s}{s+t}\right\rfloor\right)$$

$$=\max\left(s\left\lceil\frac{t}{s+t}\left(n-\frac{s}{t}\right)\right\rceil,t\left\lfloor\frac{s}{s+t}(n+1)\right\rfloor\right)$$

$$=\max\left(s\left\lceil\frac{t}{s+t}\left(\lfloor n\rfloor-\frac{s}{t}\right)\right\rceil,t\left\lfloor\frac{s}{s+t}(\lfloor n\rfloor+1)\right\rfloor\right)$$

$$=\max\left(s\left\lceil\frac{t}{s+t}\left(\left\lfloor\frac{n(s+t)}{s+t}\right\rfloor-\frac{s}{t}\right)\right\rceil,t\left\lfloor\frac{s}{s+t}\left(\left\lfloor\frac{n(s+t)}{s+t}\right\rfloor+1\right)\right\rfloor\right)$$

hence with $\Theta=0$ and $n=N$ we get:

$$u_N(s,t)=\max\left(s\left\lceil\frac{t}{s+t}\left(\left\lfloor\frac{N(s+t)}{s+t+\Theta}\right\rfloor-\frac{s}{t}\right)\right\rceil,t\left\lfloor\frac{s}{s+t}\left(\left\lfloor\frac{N(s+t)}{s+t+\Theta}\right\rfloor+1\right)\right\rfloor\right) \tag{8}$$

4.5 Unifed formula

Formulas (7), (8) are identical and both cover the entire $\mathbb{R}^+$ domain, hence they are unified formula. Formula [2](5) is a special case with natural s,t noted as a,b.

It is worth noting that unifed (8) is more difficult to calculate because it usually contains many fractions with real nominators and denominators, while (2) and (4) have natural numbers in these places.

For all formulas (2), (4) and (8) with at least one non-natural s or t, it is necessary to determine their commensurability and for the commensurate s,t calculation a,b.

References

[1] ZIELIŃSKI, W.: *Union of two arithmetic sequences - Basic calculation formula (1)*, viXra:1712.0636.

[2] ZIELIŃSKI, W.: *Union of two arithmetic sequences - Basic calculation formula (2)*, viXra:1807.0477.

[3] https://en.wikipedia.org/wiki/Greatest_common_divisor#Other_methods

[4] https://en.wikipedia.org/wiki/Thomae's_function