A biquaternion based generalization of the Dirac current into a Dirac current probability tensor with closed system condition

E.P.J. de Haas

Nijmegen
The Netherlands

haas2u@gmail.com

By taking spin away from particles and putting it in the metric, thus following Dirac's vision, I start my attempt to formulate an alternative math-phys language, biquaternion based and incorporating Clifford algebra. At the Pauli level of two by two matrix representation of biquaternion space, a dual base is applied, a space-time and a spin-norm base. The chosen space-time base comprises what Synge called the minquats and in the same spirit I call their spin-norm dual the pauliquats. Relativistic mechanics, electrodynamics and quantum mechanics are analyzed using this approach, with a generalized Poynting theorem as the most interesting result. Then moving onward to the Dirac level, the Möbius doubling of the minquat/pauliquat basis allows me to formulate a generalization of the Dirac current into a Dirac probability/field tensor with connected closed system condition. This closed system condition includes the Dirac current continuity equation as its time-like part. A generalized Klein Gordon equation that includes this Dirac current probability tensor is formulated and analyzed. The usual Dirac current based Lagrangians of relativistic quantum mechanics are generalized using this Dirac probability/field tensor. The Lorentz transformation properties the generalized equation and Lagrangian is analyzed.

Keywords: Relativistic Quantum Mechanics, Electron, Quaternions

1. Introduction

In this paper, biquaternions are used to deal with relativist physics, including mechanics, electrodynamics and quantum mechanics. The paper is divided in two parts, the first on the level of the Pauli spin matrices and the second on that of the Dirac matrices as a double version of the Pauli ones. The biquaternion basis is represented by two by two complex matrices in a dual space-time and spin-norm version.

The first part is rather familiar in the context of the many biquaternion approaches that have been proposed the last hundred or so years. Slight differences are present, making the biquaternion expose on the Pauli matrices level interesting on its own. Especially the generalized Poynting theorem shows the power of the biquaternion approach, once the usual pitfalls have been circumvented. The closed system condition, going back to von Laue, in biquaternion formulation is the red line throughout the paper. The Klein Gordon equation is one of its appearances. After concluding that the Pauli matrices in the Klein Gordon equation analysis aren't what is preventing a true relativistic approach, the culprit is found in the spinors.

In the second part, matrices and spinors are treated on the Dirac level in order to arrive at the relativistic core of quantum mechanics. The Dirac matrices are presented as dual versions of the Pauli ones. It is shown that spinors in the Dirac representation can only be Lorentz transformed on the Dirac level and not on the Pauli level. The analogy between Poynting's theorem and the closed system condition for the Dirac current is used to show that the Dirac current is in reality part of a Dirac probability/field tensor and that the closed system condition for this tensor contains the continuity equation for the Dirac current as it's time-like part. In the meanwhile, the Klein Gorden equation is generalized using this Dirac tensor environment. The Lorentz transformation of the Dirac matrices is being simplified due to the method developed in the first Pauli level part of the paper. This simplification allows for a much shorten and more transparent way to demonstrate the Lorentz invariance and covariance of the equations and products. The Lorentz transformation properties of spinors is critically assessed.

2. The Pauli spin level

2.1. *A complex quaternion basis for the metric*

Quaternions can be represented by the basis $(\hat{\mathbf{1}},\hat{\mathbf{I}},\hat{\mathbf{J}},\hat{\mathbf{K}})$. This basis has the properties $\hat{\mathbf{I}}\hat{\mathbf{I}} = \hat{\mathbf{J}}\hat{\mathbf{J}} = \hat{\mathbf{K}}\hat{\mathbf{K}} = -\hat{\mathbf{1}}$ and $\hat{\mathbf{1}}\hat{\mathbf{1}} = \hat{\mathbf{1}}$; $\hat{\mathbf{1}}\hat{\mathbf{K}} = \hat{\mathbf{K}}\hat{\mathbf{1}} = \hat{\mathbf{K}}$ for $\hat{\mathbf{I}},\hat{\mathbf{J}},\hat{\mathbf{K}}$; $\hat{\mathbf{I}}\hat{\mathbf{J}} = -\hat{\mathbf{J}}\hat{\mathbf{I}} = \hat{\mathbf{K}}$; $\hat{\mathbf{J}}\hat{\mathbf{K}} = -\hat{\mathbf{K}}\hat{\mathbf{J}} = \hat{\mathbf{I}}$; $\hat{\mathbf{K}}\hat{\mathbf{I}} = -\hat{\mathbf{I}}\hat{\mathbf{K}} = \hat{\mathbf{J}}$. A quaternion number in its summation representation is given by $A = a_0\hat{\mathbf{1}} + a_1\hat{\mathbf{I}} + a_2\hat{\mathbf{J}} + a_3\hat{\mathbf{K}}$, in which the a_μ are real numbers . Bi-quaternions or complex quaternions are given by $C = A + \mathbf{i}B = c_0\hat{\mathbf{1}} + c_1\hat{\mathbf{I}} + c_2\hat{\mathbf{J}} + c_3\hat{\mathbf{K}}$ in which the $c_\mu = a_\mu + \mathbf{i}b_\mu$ are complex numbers and the a_μ and b_μ are real numbers.

This standard biquaternion basis $(\hat{\mathbf{1}},\hat{\mathbf{I}},\hat{\mathbf{J}},\hat{\mathbf{K}})$ can be used to provide a basis for relativistic 4-D space-time. One way to do this is by making the time coordinate $c_0 = b_0\mathbf{i}$ complex only and the space coordinates $(c_1,c_2,c_3) = (a_1,a_2,a_3)$ real only, see [1]. Synge called these objects Minkowski quaternions or 'minquats', Silberstein called them 'physical quaternions' [1]. This however produces confusion regarding the time-like complex number as the physics gets more complicated. As Synge put it, *the intrusion of the imaginary element is not trivial* [1]. The main reason is that minquats do not form a closed algebra under addition and multiplication as a subspace inside the wider biquaternion space, due to the multiplication operation. The reason they are used nevertheless is given by Synge. *For the application of quaternions to Lorentz transformations it is essential to introduce Minkowskian quaternions* [1].

The use of minquats produces language conflicts with almost all of modern physics, that is Quantum Mechanics and Special and General Relativity, where the space-time coordinates are always a set of four real numbers. So for several reasons, I choose to insert the time-like complex number of $c_0 = b_0\mathbf{i}$ in the basis instead of in the coordinate. So by using $c_0\hat{\mathbf{1}} = b_0\mathbf{i}\hat{\mathbf{1}} = b_0\hat{\mathbf{T}}$ the space-time basis is then given by $(\hat{\mathbf{T}},\hat{\mathbf{I}},\hat{\mathbf{J}},\hat{\mathbf{K}})$. In this way, the coordinates are always a set of real numbers $\in \mathbb{R}$. The space-time basis $(\hat{\mathbf{T}},\hat{\mathbf{I}},\hat{\mathbf{J}},\hat{\mathbf{K}})$, (a disguised minquat basis) is not closed under multiplications, as already mentioned by Synge.

A set of four numbers $\in \mathbb{R}$ is given by $A^\mu = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$, or by $A_\mu = [a_0,a_1,a_2,a_3]$. In this way, the raising or lowering of the index doesn't change any sign. A^μ simply is the transpose of A_μ and vise versa. The biquaternion basis can be given as a set $\mathbf{K}_\mu = (\hat{\mathbf{T}}, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$. Then a biquaternion space-time vector can be written as the product

$$A = A_\mu \mathbf{K}^\mu = [a_0, a_1, a_2, a_3] \begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{bmatrix} = a_0 \hat{\mathbf{T}} + a_1 \hat{\mathbf{I}} + a_2 \hat{\mathbf{J}} + a_3 \hat{\mathbf{K}} \tag{1}$$

I apply this to the space-time four vector of relativistic bi-quaternion 4-space R with the four numbers $R_\mu = (r_0, r_1, r_2, r_3) = (ct, r_1, r_2, r_3)$, so with $r_0, r_1, r_2, r_3 \in \mathbb{R}$. Then I have the space-time four-vector as the product of the coordinate set and the basis $R = R_\mu \mathbf{K}^\mu = r_0 \hat{\mathbf{T}} + r_1 \hat{\mathbf{I}} + r_2 \hat{\mathbf{J}} + r_3 \hat{\mathbf{K}} = ct\hat{\mathbf{T}} + \mathbf{r} \cdot \mathbf{K}$. I use the three-vector analogue of $R_\mu \mathbf{K}^\mu$ when I write $\mathbf{r} \cdot \mathbf{K}$. In this notation I have $R^T = -r_0 \hat{\mathbf{T}} + r_1 \hat{\mathbf{I}} + r_2 \hat{\mathbf{J}} + r_3 \hat{\mathbf{K}} = -r_0 \hat{\mathbf{T}} + \mathbf{r} \cdot \mathbf{K}$ for the time reversal operator and $R^P = r_0 \hat{\mathbf{T}} - r_1 \hat{\mathbf{I}} - r_2 \hat{\mathbf{J}} - r_3 \hat{\mathbf{K}} = r_0 \hat{\mathbf{T}} - \mathbf{r} \cdot \mathbf{K}$ for the space point mirror or parity operator, with $R^P = -R^T$. In this notation, the transpose of a matrix will be given by the suffix 't', so $R^t_\mu = R^\mu$. The complex transpose of spinors is given by the dagger symbol, as in $\psi^\dagger$. The complex conjugate of a spinor is given by ψ^*. In this language, the operators T and P take the role of raising and lowering of indexes in the General Relativity convention.

2.2. *Matrix representation of the quaternion basis*

Quaternions can be represented by 2x2 matrices. Several representations are possible, but most of those choices result in conflict with the standard approach in physics. My choice is

$$\hat{\mathbf{1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \hat{\mathbf{T}} = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix}, \hat{\mathbf{I}} = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}, \hat{\mathbf{J}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \hat{\mathbf{K}} = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{2}$$

I can compare these to the Pauli spin matrices $\boldsymbol{\sigma}_P = (\sigma_x, \sigma_y, \sigma_z)$.

$$\boldsymbol{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \boldsymbol{\sigma}_y = \begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, \boldsymbol{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3}$$

If I exchange the σ_x and the σ_z, I get $\mathbf{K} = \mathbf{i}\boldsymbol{\sigma}$ and $\mathbf{K}_\mu = \mathbf{i}(\hat{\mathbf{1}}, \boldsymbol{\sigma})$. So in my use of the Pauli matrices, I use $\boldsymbol{\sigma} \equiv (\sigma_I, \sigma_J, \sigma_K) = (\sigma_z, \sigma_y, \sigma_x)$. So also $\hat{\mathbf{I}} = \hat{\mathbf{T}}\boldsymbol{\sigma}_I, \hat{\mathbf{J}} = \hat{\mathbf{T}}\boldsymbol{\sigma}_J, \hat{\mathbf{K}} = \hat{\mathbf{T}}\boldsymbol{\sigma}_K$ and $\boldsymbol{\sigma}_I = -\hat{\mathbf{T}}\hat{\mathbf{I}}, \boldsymbol{\sigma}_J = -\hat{\mathbf{T}}\hat{\mathbf{J}}, \boldsymbol{\sigma}_K = -\hat{\mathbf{T}}\hat{\mathbf{K}}$.

With this choice of matrices, a four-vector R can be written as

$$R = r_0 \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix} + r_1 \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{4}$$

This can be compacted into a matrix representation of R:

$$R = \begin{bmatrix} r_0\mathbf{i} + \mathbf{i}r_1 & r_2 + \mathbf{i}r_3 \\ -r_2 + \mathbf{i}r_3 & r_0\mathbf{i} - \mathbf{i}r_1 \end{bmatrix} = \begin{bmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{bmatrix} \tag{5}$$

with the numbers $R_{00}, R_{01}, R_{10}, R_{11} \in \mathbb{C}$.

2.3. *Multiplication of vectors as matrix multiplication adds pauliquats to minquats*

In general, multiplication of two vectors A and B follows matrix multiplication, with $A_{ij}, B_{ij}, C_{ij} \in \mathbb{C}$.

$$AB = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix} = C. \tag{6}$$

So we have

$$C = AB = \begin{bmatrix} A_{00}B_{00} + A_{01}B_{10} & A_{00}B_{01} + A_{01}B_{11} \\ A_{10}B_{00} + A_{11}B_{10} & A_{10}B_{01} + A_{11}B_{11} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix}. \tag{7}$$

Of course, vectors A, B and C can be expressed with their a_μ, b_μ, c_μ coordinates $\in \mathbb{R}$ and if we use them, after some elementary but elaborate calculations and rearrangements we arrive at the following result of the multiplication expressed in the a_μ, b_μ and c_μ as

$$\begin{aligned} c_0 &= -a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 \\ c_{1K} &= a_2b_3 - a_3b_2 \\ c_{2K} &= a_3b_1 - a_1b_3 \\ c_{3K} &= a_1b_2 - a_2b_1 \\ c_{1\sigma} &= -a_0b_1 - a_1b_0 \\ c_{2\sigma} &= -a_0b_2 - a_2b_0 \\ c_{3\sigma} &= -a_0b_3 - a_3b_0 \end{aligned} \tag{8}$$

In short, if we use the three-dimensional Euclidean dot and cross products of Euclidean three-vectors in classical physics, this gives for the coordinates

$$c_0 = -a_0b_0 - \mathbf{a}\cdot\mathbf{b}$$

$$\mathbf{c}_K = \mathbf{a}\times\mathbf{b} \tag{9}$$

$$\mathbf{c}_\sigma = -a_0\mathbf{b} - \mathbf{a}b_0 \tag{10}$$

And in the quaternion notation we get

$$C = AB = (-a_0b_0 - \mathbf{a}\cdot\mathbf{b})\hat{\mathbf{1}} + (\mathbf{a}\times\mathbf{b})\cdot\mathbf{K} + (-a_0\mathbf{b} - \mathbf{a}b_0)\cdot\boldsymbol{\sigma} \tag{11}$$

This matrix multiplication, in which I used $\hat{\mathbf{T}}\hat{\mathbf{T}} = -\hat{\mathbf{1}}$ and $\hat{\mathbf{T}}\mathbf{K} = -\boldsymbol{\sigma}$, implies that the space-time basis $(\hat{\mathbf{T}}, \mathbf{K})$ is being duplicated by a spin-norm basis $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ by the multiplication operation.

The physically relevant multiplications of two four-vectors are all in the form $C = A^T B$. The difference between AB and $A^T B$ is in the sign of a_0. This results in

$$C = A^T B = (a_0b_0 - \mathbf{a}\cdot\mathbf{b})\hat{\mathbf{1}} + (\mathbf{a}\times\mathbf{b})\cdot\mathbf{K} + (a_0\mathbf{b} - \mathbf{a}b_0)\cdot\boldsymbol{\sigma} \tag{12}$$

From this it follows that the physically relevant norm of a four-vector, from a relativistic perspective, is the product $A^T A$ and not the product AA:

$$C = A^T A = (a_0a_0 - \mathbf{a}\cdot\mathbf{a})\hat{\mathbf{1}} + (\mathbf{a}\times\mathbf{a})\cdot\mathbf{K} + (a_0\mathbf{a} - \mathbf{a}a_0)\cdot\boldsymbol{\sigma} = (a_0a_0 - \mathbf{a}\cdot\mathbf{a})\hat{\mathbf{1}} = c^2a_\tau^2\hat{\mathbf{1}}. \tag{13}$$

The main quadratic form of the metric is $dR^T dR = (c^2dt^2 - d\mathbf{r}^2)\hat{\mathbf{I}} = c^2 d\tau^2 \hat{\mathbf{I}} = ds^2\hat{\mathbf{I}}$ with $ds = cd\tau$. The quadratic giving the distance of a point R to the origin of its reference system is given by $R^T R = (c^2t^2 - \mathbf{r}^2)\hat{\mathbf{I}} = c^2\tau^2\hat{\mathbf{I}} = s^2\hat{\mathbf{I}}$ with $s = c\tau$.

The multiplication of two four vectors can also be arranged as the multiplication of two tensors, a coordinate tensor times a metric tensor using that

$$(A_\mu \mathbf{K}^\mu)^T B_\nu \mathbf{K}^\nu = A_\mu B^\nu (\mathbf{K}_\mu)^T \mathbf{K}^\nu = C_\mu{}^\nu \mathbf{K}_\mu{}^\nu \tag{14}$$

with the metric tensor as

$$\mathbf{K}_\mu{}^\nu = (\mathbf{K}_\mu)^T \mathbf{K}^\nu = \begin{bmatrix} -\hat{\mathbf{T}}, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{bmatrix} = \tag{15}$$

$$\begin{bmatrix} -\hat{\mathbf{T}}\hat{\mathbf{T}} & \hat{\mathbf{I}}\hat{\mathbf{T}} & \hat{\mathbf{J}}\hat{\mathbf{T}} & \hat{\mathbf{K}}\hat{\mathbf{T}} \\ -\hat{\mathbf{T}}\hat{\mathbf{I}} & \hat{\mathbf{I}}\hat{\mathbf{I}} & \hat{\mathbf{J}}\hat{\mathbf{I}} & \hat{\mathbf{K}}\hat{\mathbf{I}} \\ -\hat{\mathbf{T}}\hat{\mathbf{J}} & \hat{\mathbf{I}}\hat{\mathbf{J}} & \hat{\mathbf{J}}\hat{\mathbf{J}} & \hat{\mathbf{K}}\hat{\mathbf{J}} \\ -\hat{\mathbf{T}}\hat{\mathbf{K}} & \hat{\mathbf{I}}\hat{\mathbf{K}} & \hat{\mathbf{J}}\hat{\mathbf{K}} & \hat{\mathbf{K}}\hat{\mathbf{K}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{1}} & -\sigma_I & -\sigma_J & -\sigma_K \\ \sigma_I & -\hat{\mathbf{1}} & -\hat{\mathbf{K}} & \hat{\mathbf{J}} \\ \sigma_J & \hat{\mathbf{K}} & -\hat{\mathbf{1}} & -\hat{\mathbf{I}} \\ \sigma_K & -\hat{\mathbf{J}} & \hat{\mathbf{I}} & -\hat{\mathbf{1}} \end{bmatrix}. \tag{16}$$

This multiplication product has a norm $\hat{\mathbf{1}}$ part, a space $\mathbf{K}$ part and a spin $\boldsymbol{\sigma}$ part. So the multiplication of two four vectors $A^T B = C$ has this multiplication matrix. The multiplication combines the properties of symmetric and anti-symmetric in one product.

The inevitable appearance of the spin-norm basis in the multiplication of two Synge minquats or Silberstein physical quaternions is why the minquats do not form a closed algebra under multiplication [1]. In my approach, the space-time basis $(\hat{\mathbf{T}}, \mathbf{K})$ doesn't form a closed algebra under multiplications, it needs a spin-norm complex dual $(\hat{\mathbf{T}}, \mathbf{K}) = \mathbf{i}(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ to cover all of biquaternion space, *while allowing real coordinates for* R_μ *and* P_μ *only in* $R_\mu \boldsymbol{K}^\mu$ *and* $P_\mu \boldsymbol{\sigma}^\mu$. The obligation, chosen freely in a Kantian way, to use real coordinates only produces the dual basis in a unique way.

The physical sphere, the cosmos so to speak, then obtains a dual space-time/spin-norm basis. This duality will prove to mirror real physics with electric charges or monopoles as part of space-time and hypothetical magnetic monopoles as spin-norm entities, if at all possible. Electric currents exist in real space-time $(\hat{\mathbf{T}}, \mathbf{K})$ and magnetic monopole currents can only, if at all, exist in the 'imaginary' spin-norm $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ sphere as will be shown further on in this paper. If Synge's minquats are $R_\mu \mathbf{K}^\mu$ biquaternions, then $P_\mu \boldsymbol{\sigma}^\mu$ are pauliquats. The sum of minquats and pauliquats cover the whole of biquaternion space. The multiplication of a minquat with a minquat produces a minquat and a pauliquat. Electric currents must be represented by minquats and magnetic current by pauliquats, if at all. Intrinsic spin is a pauliquat, its Lorentz dual intrinsic polarization is a minquat. The existence of minquats and pauliquats defies electromagnetic super-symmetry as is striven for by the magnetic monopole research community.

2.4. *The Lorentz transformation*

A normal Lorentz transformation between two reference frames connected by a relative velocity v in the $x-$ or $\hat{\mathbf{I}}$-direction, with the usual $\gamma = 1/\sqrt{1 - v^2/c^2}$, $\beta = v/c$ and $r_0 = ct$, can be expressed as

$$\begin{bmatrix} r'_0 \\ r'_1 \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} \gamma r_0 - \beta\gamma r_1 \\ \gamma r_1 - \beta\gamma r_0 \end{bmatrix}. \tag{17}$$

We want to connect this to our matrix representation of R as in Eq.(5) which gives

$$R'_{00} = \mathbf{i}r'_0 + \mathbf{i}r'_1 = \mathbf{i}\gamma r_0 - \mathbf{i}\beta\gamma r_1 + \mathbf{i}\gamma r_1 - \mathbf{i}\beta\gamma r_0 \tag{18}$$

$$R'_{11} = \mathbf{i}r'_0 - \mathbf{i}r'_1 = \mathbf{i}\gamma r_0 - \mathbf{i}\beta\gamma r_1 - \mathbf{i}\gamma r_1 + \mathbf{i}\beta\gamma r_0. \tag{19}$$

Now we want to introduce rapidity or hyperbolic Special Relativity in order to integrate Lorentz transformations into our matrix metric. In [2] I gave a brief history of rapidity in its relation to the Thomas precession and the geodesic precession. For this paper we only need elementary rapidity definitions. If we use the rapidity ψ as $e^{\psi} = \cosh\psi + \sinh\psi = \gamma + \beta\gamma$, the previous transformations can be rewritten as

$$R'_{00} = \mathbf{i}r'_0 + \mathbf{i}r'_1 = (\gamma - \beta\gamma)(\mathbf{i}r_0 + \mathbf{i}r_1) = R_{00}e^{-\psi} \tag{20}$$

$$R'_{11} = \mathbf{i}r'_0 - \mathbf{i}r'_1 = (\gamma + \beta\gamma)(\mathbf{i}r_0 - \mathbf{i}r_1) = R_{11}e^{\psi}. \tag{21}$$

As a result we have

$$R^L = \begin{bmatrix} R'_{00} & R'_{01} \\ R'_{10} & R'_{11} \end{bmatrix} = \begin{bmatrix} R_{00}e^{-\psi} & R_{01} \\ R_{10} & R_{11}e^{\psi} \end{bmatrix} = U^{-1}RU^{-1}. \tag{22}$$

In the expression $R^L = U^{-1}RU^{-1}$ we used the matrix U as

$$U = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix}. \tag{23}$$

But this means that we can write the result of a Lorentz transformation on R with a Lorentz velocity in the $\hat{\mathbf{I}}$-direction between the two reference systems as

$$R^L = r_0 \begin{bmatrix} \mathbf{i}e^{-\psi} & 0 \\ 0 & \mathbf{i}e^{\psi} \end{bmatrix} + r_1 \begin{bmatrix} \mathbf{i}e^{-\psi} & 0 \\ 0 & -\mathbf{i}e^{\psi} \end{bmatrix} + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{24}$$

This can be written as

$$R^L = r_0 U^{-1}\hat{\mathbf{T}}U^{-1} + r_1 U^{-1}\hat{\mathbf{I}}U^{-1} + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}} = r_0\hat{\mathbf{T}}^L + r_1\hat{\mathbf{I}}^L + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}}. \tag{25}$$

But because we started with Eq.(17), we now have two equivalent options to express the result of a Lorentz transformation

$$R^L = r'_0\hat{\mathbf{T}} + r'_1\hat{\mathbf{I}} + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}} = r_0\hat{\mathbf{T}}^L + r_1\hat{\mathbf{I}}^L + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}}, \tag{26}$$

either as a coordinate transformation or as a basis transformation.

This result only works for Lorentz transformation between v_x-, v_1- or $\hat{\mathbf{I}}$-aligned reference systems. Reference systems which do not have their relative Lorentz velocity aligned in the $\hat{\mathbf{I}}$-direction will have to be space rotated into such an alignment before the Lorentz transformation in the form $R^L = U^{-1}RU^{-1}$ is applied. In principle, such a rotation in order to achieve the $\hat{\mathbf{I}}$ alignment of the primary reference frame to a secondary reference frame is always possible as an operation prior to a Lorentz transformation. This unique alignment between two frames of reference S and S', needed to match the physics with the algebra, is analyzed by Synge in [1, p. 41-48] and focuses on the concept of a communal photon. The requirement of reference system alignment is also the reason for the appearance of the Thomas precession and the Thomas-Wigner rotation if the axis are not aligned; the notion that two Lorentz transformations in different directions in space can always be substituted by the subsequent application of one space rotation and one single Lorentz transformation, see [2]. The communal photon of Synge is the one for which the relativistic Doppler shift between S and S'results in $\nu' = \nu e^{\pm\psi}$. The minquat algebra requires inertial observers to align their principal axis along such a communal photon, my notation the $\hat{\mathbf{I}}$ axis.

The Lorentz transformation of the coordinates can be written as

$$(R^\mu)^L = \begin{bmatrix} r'_0 \\ r'_1 \\ r'_2 \\ r'_3 \end{bmatrix} = \Lambda_\nu^\mu R^\nu = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \gamma r_0 - \beta\gamma r_1 \\ \gamma r_1 - \beta\gamma r_0 \\ r_2 \\ r_3 \end{bmatrix}$$

So the Lorentz transformation of $R = R_\mu \mathbf{K}^\mu = \mathbf{K}_\mu R^\mu$ can be presented as

$$R^L = \mathbf{K}_\mu (R^\mu)^L = \mathbf{K}_\mu \Lambda_\nu^\mu R^\nu = (\mathbf{K}_\mu \Lambda_\nu^\mu) R^\nu = (\mathbf{K}_\nu)^L R^\nu = U^{-1}\mathbf{K}_\nu U^{-1} R^\nu = U^{-1}\mathbf{K}_\nu R^\nu U^{-1} = U^{-1} R U^{-1} \tag{27}$$

This implies the identity $\mathbf{K}_\mu \Lambda_\nu^\mu = U^{-1}\mathbf{K}_\nu U^{-1}$, an identity that isn't possible for the coordinates only. The matrix representation of the basis is key to this identity, because the relativistic Doppler factor $e^{\pm\psi}$ appears differently attached to the matrix elements. As is the $\hat{\mathbf{I}}$ alignment of the two involved reference frames during the Lorentz transformation. Given that $\mathbf{K}_\mu = \mathbf{i}\boldsymbol{\sigma}_\mu$, the identity $\mathbf{K}_\mu \Lambda_\nu^\mu = U^{-1}\mathbf{K}_\nu U^{-1}$ can also be seen as an instruction for the Lorentz transformation of the Pauli spin matrices as a norm-spin four set $\boldsymbol{\sigma}_\mu = (\hat{\mathbf{1}}, \boldsymbol{\sigma})$.

The Lorentz transformation of A^T is also interesting, due to the importance of the product $C = A^T B$ and therefore the Lorentz transformation C^L. Given the inverse Lorentz transformation as

$$A^{L^{-1}} \equiv UAU \tag{28}$$

one can prove

$$\left(A^T\right)^{L^{-1}} = U\left(A^T\right)U = \left(U^{-1}AU^{-1}\right)^T = \left(A^L\right)^T \tag{29}$$

and

$$\left(A^T\right)^L = U^{-1}\left(A^T\right)U^{-1} = (UAU)^T = \left(A^{L^{-1}}\right)^T. \tag{30}$$

The result $\left(A^L\right)^T = UA^TU$ will be used in several important derivations in this paper, when the Lorentz transformation of a product and the possible invariance or Lorentz covariance has to be investigated, as in the next example.

Given A and B in reference system S_1 and their product in S_1 as $C = A^T B$. Then in reference system S_2 one has A^L and B^L and their product $C^L = \left(A^L\right)^T B^L$. We then have

$$C^L = \left(A^L\right)^T B^L = \left(A^T\right)^{L^{-1}} B^L = U\left(A^T\right)UU^{-1}BU^{-1} = UA^TBU^{-1} = UCU^{-1}. \tag{31}$$

As a result, it is easy to prove that the quadratic $A^T A = c^2 a_\tau^2 \hat{\mathbf{1}}$ is Lorentz invariant. We have

$$\begin{aligned}(A^L)^T A^L = (A^T)^{-L} A^L = U A^T U U^{-1} A U^{-1} = U A^T A U^{-1} = \\ U(c^2 a_\tau^2)\hat{\mathbf{1}} U^{-1} = U U^{-1}(c^2 a_\tau^2)\hat{\mathbf{1}} = c^2 a_\tau^2 \hat{\mathbf{1}} = A^T A.\end{aligned} \tag{32}$$

So both quadratics $R^T R$ and $dR^T dR$ are Lorentz invariant scalars, as has been shown for every quadratic of four-vectors.

2.5. *Adding the dynamic vectors*

If we want to apply the previous to relativistic electrodynamics and to quantum physics, we need to develop the mathematical language further. We start by adding the most relevant dynamic four vectors. The basic definitions we use are quite common in the formulations of relativistic dynamics, see for example [3]. We start with a particle with a given three vector velocity as $\mathbf{v}$, a rest mass as m_0 and an inertial mass $m_i = \gamma m_0$, with the usual $\gamma = (\sqrt{1 - v^2/c^2})^{-1}$. We use the Latin suffixes as abbreviations for words, not for numbers. So m_i stands for inertial mass and U_p for potential energy. The Greek suffixes are used as indicating a summation over the numbers 0, 1, 2 and 3. So P_μ stands for a momentum four-vector coordinate row with components $(p_0 = \frac{1}{c} U_i, p_1, p_2, p_3)$. The momentum three-vector is written as $\mathbf{p}$ and has components (p_1, p_2, p_3).

We define the coordinate velocity four vector as

$$V = V_\mu \mathbf{K}^\mu = \frac{d}{dt} R_\mu \mathbf{K}^\mu = c\hat{\mathbf{T}} + \mathbf{v} \cdot \mathbf{K} = v_0 \hat{\mathbf{T}} + \mathbf{v} \cdot \mathbf{K}. \tag{33}$$

The proper velocity four vector on the other hand will be defined using the proper time $\tau = t_0$, with $t = \gamma t_0 = \gamma \tau$, as

$$U = U_\mu \mathbf{K}^\mu = \frac{d}{d\tau} R_\mu \mathbf{K}^\mu = \frac{d}{\frac{1}{\gamma} dt} R_\mu \mathbf{K}^\mu = \gamma V_\mu \mathbf{K}^\mu = u_0 \hat{\mathbf{T}} + \mathbf{u} \cdot \mathbf{K}. \tag{34}$$

The momentum four vector will be, at least when we have the symmetry condition $\mathbf{p} = m_i \mathbf{v}$,

$$P = P_\mu \mathbf{K}^\mu = m_i V_\mu \mathbf{K}^\mu = m_i V = m_0 U_\mu \mathbf{K}^\mu = m_0 U. \tag{35}$$

The four vector partial derivative $\partial = \partial_\mu \mathbf{K}^\mu$ will be defined using the coordinate four set

$$\partial_\mu = \left[-\frac{1}{c} \partial_t, \nabla_1, \nabla_2, \nabla_3 \right] = [\partial_0, \partial_1, \partial_2, \partial_3]. \tag{36}$$

The electrodynamic potential four vector $A = A_\mu \mathbf{K}^\mu$ will be defined by the coordinate four set

$$A_\mu = \left[\frac{1}{c} \phi, A_1, A_2, A_3 \right] = [A_0, A_1, A_2, A_3] \tag{37}$$

The electric four current density vector $J = J_\mu \mathbf{K}^\mu$ will be defined by the coordinate four set

$$J_\mu = [c\rho_e, J_1, J_2, J_3] = [J_0, J_1, J_2, J_3], \tag{38}$$

with ρ_e as the electric charge density. The electric four current with a charge q will be also be written as J_μ and the context will indicate which one is used.

Although we defined these fourvectors using the coordinate column notation, we will often use the matrix or summation notation, as for example with $P = P_\mu \mathbf{K}^\mu$, written as

$$\begin{aligned} P &= p_0\hat{\mathbf{T}} + p_1\hat{\mathbf{I}} + p_2\hat{\mathbf{J}} + p_3\hat{\mathbf{K}} = p_0\hat{\mathbf{T}} + \mathbf{p}\cdot\mathbf{K} \\ &= \begin{bmatrix} \mathbf{i}p_0 + \mathbf{i}p_1 & p_2 + \mathbf{i}p_3 \\ -p_2 + \mathbf{i}p_3 & \mathbf{i}p_0 - \mathbf{i}p_1 \end{bmatrix} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}. \end{aligned} \tag{39}$$

The flexibility to use either of these notations is a strength of the math-phys language as developed in this paper. There are cases where one needs to go all the way to the internal scalar matrix notation to solve issues as for example the product rule in calculating a derivative, after which one returns to the more compact notation to evaluate the outcome.

2.6. *The EM field in our language*

I we apply the matrix multiplication rules to the electromagnetic field with four derivative ∂ and four potential A, with $\partial_0 = -\frac{1}{c}\partial_t$ and $A_0 = \frac{1}{c}\phi$, we get $B = \partial^T A$ as

$$B = \partial^T A = (-\frac{1}{c^2}\partial_t\phi - \boldsymbol{\nabla}\cdot\boldsymbol{A})\hat{\mathbf{I}} + (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\mathbf{K} + \frac{1}{c}(-\partial_t\boldsymbol{A} - \boldsymbol{\nabla}\phi)\cdot\boldsymbol{\sigma}. \tag{40}$$

If we apply the Lorenz gauge $\mathbb{B}_0 = -\frac{1}{c^2}\partial_t\phi - \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ and the usual EM definitions of the fields in terms of the potentials we get

$$B = \partial^T A = \boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}. \tag{41}$$

Using $\boldsymbol{\sigma} = -\hat{\mathbf{T}}\mathbf{K} = -\mathbf{i}\mathbf{K}$, this can also be written as

$$B = \partial^T A = (\boldsymbol{B} - \mathbf{i}\frac{1}{c}\boldsymbol{E})\cdot\mathbf{K} = \overrightarrow{\mathbb{B}}\cdot\mathbf{K}. \tag{42}$$

The use of $\mathbb{B} = \boldsymbol{B} - \mathbf{i}\frac{1}{c}\boldsymbol{E}$ dates back to Minkowski's 1908 treatment of the subject [4].

Using $\mathbb{B}$ we can write B as

$$B = \mathbb{B}_1\hat{\mathbf{I}} + \mathbb{B}_2\hat{\mathbf{J}} + \mathbb{B}_3\hat{\mathbf{K}} = \overrightarrow{\mathbb{B}}\cdot\mathbf{K} = \begin{bmatrix} \mathbf{i}\mathbb{B}_1 & \mathbb{B}_2 + \mathbf{i}\mathbb{B}_3 \\ -\mathbb{B}_2 + \mathbf{i}\mathbb{B}_3 & -\mathbf{i}\mathbb{B}_1 \end{bmatrix} = \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix}. \tag{43}$$

For the Lorentz transformation of B we can apply the result of the previous section to get $B^L = (\partial^L)^T A^L = (\partial^T)^{-L} A^L = U(\partial^T)UU^{-1}AU^{-1} = U(\partial^T A)U^{-1} = UBU^{-1}$, so

$$B^L = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix} \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} \begin{bmatrix} e^{-\frac{\psi}{2}} & 0 \\ 0 & e^{\frac{\psi}{2}} \end{bmatrix} = \begin{bmatrix} B_{00} & B_{01}e^{\psi} \\ B_{10}e^{-\psi} & B_{11} \end{bmatrix} \tag{44}$$

which, when written out with $\boldsymbol{E}$ and $\boldsymbol{B}$ leads to the usual result for the Lorentz transformation of the EM field with the Lorentz velocity in the x-direction. But it can also be written as a transformation of the basis, while leaving the coordinates invariant:

$$\begin{aligned} B^L = UBU^{-1} &= \mathbb{B}_1 U\hat{\mathbf{I}}U^{-1} + \mathbb{B}_2 U\hat{\mathbf{J}}U^{-1} + \mathbb{B}_3 U\hat{\mathbf{K}}U^{-1} = \\ \mathbb{B}_1\hat{\mathbf{I}} + \mathbb{B}_2\hat{\mathbf{J}}^L + \mathbb{B}_3\hat{\mathbf{K}}^L &= \mathbb{B}_1 \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} + \mathbb{B}_2 \begin{bmatrix} 0 & e^{\psi} \\ -e^{-\psi} & 0 \end{bmatrix} + \mathbb{B}_3 \begin{bmatrix} 0 & \mathbf{i}e^{\psi} \\ \mathbf{i}e^{-\psi} & 0 \end{bmatrix}. \end{aligned} \tag{45}$$

The Lorentz transformation of the EM field can be performed by internally twisting the $(\hat{\mathbf{J}}, \hat{\mathbf{K}})$-surface perpendicular to the Lorentz velocity and in the process leaving the EM-coordinates invariant.

That the above equals the usual Lorentz transformation of the EM field can be shown by going back to [4], where he wrote the transformation in a form equivalent to

$$\begin{bmatrix} \mathbb{B}'_1 \\ \mathbb{B}'_2 \\ \mathbb{B}'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma & \mathbf{i}\beta\gamma \\ 0 & -\mathbf{i}\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} \mathbb{B}_1 \\ \mathbb{B}_2 \\ \mathbb{B}_3 \end{bmatrix} = \begin{bmatrix} \mathbb{B}_1 \\ \gamma\mathbb{B}_2 + \mathbf{i}\beta\gamma\mathbb{B}_3 \\ \gamma\mathbb{B}_3 - \mathbf{i}\beta\gamma\mathbb{B}_2 \end{bmatrix} \tag{46}$$

So we have

$$B'_{01} = \mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = \gamma\mathbb{B}_2 + \mathbf{i}\beta\gamma\mathbb{B}_3 + \mathbf{i}\gamma\mathbb{B}_3 + \beta\gamma\mathbb{B}_2 \tag{47}$$

and

$$B'_{10} = -\mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = -\gamma\mathbb{B}_2 - \mathbf{i}\beta\gamma\mathbb{B}_3 + \mathbf{i}\gamma\mathbb{B}_3 + \beta\gamma\mathbb{B}_2. \tag{48}$$

If we use the rapidity ψ as $e^{\psi} = \cosh\psi + \sinh\psi = \gamma + \beta\gamma$, this can be rewritten as

$$B'_{01} = \mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = (\gamma + \beta\gamma)(\mathbb{B}_2 + \mathbf{i}\mathbb{B}_3) = B_{01}e^{\psi} \tag{49}$$

and

$$B'_{10} = -\mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = (\gamma - \beta\gamma)(-\mathbb{B}_2 + \mathbf{i}\mathbb{B}_3) = B_{10}e^{-\psi}, \tag{50}$$

which leads to Eqn. (44).

2.7. *The Maxwell Equations and the Lorentz force law*

The Maxwell equations in our language can be given as, using $J = \rho V$, $\partial B = \mu_0 J$ and the Lorentz force law, with a four force density $\mathscr{F}$, as $JB = \mathscr{F}$. Maxwell's inhomogeneous wave equations can be written as $(-\partial^T\partial)B = -\mu_0\partial^T J$ and with the Lorentz invariant quadratic derivative,

$$-\partial^T\partial = (\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_t^2)\hat{\mathbf{I}} \tag{51}$$

we get the homogeneous wave equations of the EM field in free space in the familiar form as

$$(-\partial^T\partial)B = \boldsymbol{\nabla}^2 B - \frac{1}{c^2}\partial_t^2 B = 0. \tag{52}$$

I will look at $\partial B = \mu_0 J$ first. The underlying structure then also applies to the Lorentz Force Law and the inhomogeneous part of the wave equation. I start with

$$B = \partial^T A = \boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}. \tag{53}$$

Then ∂B is given by

$$\partial B = \left(-\frac{1}{c}\partial_t\hat{\mathbf{T}} + \boldsymbol{\nabla}\cdot\mathbf{K}\right)\left(\boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}\right) =$$

$$-(\boldsymbol{\nabla}\cdot\boldsymbol{B})\hat{\mathbf{I}} + \frac{1}{c}(\boldsymbol{\nabla}\cdot\boldsymbol{E})\hat{\mathbf{T}} + (\boldsymbol{\nabla}\times\boldsymbol{B} - \frac{1}{c^2}\partial_t\boldsymbol{E})\cdot\mathbf{K} + \frac{1}{c}(\boldsymbol{\nabla}\times\boldsymbol{E} + \partial_t\boldsymbol{B})\cdot\boldsymbol{\sigma} \tag{54}$$

If we interpret this result using the knowledge regarding the inhomogeneous Maxwell equations, we get an interesting result. First of all, the part of the Maxwell Equation with the dimension of the norm $\hat{\mathbf{1}}$ is zero and so is the part with the dimension of spin $\boldsymbol{\sigma}$. The space-time parts $\mathbf{K}$ and $\hat{\mathbf{T}}$ equal the space-time parts of the four current $\mu_0 J$. So we get

$$\partial B = -(\boldsymbol{\nabla}\cdot\boldsymbol{B})\hat{\mathbf{1}} + \frac{1}{c}(\boldsymbol{\nabla}\cdot\boldsymbol{E})\hat{\mathbf{T}} + (\boldsymbol{\nabla}\times\boldsymbol{B} - \frac{1}{c^2}\partial_t\boldsymbol{E})\cdot\mathbf{K} + \frac{1}{c}(\boldsymbol{\nabla}\times\boldsymbol{E} + \partial_t\boldsymbol{B})\cdot\boldsymbol{\sigma} = \\ 0\hat{\mathbf{1}} + \mu_0 c\rho\hat{\mathbf{T}} + \mu_0\boldsymbol{J}\cdot\mathbf{K} + 0\boldsymbol{\sigma} = \mu_0 J. \quad (55)$$

So the spin-norm part of the Maxwell Equations equals zero and the space-time part equals the space-time four current density times μ_0. In the line of this interpretation, magnetic monopoles and the correlated magnetic monopole current should be searched in the pauliquat dimensions of spin-norm, not in the minquat dimensions of space-time.

As for the Lorentz covariance of the Maxwell Equations, this can be demonstrated quite easily. Given the four-vectors ∂, A and J in reference system S_1, with the Maxwell Equations as $\partial(\partial^T A) = \mu_0 J$, then in reference system S_2 we have the four-vectors ∂^L, A^L and J^L and the covariant Maxwell Equations given as $\partial^L(\partial^L)^T A^L = \mu_0 J^L$. In S_2 this can be proven through

$$\partial^L(\partial^L)^T A^L = \partial^L(\partial^T)^{L^{-1}} A^L = U^{-1}\partial U^{-1} U(\partial^T)UU^{-1}AU^{-1} = \\ U^{-1}\partial(\partial^T)AU^{-1} = U^{-1}\partial BU^{-1} = U^{-1}\mu_0 JU^{-1} = \mu_0 J^L. \quad (56)$$

So if we have $\partial B = \mu_0 J$ in one frame of reference, this transforms as $\partial^L B^L = \mu_0 J^L$ in another frame of reference, which means that the equation maintains its form, it is Lorentz covariant. We have form-invariance of the equations.

I will look at $JB = F$ now, with $J = qV$. The underlying structure for the Lorentz Force Law is the same as for the Maxwell equations. So JB is given by

$$JB = \left(cq\hat{\mathbf{T}} + \boldsymbol{J}\cdot\mathbf{K}\right)\left(\boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}\right) = \\ -(\boldsymbol{J}\cdot\boldsymbol{B})\hat{\mathbf{1}} + \frac{1}{c}(\boldsymbol{J}\cdot\boldsymbol{E})\hat{\mathbf{T}} + (\boldsymbol{J}\times\boldsymbol{B} + q\boldsymbol{E})\cdot\mathbf{K} + (\frac{1}{c}\boldsymbol{J}\times\boldsymbol{E} - cq\boldsymbol{B})\cdot\boldsymbol{\sigma} \quad (57)$$

If we interpret this result using the knowledge regarding the Lorentz Force Law, we get an interesting result. First of all, the part of the Lorentz force law with the dimension of the norm $\hat{\mathbf{1}}$ is zero and so is the part with the dimension of spin $\boldsymbol{\sigma}$. The space-time parts $\mathbf{K}$ and $\hat{\mathbf{T}}$ equal the space-time parts of the four force F. Thus we get

$$JB = -(\boldsymbol{J}\cdot\boldsymbol{B})\hat{\mathbf{1}} + \frac{1}{c}(\boldsymbol{J}\cdot\boldsymbol{E})\hat{\mathbf{T}} + (\boldsymbol{J}\times\boldsymbol{B} + q\boldsymbol{E})\cdot\mathbf{K} + (\frac{1}{c}\boldsymbol{J}\times\boldsymbol{E} - cq\boldsymbol{B})\cdot\boldsymbol{\sigma} = \\ 0\hat{\mathbf{1}} + \frac{1}{c}P\hat{\mathbf{T}} + \boldsymbol{F}\cdot\mathbf{K} + 0\boldsymbol{\sigma} = F. \quad (58)$$

So the spin-norm pauliquat part of the Lorentz Force Law equals zero and the space-time minquat part equals the space-time four force.

In both cases, ∂B and BJ, we get a dual spin-norm and space-time product, with the spin-norm equal zero and the non-zero space-time leading to the inhomogeneous four-vectors of current and force. Speculations about magnetic monopoles are connected to these spin-norm parts, the set spanned by pauliquats. In my analysis, if spin-norm is the twin dual of space-time and as such an integral aspect of the metric as foreseen in [5], then searches for magnetic monopoles should focus on this spin-norm aspect of the vacuum.

But from a purely geometric perspective, the product of three four-vectors like in $BJ = \partial^T AJ = F$, we can separate the coordinate four sets ∂_μ, A^ν, and J^μ from the metric basis, as in , $BJ = ((\partial_\mu A^\nu)J^\mu)((\mathbf{K}^T_\mu \mathbf{K}^\nu)\mathbf{K}^\mu)$ and focus on the metric product alone. We then get

$$\mathbf{K}_\mu{}^\nu \mathbf{K}^\mu = (\mathbf{K}^T_\mu \mathbf{K}^\nu)\mathbf{K}^\mu = \begin{bmatrix} -\hat{\mathbf{T}}\hat{\mathbf{T}} & \hat{\mathbf{I}}\hat{\mathbf{T}} & \hat{\mathbf{J}}\hat{\mathbf{T}} & \hat{\mathbf{K}}\hat{\mathbf{T}} \\ -\hat{\mathbf{T}}\hat{\mathbf{I}} & \hat{\mathbf{I}}\hat{\mathbf{I}} & \hat{\mathbf{J}}\hat{\mathbf{I}} & \hat{\mathbf{K}}\hat{\mathbf{I}} \\ -\hat{\mathbf{T}}\hat{\mathbf{J}} & \hat{\mathbf{I}}\hat{\mathbf{J}} & \hat{\mathbf{J}}\hat{\mathbf{J}} & \hat{\mathbf{K}}\hat{\mathbf{J}} \\ -\hat{\mathbf{T}}\hat{\mathbf{K}} & \hat{\mathbf{I}}\hat{\mathbf{K}} & \hat{\mathbf{J}}\hat{\mathbf{K}} & \hat{\mathbf{K}}\hat{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{bmatrix} = \tag{59}$$

$$\begin{bmatrix} -\hat{\mathbf{T}}\hat{\mathbf{T}}\hat{\mathbf{T}} + \hat{\mathbf{I}}\hat{\mathbf{T}}\hat{\mathbf{I}} + \hat{\mathbf{J}}\hat{\mathbf{T}}\hat{\mathbf{J}} + \hat{\mathbf{K}}\hat{\mathbf{T}}\hat{\mathbf{K}} \\ -\hat{\mathbf{T}}\hat{\mathbf{I}}\hat{\mathbf{T}} + \hat{\mathbf{I}}\hat{\mathbf{I}}\hat{\mathbf{I}} + \hat{\mathbf{J}}\hat{\mathbf{I}}\hat{\mathbf{J}} + \hat{\mathbf{K}}\hat{\mathbf{I}}\hat{\mathbf{K}} \\ -\hat{\mathbf{T}}\hat{\mathbf{J}}\hat{\mathbf{T}} + \hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{I}} + \hat{\mathbf{J}}\hat{\mathbf{J}}\hat{\mathbf{J}} + \hat{\mathbf{K}}\hat{\mathbf{J}}\hat{\mathbf{K}} \\ -\hat{\mathbf{T}}\hat{\mathbf{K}}\hat{\mathbf{T}} + \hat{\mathbf{I}}\hat{\mathbf{K}}\hat{\mathbf{I}} + \hat{\mathbf{J}}\hat{\mathbf{K}}\hat{\mathbf{J}} + \hat{\mathbf{K}}\hat{\mathbf{K}}\hat{\mathbf{K}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{T}} - \hat{\mathbf{T}} - \hat{\mathbf{T}} - \hat{\mathbf{T}} \\ \hat{\mathbf{I}} - \hat{\mathbf{I}} + \hat{\mathbf{I}} + \hat{\mathbf{I}} \\ \hat{\mathbf{J}} + \hat{\mathbf{J}} - \hat{\mathbf{J}} + \hat{\mathbf{J}} \\ \hat{\mathbf{K}} + \hat{\mathbf{K}} + \hat{\mathbf{K}} - \hat{\mathbf{K}} \end{bmatrix}, \tag{60}$$

with no norm-spin $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ product in the end result. The product of three four-vectors in this metric/geometry environment should produce a space-time four vector only, as is reflected in the Maxwell equations and the Lorentz Force Law. In other words, the multiplication of three minquats produces a pure minquat, not a pauliquat or a sum of a pauliquat and a minquat. Looking for magnetic monopoles as ‘symmetric completion’ of the Maxwell Equations and the Lorentz Force Law makes no sense in the metric/geometry developed in this paper because it implies looking for non-zero $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ results from $\mathbf{K}_\mu{}^\nu \mathbf{K}^\mu$. The metric $(\hat{\mathbf{T}}, \mathbf{K}); (\hat{\mathbf{1}}, \boldsymbol{\sigma})$ dimensionality analysis implies that only non-zero $(\hat{\mathbf{T}}, \mathbf{K})$ results are possible and that excludes magnetic monopole four forces and four currents.

2.8. *Invariant EM field energies and the generalized Poynting theorem*

As for the electromagnetic energy density of a pure EM field, we have the two products BB and $B^T B$. These product are structurally different from the previous $\partial^T A$ and $\partial B = \partial\partial^T A$ because it now involves the multiplication of four four-vectors as in $BB = \partial^T A \partial^T A$.

For BB the antisymmetric part eliminates and we get the norm-time product

$$BB = \left(\boldsymbol{B} \cdot \mathbf{K} + \frac{1}{c}\boldsymbol{E} \cdot \boldsymbol{\sigma}\right)\left(\boldsymbol{B} \cdot \mathbf{K} + \frac{1}{c}\boldsymbol{E} \cdot \boldsymbol{\sigma}\right) = (\frac{1}{c^2}\boldsymbol{E}^2 - \boldsymbol{B}^2)\hat{\mathbf{1}} + (2\frac{1}{c}\boldsymbol{B} \cdot \boldsymbol{E})\hat{\mathbf{T}}, \tag{61}$$

which, with a complex $u_{EB} = u_E - u_B + 2\mathbf{i}\sqrt{u_B u_E}$, can be written as

$$\frac{1}{2\mu_0} BB = (u_E - u_B)\hat{\mathbf{1}} + (2\sqrt{u_B u_E})\hat{\mathbf{T}} = u_{EB}\hat{\mathbf{1}}. \tag{62}$$

The fact that the product BB is Lorentz invariant follows from $B^L = UBU^{-1}$ and the fact that BB result in a complex scalar value, so

$$B^L B^L = UBU^{-1}UBU^{-1} = UBBU^{-1} = 2\mu_0 u_{EB} U\hat{\mathbf{1}}U^{-1} = 2\mu_0 u_{EB}\hat{\mathbf{1}} = BB. \tag{63}$$

We also have the interesting product $2\partial u_{EB} = \partial(\frac{1}{\mu_0}BB)$, the four divergence of this Lorentz invariant EM energy related product. Using the Maxwell equations $\partial B = \mu_0 J$ and the Lorentz force density law $JB = \mathscr{F}$, we get

$$\partial u_{EB} = \partial(\frac{1}{2\mu_0}BB) \simeq \frac{2}{2\mu_0}(\partial B)B = JB = \mathscr{F}, \tag{64}$$

resulting in $\partial u_{EB} = \mathscr{F}$.

For the second EM energy related product $B^T B$ the antisymmetric part survives and we get the spin-norm product

$$B^T B = \left(\boldsymbol{B}\cdot\mathbf{K} - \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}\right)\left(\boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}\right) = -(\frac{1}{c^2}\boldsymbol{E}^2 + \boldsymbol{B}^2)\hat{\mathbf{1}} - (2\frac{1}{c}\boldsymbol{E}\times\boldsymbol{B})\cdot\boldsymbol{\sigma} =$$
$$-2\mu_0 u_{EM}\hat{\mathbf{1}} - 2\mu_0\frac{1}{c}\boldsymbol{S}\cdot\boldsymbol{\sigma} = -2\mu_0 c\left(\frac{1}{c}u_{EM}\hat{\mathbf{1}} + \frac{1}{c^2}\boldsymbol{S}\cdot\boldsymbol{\sigma}\right) = -2\mu_0 c\left(\frac{1}{c}u_{EM}\hat{\mathbf{1}} + \boldsymbol{g}\cdot\boldsymbol{\sigma}\right). \tag{65}$$

In the last equation, I used the Poynting vector $\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}$, the EM momentum density $\boldsymbol{g} = \frac{1}{c^2}\boldsymbol{S}$ and the EM energy as $2\mu_0 u_{EM} = \boldsymbol{B}^2 + \frac{1}{c^2}\boldsymbol{E}^2$. The last part can also be written as

$$B^T B = -2\mu_0 c\left(\frac{1}{c}u_{EM}\hat{\mathbf{1}} + \boldsymbol{g}\cdot\boldsymbol{\sigma}\right) = 2\mathbf{i}\mu_0 c\left(\frac{1}{c}u_{EM}\hat{\mathbf{T}} + \boldsymbol{g}\cdot\mathbf{K}\right) = 2\mathbf{i}\mu_0 cG. \tag{66}$$

Thus we get the usual EM four momentum density G and the four EM energy current density S as

$$G = \frac{1}{c^2}S = \frac{-\mathbf{i}}{2\mu_0 c}B^T B = \frac{1}{c}u_{EM}\hat{\mathbf{T}} + \boldsymbol{g}\cdot\mathbf{K}, \tag{67}$$

in which G has the appearance of a good relativistic space-time four vector. But according to our analysis it isn't a space-time four vector but a spin-norm four vector. That makes this product an interesting case for studying the characteristics of the spin-norm dual or twin dimension of space-time, as manifesting aspects of the Dirac vacuum or Dirac Æther.

For the Lorentz transformation of $B^T B$, one has to go to the Lorentz transformation of the primary constituting four vectors. We have $B^T B = (\partial^T A)^T(\partial^T A) = (\partial A^T)(\partial^T A)$. The Lorentz transformation of $B^T B$ then results in

$$\partial^L (A^L)^T (\partial^L)^T A^L = \partial^L (A^T)^{L^{-1}} (\partial^T)^{L^{-1}} A^L = U^{-1}\partial U^{-1} U(A^T)UU(\partial^T)UU^{-1}AU^{-1} =$$
$$U^{-1}(\partial A^T)UU(\partial^T A)U^{-1} = U^{-1}B^T UUBU^{-1} = (B^T)^{L^{-1}}B^L = (B^L)^T B^L. \tag{68}$$

This means that we have Lorentz covariance for the equation $G = -\frac{\mathbf{i}}{2\mu_0 c}B^T B$. So in Eqn.(67) the EM four momentum density G and the four EM energy current density S are defined in a Lorentz covariant way.

The product $\partial^T G$ is interesting too, being the divergence of the EM momentum density $B^T B$. It brings us at the level of the product of five original four-vectors. It should give a Maxwell-Lorentz structured complex force. We get

$$\partial^T G = \frac{-\mathbf{i}}{2\mu_0 c}\partial^T B^T B = \frac{-\mathbf{i}}{2\mu_0 c}(\partial B B^T)^T \simeq \frac{-\mathbf{i}}{2\mu_0 c}(2\mu_0 J B^T)^T = \frac{-\mathbf{i}}{c}(J^T B) = \mathscr{F} \tag{69}$$

implying that we returned to a product of three four-vectors with as a necessary result a Maxwell Equation, Lorentz Force Law structured outcome. The main difference is in the appearance of the complex number $\mathbf{i}$, $J^T B = \mathbf{i}c\mathscr{F}$, stemming from $\hat{\mathbf{T}} = \mathbf{i}\hat{\mathbf{1}}$, which turns space-time into spin-norm and vice versa. The second related difference is coming from the time reversal in J in $J^T B$.

Calculating $\frac{-\mathbf{i}}{c} J^T B$ gives

$$\frac{-\mathbf{i}}{c} J^T B = \frac{-\mathbf{i}}{c}\left(-cq\hat{\mathbf{T}} + \boldsymbol{J}\cdot\mathbf{K}\right)\left(\boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}\right) =$$

$$\frac{1}{c}(\boldsymbol{J}\cdot\boldsymbol{B})\hat{\mathbf{T}} + \frac{1}{c^2}(\boldsymbol{J}\cdot\boldsymbol{E})\hat{\mathbf{1}} + \frac{1}{c}(\boldsymbol{J}\times\boldsymbol{B} - q\boldsymbol{E})\cdot\boldsymbol{\sigma} - (\frac{1}{c^2}\boldsymbol{J}\times\boldsymbol{E} + q\boldsymbol{B})\cdot\mathbf{K} \tag{70}$$

We see that the spin-norm and space-time switch places due to $\mathbf{i}$ and that the sign of q changes due to T (not the sign of $\boldsymbol{J}$). The other part $\partial^T G$ leads to

$$\partial^T G = (-\frac{1}{c^2}\partial_t u_{EM} - \boldsymbol{\nabla}\cdot\mathbf{g})\hat{\mathbf{1}} + (\boldsymbol{\nabla}\times\mathbf{g})\cdot\mathbf{K} + \frac{1}{c}(\partial_t\mathbf{g} + \boldsymbol{\nabla}u_{EM})\cdot\boldsymbol{\sigma}. \tag{71}$$

The norm $\hat{\mathbf{1}}$ part of the equation $\partial^T G = \frac{-\mathbf{i}}{c}(J^T B)$ contains the relativistic Poynting's theorem:

$$\frac{1}{c^2}\partial_t u_{EM} + \boldsymbol{\nabla}\cdot\mathbf{g} = -\frac{1}{c^2}\boldsymbol{J}\cdot\boldsymbol{E} \tag{72}$$

so using $\mathbf{S} = c^2\mathbf{g}$ we get the relativistic Poynting theorem for EM energy density conservation

$$\partial_t u_{EM} + \boldsymbol{\nabla}\cdot\mathbf{S} = -\boldsymbol{J}\cdot\boldsymbol{E}. \tag{73}$$

The equation $\partial^T G = -\frac{\mathbf{i}}{c}(J^T B)$ can be perceived as the generalizes Poynting theorem. In the derivation, the step $(\partial BB^T)^T \simeq (2(\partial B)B^T)^T = (2\mu_0 JB^T)^T$ does need further evaluation, but that is a topic for another time. It's details don't influence the result regarding the presented derivation of the Poynting theorem.

Two issues are relevant for the present paper. The first point to make is that the Poynting continuity equation refers to an open system when a charge is moving in an electric field. Without the current one has the EM field energy density continuity equation for a closed system

$$\partial^T_\mu S^\mu = \partial_t u_{EM} + \boldsymbol{\nabla}\cdot\mathbf{S} = 0. \tag{74}$$

The second issue is that this continuity equation has its origin in the norm-like part of the momentum closed system condition $\partial^T G = 0$ of Eqn.(71)

$$\partial^T_\mu G^\mu = \frac{1}{c^2}\partial_t u_{EM} + \boldsymbol{\nabla}\cdot\mathbf{g} = 0. \tag{75}$$

The other closed system conditions are the space-like absence of vorticity condition

$$\boldsymbol{\nabla}\times\mathbf{g} = 0 \tag{76}$$

and the spin-like

$$\partial_t\mathbf{g} + \boldsymbol{\nabla}u_{EM} = 0. \tag{77}$$

The last part can be written as the spin-like conserved force condition

$$\partial_t \mathbf{g} = -\boldsymbol{\nabla} u_{EM}. \tag{78}$$

This pattern will repeat itself for the Dirac current. In the Dirac level part of this paper, the Dirac current will shown to be a probability/field tensor and the continuity equation for the Dirac current will turn out to be the time-like part of the closed system condition for this probability/field tensor.

As for the Lorentz covariance of invariance of Poynting's theorem including the $\boldsymbol{J}\cdot\boldsymbol{E}$ term, it clearly isn't. What is Lorentz covariant is the generalized Poynting theorem or law $\partial^T G = -\frac{\mathbf{i}}{c}(J^T B)$. In [12], Meyers stated that Poynting's theorem should be Lorentz covariant. This requirement is too strict and obviously runs into trouble, as shown in [12]. See [13] for an updated critical analysis of the Lorentz transformation properties of Poynting's theorem.

2.9. *Relativistic mechanics*

Angular momentum is given by $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$ so let's try to generalize it with the four vector action product $R^T P$. We get

$$R^T P = (U_i t - \boldsymbol{r}\cdot\boldsymbol{p})\hat{\mathbf{1}} + (\boldsymbol{r}\times\boldsymbol{p})\cdot\mathbf{K} + (ct\boldsymbol{p} - \frac{1}{c}U_i\boldsymbol{r})\cdot\boldsymbol{\sigma} = S\hat{\mathbf{1}} + \boldsymbol{L}\cdot\mathbf{K} + \boldsymbol{Z}\cdot\boldsymbol{\sigma} = S\hat{\mathbf{1}} + (\boldsymbol{Z} - \mathbf{i}\boldsymbol{L})\cdot\boldsymbol{\sigma}. \tag{79}$$

In this one single product we can recognize the scalar action S, the Pauli-level spin-orbit interaction $\mathbf{i}\boldsymbol{L}\cdot\boldsymbol{\sigma}$, but also the angular momentum six-vector $(\boldsymbol{Z} - \mathbf{i}\boldsymbol{L})$ of Frenkel, de Broglie, Kramer and Dirac [6,7,8,9]. Clearly $\boldsymbol{Z} = ct\boldsymbol{p} - m_i c\boldsymbol{r}$ represents the barycentric momentum of de Broglie and the sixvector completion of angular momentum with Frenkel, Kramers and Dirac, as a part of the six-vector $\boldsymbol{L} - \mathbf{i}\boldsymbol{Z}$.

In SR and GR, Laue's condition for the conservation of energy-momentum in a closed system is $\partial_\nu T^\nu{}_\mu = 0$ [10,11,3]. In our language we have a comparable but not identical $\partial^T P = 0$ condition as a starting point of our alternative relativistic mechanics. In the case of electrodynamics, when we have the canonical $P = qA$, we have $\partial^T A = B \neq 0$. So in circumstances analogous to a nonzero anti-symmetric EM field, the condition $\partial^T P = q\partial^T A = qB = 0$ is not fulfilled. In the previous section, we saw other conditions in the EM context where the closed system condition is not satisfied due to charges (moving) in EM fields.

The mechanic condition $\partial^T P = 0$ leads to

$$\partial^T P = (-\frac{1}{c^2}\partial_t U_i - \boldsymbol{\nabla}\cdot\mathbf{p})\hat{\mathbf{1}} + (\boldsymbol{\nabla}\times\mathbf{p})\cdot\mathbf{K} + \frac{1}{c}(\partial_t\mathbf{p} + \boldsymbol{\nabla}U_i)\cdot\boldsymbol{\sigma} = 0. \tag{80}$$

so to three subconditions

$$\frac{1}{c^2}\partial_t U_i + \boldsymbol{\nabla}\cdot\mathbf{p} = 0 \tag{81}$$

$$\boldsymbol{\nabla}\times\mathbf{p} = 0 \tag{82}$$

$$\partial_t\mathbf{p} = -\boldsymbol{\nabla}U_i. \tag{83}$$

The first one is the continuity equation, the second means that we have zero vorticity and the third that the related force field can be connected to a potential energy. Due to the second condition, the time derivative of $\boldsymbol{\nabla}\times\mathbf{p}$ must be zero, giving the secondary conserved force field condition

$$\boldsymbol{\nabla}\times\mathbf{F}=0. \tag{84}$$

The first condition can also be written as

$$\partial_t m_i+\boldsymbol{\nabla}\cdot(m_i\mathbf{v})=0, \tag{85}$$

so the continuity equation for inertial mass.

If we have $\partial^T P=0$ in one system of reference, then in another system of reference we have

$$(\partial^L)^T P^L=(\partial^T)^{L^{-1}}P^L=U\partial^T UU^{-1}PU^{-1}=U\partial^T PU^{-1}=0, \tag{86}$$

proving that the condition is Lorentz covariant.

With $\partial^T P=0$ we have a relativistic condition of a mechanical system representing a central force. It is best characterized as the extended continuity condition, it's relativistic completion: the generalized continuity equation. It has as a norm $\hat{\mathbf{1}}$ condition the continuity equation, as a space $\mathbf{K}$ condition the absence of vorticity and as a spin $\boldsymbol{\sigma}$ condition the conserved force condition. This will become crucial in relativistically extending the conserved Dirac current condition in RQM.

In the Laue condition $\partial_\nu T^\nu{}_\mu=0$ the stress-energy density tensor is $T^\nu{}_\mu=V^\nu G_\mu$. In our math-phys language we would get the not exact analog $T=V^T G$ and $\partial T=0$, but that would imply a full homogeneous Maxwell-Lorentz structure with the product $\partial V^T G=0$. Our stress energy density 'tensor' T is given by

$$T=V^T G=(u_i-\mathbf{v}\cdot\mathbf{g})\hat{\mathbf{1}}+(\mathbf{v}\times\mathbf{g})\cdot\mathbf{K}+c(\mathbf{g}-\frac{1}{c^2}u_i\mathbf{v})\cdot\boldsymbol{\sigma}. \tag{87}$$

This tensor analog contains all the elements of $T^\nu{}_\mu=V^\nu G_\mu$, with the difference that the cross product $\mathbf{v}\times\mathbf{g}$ appears directly in our $T=V^T G$ whereas ony half of it is in the usual tensor and the anti-symmetric tensor product is needed to get the full cross product.

In the case of a symmetric situation $\mathbf{v}$ has the same direction as $\mathbf{g}$, resulting in

$$T=(u_i-\mathbf{v}\cdot\mathbf{g})\hat{\mathbf{1}}=u_0\hat{\mathbf{1}} \tag{88}$$

$$\mathbf{v}\times\mathbf{g}=0 \tag{89}$$

$$\mathbf{g}=\frac{1}{c^2}u_i\mathbf{v}. \tag{90}$$

The third equation contains the mass-energy density equivalence $u_i=\rho_i c^2$, but it also implies the absence of linear stresses. The second equation implies the absence internal pressures. The first equation equals the scalar Lagrangian density, the trace of the Laue mechanical stress-energy density tensor. A symmetric T can be written as $T=\frac{1}{\rho}G^T G$ in the mass density formulation and as $T=\frac{1}{m_0}P^T P$ in the mass formulation.

The divergence of the symmetric T has the space-like part and the spin-like part equal to zero and only the norm-like part possibly non-zero. This leads to a four force density as

$$\mathscr{F} = -\partial T = -\partial \frac{1}{\rho} G^T G = -\partial u_0 \hat{\mathbf{1}} = \frac{1}{c}\partial_t u_0 \hat{\mathbf{T}} - \boldsymbol{\nabla} u_0 \cdot \mathbf{K}. \tag{91}$$

The direct parallel in electromagnetics would be that $\overrightarrow{\mathbb{B}} = 0$ with a Coulomb gauge for the field and that $T_{EM} = J^T A = \rho_0 \phi_0 \hat{\mathbf{1}}$, with $\mathscr{F} = -\partial \rho_0 \phi_0$. In the rest system this would produce a Coulomb force density and a Coulomb force power, which, for a static potential, would be zero. Thus in our relativistic dynamics, in the symmetric case the electromagnetic parallel would only produce a Coulomb force situation.

Only if $\mathbf{v}$ doesn't have the same direction as $\mathbf{g}$ will there be an anti-symmetric component present that is analog to the structure of the Maxwell-Lorentz electromagnetic field/force situation. The Lorentz force is given as $JB = F$, which can be written as $qV\partial^T A = F$ which, by using $P = qA$, results in the mechanical analog $V\partial^T P = F$. This still isn't the full $\partial V^T P = -F$. The Lorentz force law analog in our relativistic dynamics implies that $\partial^T P \neq 0$, so that $m_0 \partial^T U \neq 0$. If we look closer at $V\partial^T$, we see that it contains the three parts, norm $\hat{\mathbf{1}}$, spin $\boldsymbol{\sigma}$ and space $\mathbf{K}$ respectively,

$$(-\frac{\partial}{\partial t} - \mathbf{v}\cdot\boldsymbol{\nabla})\hat{\mathbf{1}} \equiv -\frac{d}{dt}\hat{\mathbf{1}} \tag{92}$$

$$\mathbf{v}\times\boldsymbol{\nabla} \tag{93}$$

$$c\boldsymbol{\nabla} + \frac{1}{c}\mathbf{v}\partial_t. \tag{94}$$

So the product $-V\partial^T$ is our variant of the absolute derivative, with $\frac{d}{dt}\hat{\mathbf{1}}$ as the scalar norm $\hat{\mathbf{1}}$ part of it. Thus if we go from $\partial^T P = 0$ to $V\partial^T P = F$, we move in our relativistic mechanics from a closed Coulomb only force structure or environment to an open Lorentz Force Law one, related to a move from a partial four derivative ∂ to an absolute four derivative product $V\partial^T$. That area of mechanics is outside the scope of this paper.

2.10. *The Quantum, Pauli level Klein-Gordon condition*

The basic scalar Klein-Gordon wave equation in Quantum Mechanics is

$$(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_t^2)\Psi = 0 \tag{95}$$

In our environment it can be written as

$$-\partial^T \partial \Psi = (\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_t^2)\hat{\mathbf{1}}\Psi = 0 \tag{96}$$

but then we have a two column spinor as wave-function

$$\Psi = \begin{bmatrix} \Psi_0 \\ \Psi_1 \end{bmatrix} \tag{97}$$

instead of the scalar spinor of Schrödinger- and standard Klein-Gordon QM. But it would result in two identical equations, so a degenerate situation in which the two valued spinor equation can be reduced to a single one.

Thus far, only Lorentz transformation could act on the matrix internal aspect of our basis. And even then, a coordinate interpretation was always possible, leaving the basis inert. So up until now, the matrix part of the basis has been practical but not essential. Spinors on the Pauli and Dirac level change that situation. Spinor wave functions interact with the internal elements, the matrix aspect, of the metric $(\hat{\mathbf{T}},\hat{\mathbf{I}},\hat{\mathbf{J}},\hat{\mathbf{K}})$.

The Klein-Gordon Equation has its roots in the quadratic energy-momentum condition

$$P^T P = (\frac{1}{c^2}U_i^2 - p^2)\hat{\mathbf{1}} = \frac{1}{c^2}U_0^2\hat{\mathbf{1}}, \tag{98}$$

which, as done by [14,21] in the Dirac context, can be linked to the symmetric energy-momentum matrix

$$T = V^T P = \frac{1}{\gamma m_0}P^T P = \frac{1}{\gamma}U_0\hat{\mathbf{1}} = -L\hat{\mathbf{1}}. \tag{99}$$

If you take the density version, by dividing it by a volume, this volume has one of its lengths Lorentz contracted, which then compensates for the γ in L to produce a Lorentz invariant rest-energy density. In Quantum Mechanics this volume is included in the probability density so $\Psi^\dagger L\Psi = u_0$

In Wave Mechanics this is the basis for the introduction of the eigenvalue wave equation

$$P^T P\Psi = (\frac{1}{c^2}U_i^2 - p^2)\hat{\mathbf{1}}\Psi = \frac{1}{c^2}U_0^2\hat{\mathbf{1}}\Psi. \tag{100}$$

With the operator convention $\hat{P} = -\mathbf{i}\hbar\partial$ we can switch from energy-eigenvalue condition to operator-wave equation

$$\hat{P}^T\hat{P}\Psi = \frac{1}{c^2}U_0^2\hat{\mathbf{1}}\Psi. \tag{101}$$

We can make this canonical by applying the replacement $P \to P + qA$ and $\hat{P} \to \hat{P} + qA$ or $\partial \to D = \partial + \mathbf{i}\frac{q}{\hbar}A$. We get the canonical Klein-Gordon wave equation in a biquaternion metric

$$D^T D\Psi = \frac{U_0^2}{c^2\hbar^2}\hat{\mathbf{1}}\Psi. \tag{102}$$

This equation includes the Pauli-spin EM-field interaction term. One issue with the canonical version is the rest energy term U_0 is the question what it should all include. For the moment that question is ignored. But the issue is related to the open or closed system context. A closed system has constant rest energy and thus it has

$$\partial\frac{1}{m_0}P^T P = 0. \tag{103}$$

An open system doesn't have its divergence equal zero. Electromagnetic fields with moving charges are notoriously open systems. That affects the canonical wave equations of Quantum Mechanics.

The $D^T D\Psi$ part can be expanded as

$$D^T D\Psi = \partial^T\partial\Psi + \mathbf{i}\frac{q}{\hbar}\partial^T A\Psi + \mathbf{i}\frac{q}{\hbar}A^T\partial\Psi - \frac{q^2}{\hbar^2}A^T A\Psi. \tag{104}$$

Now, the first and the last terms give scalar quadratics but the two middle terms must be examined more carefully. By writing out the two matrix products and applying the standard differentiation rule to the scalars in these matrixes, one can show that

$$\partial^T A\Psi + A^T\partial\Psi = B\Psi + 2(\frac{1}{c^2}\phi\partial_t + \mathbf{A}\cdot\boldsymbol{\nabla})\hat{\mathbf{1}}\Psi \tag{105}$$

This gives us for $D^T D\Psi = \frac{U_0^2}{c^2\hbar^2}\hat{\mathbf{1}}\Psi$ the equation

$$\partial_\mu^T\partial^\mu\hat{\mathbf{1}}\Psi - \frac{q^2}{\hbar^2}A_\mu^T A^\mu\hat{\mathbf{1}}\Psi - 2\mathbf{i}\frac{q}{\hbar}A_\mu^T\partial^\mu\hat{\mathbf{1}}\Psi = -\frac{U_0^2}{c^2\hbar^2}\hat{\mathbf{1}}\Psi + \mathbf{i}\frac{q}{\hbar}B\Psi, \tag{106}$$

with

$$\partial^T\partial = (\frac{1}{c^2}\partial_t^2 - \boldsymbol{\nabla}^2)\hat{\mathbf{1}} = -\partial_\mu^T\partial^\mu\hat{\mathbf{1}}, \tag{107}$$

$$A^T A = (\frac{1}{c^2}\phi^2 - \mathbf{A}^2)\hat{\mathbf{1}} = -A_\mu^T A^\mu\hat{\mathbf{1}}, \tag{108}$$

$$A_\mu^T\partial^\mu = (\frac{1}{c^2}\phi\partial_t + \mathbf{A}\cdot\boldsymbol{\nabla}). \tag{109}$$

The only non-degenerate part in this equation is $\mathbf{i}\frac{q}{\hbar}B\Psi$. In our units we have the Bohr magneton $\mu_B = \frac{e\hbar}{2m_0}$ and if we multiply the equation by $\frac{\hbar^2}{2m_0}$ we get the non-degenerate term as $\mathbf{i}\mu_B B\Psi$. This can be written as

$$\mathbf{i}\mu_B B\Psi = \mathbf{i}\mu_b\overrightarrow{\mathbb{B}}\cdot\mathbf{K}\Psi = -\mu_b\overrightarrow{\mathbb{B}}\cdot\boldsymbol{\sigma}\Psi = -\mu_b\mathbf{B}\cdot\boldsymbol{\sigma}\Psi + \mathbf{i}\mu_b\frac{1}{c}\mathbf{E}\cdot\boldsymbol{\sigma}\Psi, \tag{110}$$

with the remark that we exchanged the Pauli σ_x and σ_z, as $\sigma_I = \sigma_z$ and $\sigma_K = \sigma_x$. So by putting spin in the metric we get a canonical Klein Gordon equation that includes Pauli-spin EM field interaction terms. Now, we have the spin magnetic moment $\boldsymbol{\mu}_s = \mu_B\boldsymbol{\sigma}$. We can further interpret the relativistic companion of the intrinsic magnetic moment as the intrinsic zitter-effect polarization $\boldsymbol{\pi}_s = \frac{e\lambda_c}{2}\boldsymbol{\sigma}$, we get

$$\mathbf{i}\mu_B B\Psi = -\mathbf{B}\cdot\boldsymbol{\mu}_s\Psi + \mathbf{i}\mathbf{E}\cdot\boldsymbol{\pi}_s\Psi \tag{111}$$

The complete wave equation on the Pauli-spin spinor-level, in which a spinor consists of two complex variables, will then be

$$-\frac{\hbar^2}{2m_0}\hat{\mathbf{1}}\boldsymbol{\nabla}^2\Psi + \frac{\hbar^2}{2m_0c^2}\hat{\mathbf{1}}\partial_t^2\Psi + \frac{q^2}{2m_0}\mathbf{A}^2\hat{\mathbf{1}}\Psi - \frac{q^2\phi^2}{2m_0c^2}\hat{\mathbf{1}}\Psi \tag{112}$$

$$+\frac{\mathbf{i}\hbar q\phi}{m_0c^2}\hat{\mathbf{1}}\partial_t\Psi + \frac{\mathbf{i}q\hbar}{m_0}\mathbf{A}\cdot\boldsymbol{\nabla}\hat{\mathbf{1}}\Psi \tag{113}$$

$$= \frac{U_0}{2}\hat{\mathbf{1}}\Psi + \mathbf{B}\cdot\boldsymbol{\mu}_s\Psi - \mathbf{i}\mathbf{E}\cdot\boldsymbol{\pi}_s\Psi, \tag{114}$$

The first term with $\boldsymbol{\nabla}^2$ is the kinetic term, the $\boldsymbol{A}^2$ part is know as the diamagnetic part of the Pauli equation, the $\mathbf{A}\cdot\boldsymbol{\nabla}$ part as the paramagnetic part, and with the Coulomb gauge this part can also be rearranged into the orbital or angular momentum term causing the Zeeman effect [23, p. 144 and 190]. The $\mathbf{B}\cdot\boldsymbol{\mu}_s$ term is the spin magnetic moment term connected to the anormal Zeeman effect. The other terms are either simply ignored, as for example the $\mathbf{E}\cdot\boldsymbol{\pi}_s$ term, or somehow reduced to a term for the potential and a term for the constant energy.

It is interesting to observe that we have a quadratic time derivative as is usual in the Klein-Gordon equation, but that we also have a linear time derivative. It is my impression that that linear term, together with the intrinsic zitter polarization term constitutes the relativistic complement of the $\boldsymbol{J}=\boldsymbol{L}+\boldsymbol{S}$ total angular momentum. The relativistic origin of total angular momentum $\boldsymbol{J}=\boldsymbol{L}+\boldsymbol{S}$ lies in the two cross-products of the square of the canonical momentum

$$\partial^T A\Psi + A^T\partial\Psi = B\Psi + 2(\frac{1}{c^2}\phi\partial_t + \mathbf{A}\cdot\boldsymbol{\nabla})\hat{\mathbf{1}}\Psi. \tag{115}$$

With the use of $\mathbf{i}\mu_B B\Psi = -\mathbf{B}\cdot\boldsymbol{\mu}_s\Psi + \mathbf{i}\mathbf{E}\cdot\boldsymbol{\pi}_s\Psi$, this can be split into the familiar $\boldsymbol{J}=\boldsymbol{L}+\boldsymbol{S}$ parts as

$$\mathbf{i}\mathbf{B}\cdot\boldsymbol{\mu}_s\Psi + 2\mu_B\mathbf{A}\cdot\hat{\mathbf{1}}\boldsymbol{\nabla}\Psi. \tag{116}$$

and the ignored part as

$$\mathbf{E}\cdot\boldsymbol{\pi}_s\Psi + \frac{q\phi\hbar}{m_0c^2}\hat{\mathbf{1}}\partial_t\Psi. \tag{117}$$

With the intrinsic zitter-effect polarization $\boldsymbol{\pi}_s = \frac{e\lambda_c}{2}\boldsymbol{\sigma}$ and the orbital zitter-effect Compton-level polarization as $\pi_o = e\lambda_c$ this last term can be written as

$$\mathbf{E}\cdot\boldsymbol{\pi}_s\Psi + \frac{\pi_o\phi}{c}\hat{\mathbf{1}}\partial_t\Psi \tag{118}$$

and then interpreted as the total zitter-effect Compton-level polarization.

This zitter-polarization linear in time derivative might well be the damping part of the canonical Klein-Gordon equation and then be responsible for the quantum jumps. It is also possible that these two terms, scaled to the reduced Compton wavelenght of the electron λ_c, are responsible for the electric counterparts of the normal Zeeman effect and the anormal Zeeman effect, ie linear Stark effect and anormal Stark effect. It seems outdated to just ignore the parts of the equation that one cannot connect to some physical experimental phenomena, as Dirac did with the intrinsic polarization term of his equation. But perhaps some of those terms only appear in this analysis due to the non-commutative character of the math-language used/developed.

It is also possible to interpret

$$\mathbf{E}\cdot\boldsymbol{\pi}_s\Psi + \frac{q\phi\hbar}{m_0c^2}\hat{\mathbf{1}}\partial_t\Psi. \tag{119}$$

for stationary states with constant energy as

$$\mathbf{E}\cdot\boldsymbol{\pi}_s\Psi + V\hat{\mathbf{1}}\partial_t\Psi. \tag{120}$$

with $V = q\phi$ and $\partial_t\Psi = \frac{\mathbf{i}U_0}{\hbar}\Psi$. With this stationary state interpretation, the term with the linear time derivative turns out to produce the standard potential energy term, and its energy levels are then the usual Coulomb energy levels. The $\mathbf{E}\cdot\boldsymbol{\pi}_s\Psi$ term then produces a zitter-like Compton reduced wavelength scale smearing out of the principal orbits. Such an effect has been observed for the most inner S-orbits.

So with the equation $D^T D\Psi = \frac{U_0^2}{c^2\hbar^2}\hat{\mathbf{1}}\Psi$ we are able to treat Pauli spin relativistically, provided that the spinor Ψ Lorentz transforms as $\Psi^L = U\Psi$. That however is only the case for the spinors in the Weyl representation and not for spinors in the Dirac representation. The Lorentz transformation of spinors in the Dirac representation can only be achieved at the Dirac spinor level, so with four variable spinors. A two variable Pauli spinor in the Dirac representation cannot be Lorentz transformed on its own, that is, without its Dirac twin. On the Weyl level, a Lorentz transformation of a Pauli spinor is possible, but the transformation to its Dirac representation is impossible without its Weyl twin spinor. In a modern interpretation, this implies that understanding the intrinsics of a quantum jump as a damping term effect is impossible without introducing anti-particles and the related quantum field interpretation, even in atoms. If so, then we should introduce Feynman diagram like analysis in atomic physics's attempts to grasp the intrinsics of quantum jumps of electrons in atoms. It is however impossible to prove this at the Pauli spin level. In the context of atomic physics at the Pauli level of two variable spinors, quantum jumps are and will remain a mystery, without proof why that is. Just like line beings will never understand angles and surface restricted beings will never be able to understand volumes.

3. The Dirac spin level

3.1. *The Dirac environment metric matrices*

In the nineteen twenties, the quadratic relativistic scalar Klein-Gordon wave equation couldn't be applied to the relativistic electron. Dirac linearized the Klein-Gordon equation by going to four by four matrices instead of the two by two Pauli matrices. In his two seminal 1928 papers he introduces the Clifford four set $(\beta, \boldsymbol{\alpha})$ and, using what were later called the gamma matrices, the covariant Clifford four set $(\beta, \boldsymbol{\gamma})$ [24,25]. The Pauli matrices are incorporated in these matrices. Weyl later found a third covariant Clifford four set, which relates to the Dirac covariant set as low velocity relativistic to high velocity relativistic gamma matrices Clifford four set.

All these matrices can be represented as two by two matrices of the biquaternion pauliquat basis $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$. But using the biquaternion basis $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ as a basis of the space-time metric is already highly problematic, as indicated by Synge [1]. Duplicating this spin-norm basis by going from the Pauli spinor level to the Dirac spinor level is even more so. As a consequence, using the Clifford four set gamma matrices written as $\gamma_\mu = (\gamma_0, \boldsymbol{\gamma})$, as a basis for the space-time metric or as space-time four vectors on their own right is truly questionable. It is my opinion that the $(\hat{\mathbf{T}}, \mathbf{K})$ biquaternion minquat basis will provide a more solid foundation for connecting the Clifford four sets of Relativistic Quantum Mechanics to ordinary relativistic Minkowski space-time.

3.1.1. *The Dirac and Weyl matrices in dual pauliquat norm-spin mode*

In the following I present the Dirac and Weyl matrices using my reversed order of the Pauli spin matrices, with $\sigma_I = \sigma_z$, $\sigma_J = \sigma_y$, $\sigma_K = \sigma_x$ and $\boldsymbol{\sigma} = (\sigma_I, \sigma_J, \sigma_K)$. This implies that the order of the gamma matrices are reversed correspondingly, with $\gamma_1 = \gamma_I = \gamma_z$, $\gamma_2 = \gamma_J = \gamma_y$, $\gamma_3 = \gamma_K = \gamma_x$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) = (\gamma_I, \gamma_J, \gamma_K)$).

In my $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ norm-spin basis the Dirac set $\alpha_\mu = (\not{1}, \boldsymbol{\alpha})$ can be represented as

$$\alpha_\mu = (\not{1}, \boldsymbol{\alpha}) = \left(\begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & \hat{\mathbf{1}} \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix} \right). \tag{121}$$

The most straightforward doubling of the Pauli level norm-spin set $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ is the Dirac level norm-spin set $\Sigma_\mu = (\not{1}, \boldsymbol{\Sigma})$ defined as

$$\Sigma_\mu = (\not{1}, \boldsymbol{\Sigma}) = \left(\begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & \hat{\mathbf{1}} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix} \right). \tag{122}$$

The set of gamma matrices in the Dirac representation, $\gamma_\mu = (\beta, \boldsymbol{\gamma}) = (\gamma_0, \boldsymbol{\gamma})$, can be defined as

$$\gamma_\mu = (\beta, \boldsymbol{\gamma}) = (\gamma_0, \boldsymbol{\gamma}) = \left(\begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & -\hat{\mathbf{1}} \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} \right) \tag{123}$$

The set of gamma matrices in the Weyl representation, $\gamma_\mu = (\gamma_0, \boldsymbol{\gamma})$, can be defined as

$$\gamma_\mu = (\gamma_0, \boldsymbol{\gamma}) = \left(\begin{bmatrix} 0 & \hat{\mathbf{1}} \\ \hat{\mathbf{1}} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} \right) \tag{124}$$

The last matrix we need to define in this environment is the γ_5 matrix as

$$\gamma_5 = \begin{bmatrix} 0 & \hat{\mathbf{1}} \\ -\hat{\mathbf{1}} & 0 \end{bmatrix}. \tag{125}$$

The most important product needed to understand the genius of Dirac's two 1928 Dirac equation papers is the Dirac gamma product

$$\gamma_0 \gamma_\mu = (\gamma_0 \gamma_0, \gamma_0 \boldsymbol{\gamma}) = (\hat{\mathbf{1}}, \boldsymbol{\alpha}) = \alpha_\mu. \tag{126}$$

This product is key towards understanding the Dirac four current and the related continuity equation. This product eventually leads to the definition of the Dirac adjoint as $\overline{\Psi} = \Psi^\dagger \gamma_0$, the Dirac probability current as

$$J_\mu = c\overline{\Psi}\gamma_\mu \Psi = c\Psi^\dagger \gamma_0 \gamma_\mu \Psi = c\Psi^\dagger \alpha_\mu \Psi \tag{127}$$

and the Dirac current continuity equation as

$$\partial_\mu J^\mu = c\partial_\mu \overline{\Psi}\gamma^\mu \Psi = c\partial_\mu \Psi^\dagger \gamma_0 \gamma^\mu \Psi = c\partial_\mu \Psi^\dagger \alpha^\mu \Psi = 0. \tag{128}$$

The main innovative result of the Dirac level part of this paper is the conclusion that the elements of this probability current four vector can be interpreted as part of a metric probability tensor and that the continuity equation has its origin in the time like part of the closed system condition of that metric probability tensor, as in

$$\partial_\nu \Phi_\mu{}^\nu \equiv \partial_\nu \Psi^\dagger \gamma_\mu \gamma^\nu \Psi = 0. \tag{129}$$

In order to make this consistent as a space-time metric probability condition, I need to introduce the Dirac and Weyl related matrix representations in the time-space minquat basis $(\hat{\mathbf{T}}, \mathbf{K})$, what I will call the bèta matrices, instead of gamma matrices in the norm-spin pauliquat basis $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$.

In my treatment of RQM, the Weyl representation in the time-space basis $(\hat{\mathbf{T}}, \mathbf{K})$ will prove to be like Machiavelli's "return to the banner" when coherence is fading. In my context, it is the most simple point of departure possible, from where almost all the rest can be derived. To return to the space-time basis, $(\hat{\mathbf{T}}, \mathbf{K})$ and the related Weyl β_μ as its dual-parity version will prove its strategic worth. But the Dirac representation has proven it's worth for almost all practical, experimental area's of interest, so to understand the operator that switches between them is as important.

3.1.2. *The transformation from the Dirac to the Weyl representation and vice versa*

The transformation from the Weyl to the Dirac representation and vice versa is an operator that is usually written as S. Two possible versions of S are being used. The most common one is

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{\mathbf{1}} & \hat{\mathbf{1}} \\ \hat{\mathbf{1}} & -\hat{\mathbf{1}} \end{bmatrix} \tag{130}$$

and the one I prefer is the less common

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{\mathbf{1}} & \hat{\mathbf{1}} \\ -\hat{\mathbf{1}} & \hat{\mathbf{1}} \end{bmatrix} \tag{131}$$

The reason I will only use the second version is that it has the property $\gamma_0 S = S^{-1}\gamma_0$ and the directly related $S\gamma_0 = \gamma_0 S^{-1}$.

The switch from the Weyl γ_w^ν to the Dirac γ_d^ν is then given by $\gamma_d^\nu = S\gamma_w^\nu S^{-1}$ and the switch from the Dirac to the Weyl representation by the inverse $\gamma_w^\nu = S^{-1}\gamma_d^\nu S$. This also applies to the α^ν matrices, which are almost always given in their Dirac representation, but who can also be written in the Weyl representation as

$$\alpha_w^\nu = S^{-1}\alpha_d^\nu S = \left(\begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & \hat{\mathbf{1}} \end{bmatrix}, \begin{bmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix} \right). \tag{132}$$

As a logical consequence one has

$$\gamma_0^w \gamma_\nu^w = (\gamma_0^w \gamma_0^w, \gamma_0^w \boldsymbol{\gamma}^w) = (\hat{\mathbf{1}}, \boldsymbol{\alpha}^w) = \alpha_\nu^w. \tag{133}$$

A Weyl adjoint can be defined as $\overline{\Psi}^w = \Psi^{\dagger w}\gamma_0^w$ and a Weyl current as $J_\nu^w = \overline{\Psi}^w \gamma_\nu^w \Psi^w$. This Weyl current is exactly the same as the Dirac current, due to the transformation properties of the spinors under the Dirac to Weyl representation transformation, given as $\Psi_w = S^{-1}\Psi_d$ and $\Psi_w^\dagger = \Psi_d^\dagger S$. On has

$$J_\nu^w = \overline{\Psi}^w \gamma_\nu^w \Psi^w = \Psi^{\dagger w}\gamma_0^w \gamma_\nu^w \Psi^w = \Psi^{\dagger w}\alpha_\nu^w \Psi^w = \Psi^{\dagger d} S S^{-1} \alpha_\nu^d S S^{-1} \Psi^d = \Psi^{\dagger d}\alpha_\nu^d \Psi^d = J_\nu^d. \tag{134}$$

The Dirac level norm-spin set $\Sigma_\mu = (\mathbb{1}, \boldsymbol{\Sigma})$ has it's Weyl representation given by the unchanged

$$\Sigma_\nu^w = S^{-1}(\mathbb{1}, \boldsymbol{\Sigma})S = \left(\begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & \hat{\mathbf{1}} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix} \right). \tag{135}$$

3.1.3. *The closed system condition for the Dirac probability current tensor*

The derivative of the probability density tensor in its closed system condition,

$$\partial_\nu \Phi_\mu{}^\nu \equiv \partial_\nu \Psi^\dagger \gamma_\mu \gamma^\nu \Psi = 0, \tag{136}$$

can be retraced to the Klein Gordon equation on the Dirac level as

$$\partial_\nu \Psi^\dagger \not{V} \not{P} \Psi = \partial_\nu \frac{1}{m_0} \Psi^\dagger \not{P} \not{P} \Psi = \partial_\nu \Psi^\dagger U_0 \not{1} \Psi = U_0 \partial_\nu \Psi^\dagger \Psi = 0. \tag{137}$$

which includes the proof of the closed system condition for the symmetric tensor $\not{T} = \not{V}\not{P}$ as $\partial_\nu \not{T} = 0$. This closed system condition applies to both the Dirac representation as the Weyl representation, as long as it is clear that not only γ_0 but also $\boldsymbol{\alpha}$ and $\overline{\Psi}$ have a Dirac representation and a Weyl representation. The gamma tensor $\gamma_\mu{}^\nu \equiv \gamma_\mu \gamma^\nu$ is given by

$$\gamma_\mu \gamma^\nu = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \begin{bmatrix} \gamma_0\gamma_0 & \gamma_1\gamma_0 & \gamma_2\gamma_0 & \gamma_3\gamma_0 \\ \gamma_0\gamma_1 & \gamma_1\gamma_1 & \gamma_2\gamma_1 & \gamma_3\gamma_1 \\ \gamma_0\gamma_2 & \gamma_1\gamma_2 & \gamma_2\gamma_2 & \gamma_3\gamma_2 \\ \gamma_0\gamma_3 & \gamma_1\gamma_3 & \gamma_2\gamma_3 & \gamma_3\gamma_3 \end{bmatrix} = \begin{bmatrix} \not{1} & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & -\not{1} & -\mathbf{i}\Sigma_3 & \mathbf{i}\Sigma_2 \\ \alpha_2 & \mathbf{i}\Sigma_3 & -\not{1} & -\mathbf{i}\Sigma_1 \\ \alpha_3 & -\mathbf{i}\Sigma_2 & \mathbf{i}\Sigma_1 & -\not{1} \end{bmatrix} \tag{138}$$

The probability density tensor is then given by

$$\Phi_\mu{}^\nu = \Psi^\dagger \gamma_\mu{}^\nu \Psi = \begin{bmatrix} \Psi^\dagger \not{1} \Psi & -\Psi^\dagger \alpha_1 \Psi & -\Psi^\dagger \alpha_2 \Psi & -\Psi^\dagger \alpha_3 \Psi \\ \Psi^\dagger \alpha_1 \Psi & -\Psi^\dagger \not{1} \Psi & -\Psi^\dagger \mathbf{i}\Sigma_3 \Psi & \Psi^\dagger \mathbf{i}\Sigma_2 \Psi \\ \Psi^\dagger \alpha_2 \Psi & \Psi^\dagger \mathbf{i}\Sigma_3 \Psi & -\Psi^\dagger \not{1} \Psi & -\Psi^\dagger \mathbf{i}\Sigma_1 \Psi \\ \Psi^\dagger \alpha_3 \Psi & -\Psi^\dagger \mathbf{i}\Sigma_2 \Psi & \Psi^\dagger \mathbf{i}\Sigma_1 \Psi & -\Psi^\dagger \not{1} \Psi \end{bmatrix}. \tag{139}$$

The time-like part of $\partial_\nu \Phi_\mu{}^\nu = 0$ is given by

$$\frac{1}{c}\partial_t \Psi^\dagger \not{1} \Psi + \nabla_1 \Psi^\dagger \alpha_1 \Psi + \nabla_2 \Psi^\dagger \alpha_2 \Psi + \nabla_3 \Psi^\dagger \alpha_3 \Psi = \frac{1}{c}\partial_t \Psi^\dagger \not{1} \Psi + \boldsymbol{\nabla} \Psi^\dagger \boldsymbol{\alpha} \Psi = 0 \tag{140}$$

This can be abbreviated as the Dirac current continuity equation

$$c\partial_\nu \Psi^\dagger \alpha^\nu \Psi = c\partial_\nu \overline{\Psi} \gamma^\nu \Psi = \partial_\nu J^\nu = 0. \tag{141}$$

This proves that the Klein Gordon equation on the Dirac level includes the continuity equation for the probability current as part of a much stronger closed system condition for the probability density (current-)tensor. That connects the Klein Gordon at Dirac level environment to the Laue closed system condition, which in turn is a basic axiom of or prerequisite for General Relativity's symmetric stress energy density tensors $T = VG$.

The space-like derivatives of $\partial_\nu \Phi_\mu{}^\nu = 0$ can be split into a complex part and a real part. The complex part gives

$$\boldsymbol{\nabla} \times \Psi^\dagger \boldsymbol{\Sigma} \Psi = 0. \tag{142}$$

The real part gives

$$\partial_t \Psi^\dagger \boldsymbol{\alpha} \Psi = c \boldsymbol{\nabla} \Psi^\dagger \not{1} \Psi \tag{143}$$

which can be multiplied by the constants $m_0 c$, and using the Dirac adjoint, to give

$$\partial_t m_0 c \overline{\Psi} \boldsymbol{\gamma} \Psi = \boldsymbol{\nabla} m_0 c^2 \overline{\Psi} \gamma_0 \Psi. \tag{144}$$

The last two conditions show that the closed system condition for the probability density tensor is a stronger condition than the continuity equation on its own. The above two conditions can be connected to the earlier $\boldsymbol{\nabla}\times\boldsymbol{p}=0$ and the $\partial_t\boldsymbol{p}=-\boldsymbol{\nabla}U_i$ as there probability/field analogues. The first prohibits a probability/field vorticity in the closed system condition, the second implies a conserved force-field condition for the probability/field, connecting the time-rate of change of the current to the space divergence of the related density.

Given the fact that all Lagrangians of the Standard Model's Dirac fields are based upon the Dirac current, the Dirac adjoint and the use of the Dirac equation to prove the continuity equation for the Dirac current, it's generalization into a Dirac probability or field tensor with connected much stronger closed system condition and a prove of its validity based upon the Dirac level Klein Gordon equation should have some impact. The recognition that the Dirac current is just a part of a tensor and that the Dirac current continuity equation is just the time-like part of a space-time closed system condition of that tensor will close the gap with General Relativity considerably, given the relation of both to the Laue closed system condition $\partial_\nu T_\mu{}^\nu=0$. I propose to use tensor Lagrangians based on

$$\mathscr{L}=\frac{1}{m_0}\Psi^\dagger\hat{\not{p}}\hat{\not{p}}\Psi, \tag{145}$$

which then contain the inertial probability or inertial field tensor

$$m_\mu{}^\nu c^2=m_0\Phi_\mu{}^\nu c^2=m_0\Psi^\dagger\gamma_\mu\gamma^\nu\Psi c^2, \tag{146}$$

as a relativistic generalization of the usual Dirac current with Dirac adjoint based Lagrangians of the Standard Model.

3.1.4. *The Dirac and Weyl matrices in dual minquat time-space mode as bèta matrices*

What is absent in the above treatments is the Lorentz transformation and the check if all relations that are given are Lorentz invariant or at least Lorentz covariant. The Lorentz transformation of the matrices, the four vectors and the spinors are most elementary in the time-space $(\hat{\mathbf{T}},\mathbf{K})$ Weyl representation. I will call these time-space Weyl-Dirac matrices the bèta matrices.

In my math-phys language and with a Möbius kind of doubling in mind I can define matrices through the application of parity or point reflection P and time reversal or present reflection T as

$$\begin{bmatrix} P & P \\ P^P & P^T \end{bmatrix}=\begin{bmatrix} P & P \\ -P^T & P^T \end{bmatrix}=$$
$$p_0\begin{bmatrix} \hat{\mathbf{T}} & \hat{\mathbf{T}} \\ \hat{\mathbf{T}} & -\hat{\mathbf{T}} \end{bmatrix}+p_1\begin{bmatrix} \hat{\mathbf{I}} & \hat{\mathbf{I}} \\ -\hat{\mathbf{I}} & \hat{\mathbf{I}} \end{bmatrix}+p_2\begin{bmatrix} \hat{\mathbf{J}} & \hat{\mathbf{J}} \\ -\hat{\mathbf{J}} & \hat{\mathbf{J}} \end{bmatrix}+p_3\begin{bmatrix} \hat{\mathbf{K}} & \hat{\mathbf{K}} \\ -\hat{\mathbf{K}} & \hat{\mathbf{K}} \end{bmatrix}=$$
$$p_0\begin{bmatrix} \hat{\mathbf{T}} & \hat{\mathbf{T}} \\ \hat{\mathbf{T}} & -\hat{\mathbf{T}} \end{bmatrix}+\mathbf{p}\cdot\begin{bmatrix} \mathbf{K} & \mathbf{K} \\ -\mathbf{K} & \mathbf{K} \end{bmatrix}\left(=\mathbf{i}p_0\begin{bmatrix} \hat{\mathbf{1}} & \hat{\mathbf{1}} \\ \hat{\mathbf{1}} & -\hat{\mathbf{1}} \end{bmatrix}+\mathbf{i}\mathbf{p}\cdot\begin{bmatrix} \boldsymbol{\sigma} & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & \boldsymbol{\sigma} \end{bmatrix}\right). \tag{147}$$

The norm of this matrix is simply $2P^TP = 2U_0\mathbb{1}$.

I split this into $P_\mu\beta^\mu + P_\mu\xi^\mu$ by defining

$$\not{P} = P_\mu\beta^\mu = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} = p_0\begin{bmatrix} 0 & \hat{\mathbf{T}} \\ \hat{\mathbf{T}} & 0 \end{bmatrix} + \mathbf{p}\cdot\begin{bmatrix} 0 & \mathbf{K} \\ -\mathbf{K} & 0 \end{bmatrix} = p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\beta} = \\ p_0\begin{bmatrix} 0 & \hat{\mathbf{T}} \\ \hat{\mathbf{T}} & 0 \end{bmatrix} + p_1\begin{bmatrix} 0 & \hat{\mathbf{I}} \\ -\hat{\mathbf{I}} & 0 \end{bmatrix} + p_2\begin{bmatrix} 0 & \hat{\mathbf{J}} \\ -\hat{\mathbf{J}} & 0 \end{bmatrix} + p_3\begin{bmatrix} 0 & \hat{\mathbf{K}} \\ -\hat{\mathbf{K}} & 0 \end{bmatrix} \tag{148}$$

with $\not{P} = P_\mu\beta^\mu = p_0\beta_0 + p_1\beta_1 + p_2\beta_2 + p_3\beta_3$, and

$$P_\mu\xi^\mu = \begin{bmatrix} P & 0 \\ 0 & P^T \end{bmatrix} = p_0\begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & -\hat{\mathbf{T}} \end{bmatrix} + \mathbf{p}\cdot\begin{bmatrix} \mathbf{K} & 0 \\ 0 & \mathbf{K} \end{bmatrix} = p_0\xi_0 + \mathbf{p}\cdot\boldsymbol{\xi} = \\ p_0\begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & -\hat{\mathbf{T}} \end{bmatrix} + p_1\begin{bmatrix} \hat{\mathbf{I}} & 0 \\ 0 & \hat{\mathbf{I}} \end{bmatrix} + p_2\begin{bmatrix} \hat{\mathbf{J}} & 0 \\ 0 & \hat{\mathbf{J}} \end{bmatrix} + p_3\begin{bmatrix} \hat{\mathbf{K}} & 0 \\ 0 & \hat{\mathbf{K}} \end{bmatrix} \tag{149}$$

with $P_\mu\xi^\mu = p_0\xi_0 + p_1\xi_1 + p_2\xi_2 + p_3\xi_3$.

If I use $\hat{\mathbf{T}} = \mathbf{i}\hat{\mathbf{1}}$ and $\mathbf{K} = \mathbf{i}\boldsymbol{\sigma}$ I get

$$\beta_\mu = (\beta_0, \boldsymbol{\beta}) = \left(\begin{bmatrix} 0 & \mathbf{i}\hat{\mathbf{1}} \\ \mathbf{i}\hat{\mathbf{1}} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{i}\boldsymbol{\sigma} \\ -\mathbf{i}\boldsymbol{\sigma} & 0 \end{bmatrix} \right) = (\mathbf{i}\hat{\mathbf{1}}, \mathbf{i}\boldsymbol{\gamma}) = \mathbf{i}\gamma_\mu \tag{150}$$

which relates the parity dual β_μ to the Weyl gamma representation. The Dirac representation mixes the bèta and the xi representation and thus represents a PT dual. I nevertheless, using the gamma tradition, use the bèta and Feynman slash symbols for both representations in the time-space $(\hat{\mathbf{T}}, \mathbf{K})$ basis. This gives for the Dirac bèta representation

$$\not{P} = P_\mu\beta^\mu = p_0\begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & -\hat{\mathbf{T}} \end{bmatrix} + \mathbf{p}\cdot\begin{bmatrix} 0 & \mathbf{K} \\ -\mathbf{K} & 0 \end{bmatrix} = p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\beta} = \\ p_0\begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & -\hat{\mathbf{T}} \end{bmatrix} + p_1\begin{bmatrix} 0 & \hat{\mathbf{I}} \\ -\hat{\mathbf{I}} & 0 \end{bmatrix} + p_2\begin{bmatrix} 0 & \hat{\mathbf{J}} \\ -\hat{\mathbf{J}} & 0 \end{bmatrix} + p_3\begin{bmatrix} 0 & \hat{\mathbf{K}} \\ -\hat{\mathbf{K}} & 0 \end{bmatrix}. \tag{151}$$

As with the Weyl representation, in the Dirac representation we have $\beta_\mu = \mathbf{i}\gamma_\mu$.

The transformation matrix S remains unchanged. But its interpretation can be enriched. It isn't just a neutral change of representations, it changes a parity only Weyl dual representation of space-time into a combined parity, time reversal Dirac dual representation of space-time (and vice versa). The transformation operation S adds or removes time reversal from the dual, it is a time reversal transformation.

3.2. *The Dirac and Weyl equations in the space-time bèta matrices environment*

The trick in formulating equations in the Dirac environment is that they have to be reducible to the Klein Gordon energy condition $P^TP = E^2\hat{\mathbf{1}}$ with $E = \frac{U_0}{c} = m_0c$. We have three equations that match this demand, but only the first two use a Clifford four set. The third equation uses tricks to compensate for the limitations of a Clifford three set in a 4-D environment. In the Weyl and Dirac equations we can split $-E^2\mathbb{1}$ using the ξ matrix, as

$\not{E}^2 = (E\xi)^2 = -E^2\not{1}$, with the eigen time matrix ξ, defined as

$$\xi = \begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & \hat{\mathbf{T}} \end{bmatrix}. \tag{152}$$

The Weyl or chiral equation stems from the quadratic $\not{P}\not{P} = \not{E}\not{E}$ in the space-time Weyl representation.

$$\not{P}\not{P} = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} = \begin{bmatrix} -PP^T & 0 \\ 0 & -P^T P \end{bmatrix} = \begin{bmatrix} -E^2\hat{\mathbf{1}} & 0 \\ 0 & -E^2\hat{\mathbf{1}} \end{bmatrix} = -E^2\not{1} = \not{E}\not{E} \tag{153}$$

So we have $\not{P}\not{P} - \not{E}\not{E} = 0$. This leads to $(\not{P} - \not{E})(\not{P} + \not{E}) = 0$. If we split this into two equations, $\not{P} - \not{E} = 0$ and $\not{P} + \not{E} = 0$, then only the trivial all zero solution is possible. But if we add the Dirac spinors, then non zero solutions are possible. We get $\Psi^\dagger(\not{P} - \not{E})(\not{P} + \not{E})\Psi = 0$, which can be split into $\Psi^\dagger(\not{P} - \not{E}) = 0$ and $(\not{P} + \not{E})\Psi = 0$. By interpreting the spinors as waves or wave-like fields all the solutions of those equations can be interpreted as eigenvalue solutions of related operators and we get the Weyl wave equations as

$$\hat{\not{P}}\Psi = \not{E}\Psi \tag{154}$$
$$\hat{\not{P}}\Psi = -\not{E}\Psi \tag{155}$$

if we use $\hat{\not{P}} = -\mathbf{i}\hbar\not{\partial}$ and a four column dual spinor Ψ.

The Dirac equation stems from the quadratic $(p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\beta})^2 = -E^2\not{1}$.

$$\not{P}\not{P} = \begin{bmatrix} p_0\hat{\mathbf{T}} & \mathbf{p}\cdot\mathbf{K} \\ -\mathbf{p}\cdot\mathbf{K} & -p_0\hat{\mathbf{T}} \end{bmatrix} \begin{bmatrix} p_0\hat{\mathbf{T}} & \mathbf{p}\cdot\mathbf{K} \\ -\mathbf{p}\cdot\mathbf{K} & -p_0\hat{\mathbf{T}} \end{bmatrix} = \begin{bmatrix} (-p_0^2 + \mathbf{p}^2)\hat{\mathbf{1}} & 0 \\ 0 & (-p_0^2 + \mathbf{p}^2)\hat{\mathbf{1}} \end{bmatrix} = -E^2\not{1} \tag{156}$$

This leads to the two options for the Dirac equations

$$(\hat{p}_0\beta_0 + \hat{\mathbf{p}}\cdot\boldsymbol{\beta})\Psi = E\not{1}\Psi \tag{157}$$
$$(\hat{p}_0\beta_0 + \hat{\mathbf{p}}\cdot\boldsymbol{\beta})\Psi = -E\not{1}\Psi \tag{158}$$

if we use $\hat{P} = -\mathbf{i}\hbar\partial$ and a four column spinor Ψ.

So in the space-time representation we have the Weyl $\not{P}$ as

$$\not{P}_w = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \tag{159}$$

and the Dirac $\not{P}$ as

$$\not{P}_d = \begin{bmatrix} p_0\hat{\mathbf{T}} & \mathbf{p}\cdot\mathbf{K} \\ -\mathbf{p}\cdot\mathbf{K} & -p_0\hat{\mathbf{T}} \end{bmatrix} \tag{160}$$

and the transformation between them as $\not{P}_w = S^{-1}\not{P}_d S$ and $\not{P}_d = S\not{P}_w S^{-1}$.

3.3. *Lorentz transformations of the vectors in the Dirac and Weyl representation environments*

In the Pauli level part of this paper I developed the $(\hat{\mathbf{T}}, \mathbf{K})$ relativistic approach. This resulted in the Lorentz transformation of a four vector $P = (p_0\hat{\mathbf{T}}, \boldsymbol{p}\cdot\mathbf{K})$ as $P^L = U^{-1}PU^{-1}$ and the Lorentz transformation of its time reversal P^T as $(P^L)^T = (P^T)^{L^{-1}} = UP^TU$ with U as

$$U = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix} \tag{161}$$

and the rapidity ψ. The Lorentz transformation of its time reversal P^T was $(P^L)^T = (P^T)^{L^{-1}} = UP^TU$. The quadratic P^TP then is automatically a Lorentz invariant scalar $\frac{U_0^2}{c^2}\hat{\mathbf{1}}$ with the dimension of the norm $\hat{\mathbf{1}}$. If in the space-time representation we have the Weyl $\not{P}$ in a reference system S as

$$\not{P} = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \tag{162}$$

then in reference system S' we have P^L and so also the Weyl $\not{P}^L$ as

$$\not{P}^L = \begin{bmatrix} 0 & P^L \\ -(P^L)^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & U^{-1}PU^{-1} \\ -UP^TU & 0 \end{bmatrix} \tag{163}$$

The question then is how to generate this result. The obvious answer is

$$\not{P}_w^L = \Lambda^{-1}\not{P}_w\Lambda = \begin{bmatrix} U^{-1} & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & U^{-1} \end{bmatrix} = \begin{bmatrix} 0 & U^{-1}PU^{-1} \\ -UP^TU & 0 \end{bmatrix} \tag{164}$$

with the Lorentz transformation matrix

$$\Lambda = \begin{bmatrix} U & 0 \\ 0 & U^{-1} \end{bmatrix} \tag{165}$$

and its obvious inverse Λ^{-1}.

As for the generator of Λ, we have

$$\Lambda = \begin{bmatrix} U & 0 \\ 0 & U^{-1} \end{bmatrix} = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 & 0 & 0 \\ 0 & e^{-\frac{\psi}{2}} & 0 & 0 \\ 0 & 0 & e^{-\frac{\psi}{2}} & 0 \\ 0 & 0 & 0 & e^{\frac{\psi}{2}} \end{bmatrix} =$$

$$\begin{bmatrix} \cosh\left(\frac{\psi}{2}\right) & 0 & 0 & 0 \\ 0 & \cosh\left(\frac{\psi}{2}\right) & 0 & 0 \\ 0 & 0 & \cosh\left(\frac{\psi}{2}\right) & 0 \\ 0 & 0 & 0 & \cosh\left(\frac{\psi}{2}\right) \end{bmatrix} +$$

$$\begin{bmatrix} \sinh\left(\frac{\psi}{2}\right) & 0 & 0 & 0 \\ 0 & -\sinh\left(\frac{\psi}{2}\right) & 0 & 0 \\ 0 & 0 & -\sinh\left(\frac{\psi}{2}\right) & 0 \\ 0 & 0 & 0 & \sinh\left(\frac{\psi}{2}\right) \end{bmatrix} =$$

$$\cosh\left(\frac{\psi}{2}\right) \begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & \hat{\mathbf{1}} \end{bmatrix} + \sinh\left(\frac{\psi}{2}\right) \begin{bmatrix} \sigma_I & 0 \\ 0 & -\sigma_I \end{bmatrix} =$$

$$\not{1} \cosh\left(\frac{\psi}{2}\right) - \alpha_I \sinh\left(\frac{\psi}{2}\right) = \not{1} e^{-\alpha_I\left(\frac{\psi}{2}\right)} \tag{166}$$

with the α_I in its Weyl representation. The inverse is then obviously given by $\Lambda^{-1} = \not{1} e^{\alpha_I\left(\frac{\psi}{2}\right)}$.

The Klein Gordon equation's Lorentz invariance or covariance depends on the products $\not{p}^L \not{p}^L$. Using the previous result, we have for the Lorentz transformation of the product $\not{p}\not{p}$ in the Weyl representation

$$\not{p}^L \not{p}^L = \Lambda^{-1} \not{p} \Lambda \Lambda^{-1} \not{p} \Lambda = \Lambda^{-1} \not{p}\not{p} \Lambda = \Lambda^{-1} \not{E}\not{E} \Lambda = -E^2 \not{1} \Lambda^{-1} \Lambda = -E^2 \not{1} = \not{p}\not{p}, \tag{167}$$

so a Lorentz invariant product. This proof then included that $\not{E}^L \not{E}^L = \not{E}\not{E}$. This ensures the Lorentz invariance of the Klein Gordon condition $\not{p}\not{p} = \not{E}\not{E}$ in the Weyl representation.

In the Dirac version, where $\not{P} = p_0 \beta_0 + \mathbf{p} \cdot \boldsymbol{\beta}$, things get more complicated. We have to start with the Dirac $\not{P}_d$ in the primary reference system and we want to end up with $\not{P}_d^L$ in the secondary reference system. We know how to transform between the Dirac and the Weyl representations and we know how to Lorentz transform the Weyl $\not{P}_w$. This means we have to go from Dirac to Weyl in the primary reference system, then Lorentz transform the Weyl four vector to the secondary reference system and then transform back from the Weyl to the Dirac representation, three operations in total. The total result gives

$$\not{P}_d^L = S \Lambda^{-1} S^{-1} \not{P}_d S \Lambda S^{-1}. \tag{168}$$

For the Klein Gordon equation in the Dirac representation, we get the Lorentz invariance through

$$\begin{aligned} \not{P}_d^L \not{P}_d^L &= S\Lambda^{-1}S^{-1}\not{P}_d S\Lambda S^{-1} S\Lambda^{-1}S^{-1}\not{P}_d S\Lambda S^{-1} = S\Lambda^{-1}S^{-1}\not{P}_d S\Lambda\Lambda^{-1}S^{-1}\not{P}_d S\Lambda S^{-1} = \\ & S\Lambda^{-1}S^{-1}\not{P}_d SS^{-1}\not{P}_d S\Lambda S^{-1} = S\Lambda^{-1}S^{-1}\not{P}_d\not{P}_d S\Lambda S^{-1} = S\Lambda^{-1}S^{-1}\not{E}_d\not{E}_d S\Lambda S^{-1} = \\ & -E^2\not{1} S\Lambda^{-1}S^{-1}S\Lambda S^{-1} = -E^2\not{1} S\Lambda^{-1}\Lambda S^{-1} = -E^2\not{1} SS^{-1} = -E^2\not{1} = \not{P}_d\not{P}_d \end{aligned} \tag{169}$$

The Lorentz transformation of the Dirac representation momentum four vector goes as

$$\not{p}_d^L = S\Lambda^{-1}S^{-1}\not{P}_d S\Lambda S^{-1}. \tag{170}$$

In details, with rapidity ψ, the operator $S\Lambda^{-1}S^{-1}$ is given as

$$S\Lambda^{-1}S^{-1} = \begin{bmatrix} \cosh(\frac{\psi}{2})\hat{\mathbf{1}} & \sinh(\frac{\psi}{2})\sigma_I \\ \sinh(\frac{\psi}{2})\sigma_I & \cosh(\frac{\psi}{2}\hat{\mathbf{1}} \end{bmatrix} = \not{1}\,\cosh(\frac{\psi}{2}) + \alpha_I\,\sinh(\frac{\psi}{2}) = \not{1}e^{\left(\alpha_I\frac{\psi}{2}\right)}, \tag{171}$$

with $\not{1}e^{\left(\alpha_I\frac{\psi}{2}\right)}$ as the generator of the Lorentz boost. The operator $S\Lambda S^{-1}$ is given as

$$S\Lambda S^{-1} = \begin{bmatrix} \cosh(\frac{\psi}{2})\hat{\mathbf{1}} & -\sinh(\frac{\psi}{2})\sigma_I \\ -\sinh(\frac{\psi}{2})\sigma_I & \cosh(\frac{\psi}{2}\hat{\mathbf{1}} \end{bmatrix} = \not{1}\,\cosh(\frac{\psi}{2}) - \alpha_I\,\sinh(\frac{\psi}{2}) = \not{1}e^{-\left(\alpha_I\frac{\psi}{2}\right)}. \tag{172}$$

If we look at the generator for a Lorentz transformation along the $\hat{\mathbf{I}}$-axis, containing α_I and a rapidity ψ as the boost parameter, the generator language might be generalized into the product of a matrix ${\gamma_\mu}^\nu$ and a parameter matrix ${\omega_\mu}^\nu$ resulting in the Poincaré group generator of boosts α, rotations $\mathbf{i}\Sigma$ and length gauges $\not{1}$

$$\not{1}e^{-\left({\gamma_\mu}^\nu{\omega_\mu}^\nu\right)}. \tag{173}$$

Each single Poincaré operation/transformation may contain only one non-zero parameter in ${\omega_\mu}^\nu$ at a time. And in order to avoid the Thomas precession complication, it is advised to align reference systems along the $\hat{\mathbf{I}}$-axis using rotations before performing a Lorentz boost. The generator formalism using ${\gamma_\mu}^\nu$ works identically in both the Weyl representation as in the Dirac representation, with gamma-matrices as well as with bèta matrices, the last because ${\beta_\mu}^\nu = -{\gamma_\mu}^\nu$.

In the transformation of the four vector we have $\not{P}_d = P_\mu\beta^\mu$. Because the operators only work on the matrix aspect of β^μ the Lorentz transformation can also be written as

$$\not{P}^L = e^{\left(\alpha_I\frac{\psi}{2}\right)}\not{P}e^{-\left(\alpha_I\frac{\psi}{2}\right)} = S\Lambda^{-1}S^{-1}P_\mu\beta^\mu S\Lambda S^{-1} = P_\mu S\Lambda^{-1}S^{-1}\beta^\mu S\Lambda S^{-1} \tag{174}$$

and we can focus on

$$(\beta^\mu)^L = S\Lambda^{-1}S^{-1}\beta^\mu S\Lambda S^{-1} \tag{175}$$

thus interpreting the Lorentz transformation as a boost of the metric. Using the Lorentz transformation expression of the operator combinations $S\Lambda^{-1}S^{-1}$ and $S\Lambda S^{-1}$ in terms of the rapidity and the hyperbolic trigonometric expressions, we can calculate the result on the bèta matrices of the $S\Lambda^{-1}S^{-1}$ and $S\Lambda S^{-1}$ operators. After some calculations this results in

$$(\beta^\mu)^L = S\Lambda^{-1}S^{-1}\beta^\mu S\Lambda S^{-1} = {\Lambda_\mu}^\nu\beta^\mu = \beta^\nu \tag{176}$$

with, given the usual Lorentz boost $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$,

$$(\beta^\mu)^L = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}^L = \Lambda_\mu^\nu\beta^\mu = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} \gamma\beta_0 - \beta\gamma\beta_1 \\ \gamma\beta_1 - \beta\gamma\beta_0 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \beta^\nu. \tag{177}$$

The Lorentz transformation of $\not{P}$ can then be given as

$$\not{P}^L = P_\mu S\Lambda^{-1}S^{-1}\beta^\mu S\Lambda S^{-1} = P_\nu(\Lambda_\mu^{\ \nu}\beta^\mu) = P_\nu\beta^\nu. \tag{178}$$

This result allows us to return to the original interpretation of the Lorentz transformation as a change of the coordinates against the background of a fixed metric, because

$$\not{P}^L = P_\nu(\Lambda_\mu^{\ \nu}\beta^\mu) = (P_\nu\Lambda_\mu^{\ \nu})\beta^\mu = P_\mu\beta^\mu. \tag{179}$$

In the space-time Weyl representation the results were the same, giving

$$(\beta^\mu)^L = \Lambda^{-1}\beta^\mu\Lambda = \Lambda_\mu^{\ \nu}\beta^\mu = \beta^\nu \tag{180}$$

The ease of the Lorentz transformation and proving Lorentz covariance of invariance in the developed math-phys environment can be contrasted with the usual approach as critically analyzed and alternatively presented in [15]. The relation $S\Lambda^{-1}S^{-1}\beta^\mu S\Lambda S^{-1} = \Lambda_\mu^{\ \nu}\beta^\mu$ for the Dirac matrices in this paper has been derived based on the derivation of the same relation for the Weyl bèta matrices. In the usual approach, this relationship is formulated as a requisite in order to have a Lorentz covariant Dirac equation, see for example [16, p. 147, Eqn. 5.102] and [17, p. 138, Eqn. 3.34]. The operator S in [16, p. 147, Eqn. 5.102] and equals my operator $S\Lambda^{-1}S^{-1}$. In the words of Greiner: *To find S means solving (3.34).* Stone described the procedure as *The Lorentz covariance of the Dirac equation is guaranteed if there exists a matrix representation $S(L)$ of the Lorentz group so that for any Lorentz transformation $L^\mu{}_\nu$ there exists a matrix $S(L)$ such that $S(L)\gamma^\mu S^{-1}(L) = (L^{-1})^\mu{}_\nu\gamma^\nu$ [18, p. 73].* A similar reasoning is given in [19, p. 42] and in [20, p. 93]. That means they have to solve $S\Lambda^{-1}S^{-1}\beta^\mu S\Lambda S^{-1} = \Lambda_\mu^{\ \nu}\beta^\mu$ for S, a relation that I constructed and proved instead of solved, mainly because of its very close connection to the Lorentz transformation approach in the biquaternion representation of the Pauli level physics.

My approach confirms the claim that the gamma matrices transform like a regular four-vector, as long as one realizes that in the $\not{P} = P_\mu\gamma^\mu$ notation, the Lorentz transformation is either performed on P_μ or on γ^μ. So when the dual pauliquat gamma-matrices (or my dual minquat bèta matrices) are used to Lorentz transform $\not{P}$, then the coordinates P_μ are invariant and vice versa. Either the metric is Doppler twisted or the coordinates are, but not both.

3.4. *Lorentz transformations of the spinors in the Dirac and Weyl representation environments*

As for the Lorentz transformation of a Weyl 4-spinor, we have the requirement that we want the Lagrangian density element $\mathscr{L} = \frac{1}{m_0}\Psi^\dagger\not{P}\not{P}\Psi$ to be Lorentz covariant (or invariant if possible). To arrive at this requirement of covariance I start with

$$\mathscr{L}^L = \frac{1}{m_0}(\Psi^L)^\dagger\not{P}^L\not{P}^L\Psi^L = \frac{1}{m_0}(\Psi^L)^\dagger\Lambda^{-1}\not{P}\Lambda\Lambda^{-1}\not{P}\Lambda\Psi^L =$$
$$\frac{1}{m_0}(\Psi^L)^\dagger\Lambda^{-1}\not{P}\not{P}\Lambda\Psi^L(=?)\frac{1}{m_0}(\Psi^\dagger)\not{P}\not{P}\Psi \tag{181}$$

This implies first that we would like to have

$$\Lambda\Psi^L = \Psi \tag{182}$$

so

$$\Psi^L = \Lambda^{-1}\Psi. \tag{183}$$

Inserting this into the Lagrangian's Lorentz transformation gives us

$$\mathscr{L}^L = \frac{1}{m_0}(\Psi^L)^\dagger \Lambda^{-1} \not{p}\not{p}\Lambda\Lambda^{-1}\Psi = \frac{1}{m_0}(\Psi^L)^\dagger \Lambda^{-1}\not{p}\not{p}\Psi, \tag{184}$$

which, through the detail $(\not{p}\not{p}\Psi)^L = \Lambda^{-1}\not{p}\not{p}\Psi$, shows that the spinor $\not{p}\not{p}\Psi$ behaves as Ψ under a Lorentz transformation. From $\Psi^L = \Lambda^{-1}\Psi$ we can derive the relation

$$(\Psi^L)^\dagger = (\Lambda^{-1}\Psi)^\dagger = \Psi^\dagger\Lambda^{-1}, \tag{185}$$

due to the fact that Λ is diagonal real and thus equal to its conjugate transpose. Inserting this into the result we had thus far gives

$$\mathscr{L}^L = \frac{1}{m_0}(\Psi^L)^\dagger \Lambda^{-1}\not{p}\not{p}\Psi = \frac{1}{m_0}\Psi^\dagger\Lambda^{-1}\Lambda^{-1}\not{p}\not{p}\Psi = \frac{1}{m_0}\Psi^\dagger e^{-\alpha_I\psi}\not{p}\not{p}\Psi \tag{186}$$

This result has an interesting interpretation: the factor $e^{-\alpha_I\psi}$ represents a full Doppler shift of the probability/field density. A single spinor obtains half a Doppler shift with Λ^{-1} and its square $\Psi^\dagger\Psi$ obtains a full Doppler shift as the result of a Lorentz boost. With this interpretation, a 'solution' as for example in the form of the introduction of a Dirac adjoint, isn't wanted because the outcome is what should be hoped for. We are working on the Dirac level of a dual Maxwell-Lorentz structured environment, so a full relativistic Doppler shift as the result of a Lorentz boost of wave-like density is a welcomed result that doesn't need a fix. Thus the result

$$\begin{aligned}(\Psi^\dagger\Psi)^L = (\Psi^L)^\dagger(\Psi^L) = (\Lambda^{-1}\Psi)^\dagger(\Lambda^{-1}\Psi) = \Psi^\dagger\Lambda^{-1}\Lambda^{-1}\Psi = \Psi^\dagger e^{-\alpha_I\psi}\Psi = \\ \Psi^\dagger\Psi\,\cosh(\psi) + \Psi^\dagger\alpha_I\Psi\,\sinh(\psi) = \Psi^\dagger\Psi\gamma + \Psi^\dagger\alpha_I\Psi\gamma\beta,\end{aligned} \tag{187}$$

with Lorentz boost $\gamma = \cosh(\psi)$ and $\gamma\beta = \sinh(\psi)$, is what should be expected and wanted. It doesn't need a fix. A relativistic Doppler shift of a wave phenomenon should be a desired outcome of the Lorentz boost of that phenomenon, given the Maxwell-Lorentz structured environment.

The conditon $\Psi^L = \Lambda^{-1}\Psi$ gives

$$\Psi_w^L = \Lambda^{-1}\Psi_w = \begin{bmatrix} U^{-1} & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} \Psi_w^1 \\ \Psi_w^2 \end{bmatrix} = \begin{bmatrix} U^{-1}\Psi_w^1 \\ U\Psi_w^2 \end{bmatrix}. \tag{188}$$

Important in this last equation is the result that the bispinors Ψ^1 and Ψ^2 do not mix in the Lorentz transformation in the space-time Weyl representation.

The same line of reasoning will give us the Lorentz transformation rules for the spinors in the space-time Dirac representation, respectively

$$\Psi_d^L = S\Lambda^{-1}S^{-1}\Psi_d \tag{189}$$

and

$$(\Psi^L)^\dagger_d = (\Psi^\dagger_d) S\Lambda^{-1}S^{-1}, \tag{190}$$

so with the same 'problem' for the Λ^{-1}'s but now including the S operator. The result is

$$(\Psi^\dagger_d \Psi_d)^L = (\Psi^L)^\dagger_d \Psi^L_d = (\Psi^\dagger_d) S\Lambda^{-1}S^{-1}S\Lambda^{-1}S^{-1}\Psi_d = (\Psi^\dagger_d) S\Lambda^{-2}S^{-1}\Psi_d. \tag{191}$$

In the Dirac representation, we have to calculate $S\Lambda^{-2}S^{-1}$ in order to be able to evaluate the result. In details, with rapidity ψ, the operator $S\Lambda^{-2}S^{-1}$ is given as

$$S\Lambda^{-2}S^{-1} = \begin{bmatrix} \cosh(\psi)\hat{\mathbf{1}} & \sinh(\psi)\sigma_I \\ \sinh(\psi)\sigma_I & \cosh(\psi)\hat{\mathbf{1}} \end{bmatrix} = \mathbb{1}\cosh(\psi) + \alpha_I \sinh(\psi) = \mathbb{1}e^{(\alpha_I\psi)}, \tag{192}$$

with $\mathbb{1}e^{(\alpha_I\psi)}$ as the generator of the Lorentz boost delivered Doppler shift of the probability/field density, as

$$(\Psi^\dagger\Psi)^L = \Psi^\dagger e^{(\alpha_I\psi)}\Psi = \Psi^\dagger\Psi\cosh(\psi) + \Psi^\dagger\alpha_I\Psi\sinh(\psi). \tag{193}$$

Zooming in further, we get for the Dirac representation and using $\cosh(\psi) = \gamma$ and $\sinh(\psi) = \gamma\beta$

$$(\Psi^\dagger\Psi)^L =$$

$$\begin{bmatrix} \Psi_1^* & \Psi_2^* & \Psi_3^* & \Psi_4^* \end{bmatrix} \begin{bmatrix} \gamma & 0 & \gamma\beta & 0 \\ 0 & \gamma & 0 & -\gamma\beta \\ \gamma\beta & 0 & \gamma & 0 \\ 0 & -\gamma\beta & 0 & \gamma \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{bmatrix} \tag{194}$$

$$= \gamma\Psi_1^*\Psi_1 + \gamma\beta\Psi_1^*\Psi_3 + \gamma\Psi_2^*\Psi_2 - \gamma\beta\Psi_2^*\Psi_4 \tag{195}$$

$$+\gamma\Psi_3^*\Psi_3 + \gamma\beta\Psi_3^*\Psi_1 + \gamma\Psi_4^*\Psi_4 - \gamma\beta\Psi_4^*\Psi_2. \tag{196}$$

We see that in the Dirac representation, boosting the probability density mixes the spinors and thus the particles and the anti-particles, the electrons and the positrons. In the Weyl representation, boosting the probability density doesn't mix the spinors because then we have a diagonal matrix in the Lorentz boost operator, as

$$(\Psi^\dagger\Psi)^L =$$

$$\begin{bmatrix} \Psi_1^* & \Psi_2^* & \Psi_3^* & \Psi_4^* \end{bmatrix} \begin{bmatrix} \gamma-\gamma\beta & 0 & 0 & 0 \\ 0 & \gamma+\gamma\beta & 0 & 0 \\ 0 & 0 & \gamma+\gamma\beta & 0 \\ 0 & 0 & 0 & \gamma-\gamma\beta \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{bmatrix} \tag{197}$$

$$= \gamma\Psi_1^*\Psi_1 - \gamma\beta\Psi_1^*\Psi_1 + \gamma\Psi_2^*\Psi_2 + \gamma\beta\Psi_2^*\Psi_2 \tag{198}$$

$$+\gamma\Psi_3^*\Psi_3 + \gamma\beta\Psi_3^*\Psi_3 + \gamma\Psi_4^*\Psi_4 - \gamma\beta\Psi_4^*\Psi_4 = \tag{199}$$

$$\Psi_1^*\Psi_1 e^{-\psi} + \Psi_2^*\Psi_2 e^{\psi} + \Psi_3^*\Psi_3 e^{\psi} + \Psi_4^*\Psi_4 e^{-\psi} \tag{200}$$

What we get is that the wave equations with only Ψ, so without $\Psi^\dagger$, behave as Lorentz covariant spinors. But the action of multiplying this spinor by its adjoint in order to arrive at an observable seems observer specific. The trick of the Dirac adjoint doesn't solve this issue, it just hides it deeper in the formalism. It shouldn't be that much of a surprise to find out that the observables, the observable probabilities of measuring specific eigenstates, in relativistic quantum mechanics are observer specific. So the waves are covariant spinors, the reduction of such a wave to an observable is reference frame specific, expressed mathematically in the appearance of the factor $\Lambda^2 = \not{1} e^{-\alpha_I \psi}$, representing a full Doppler shift. In terms of the structure, we have a Lorentz invariant $\not{P}\not{P}$, Lorentz covariant $\gamma_\mu{}^\nu$ and $\gamma_\mu{}^\nu \Psi$ but not a covariant $\Phi_\mu{}^\nu = \Psi^\dagger \gamma_\mu{}^\nu \Psi$. We do however have a procedure to arrive at $(\Phi_\mu{}^\nu)^L$, a recipe so to speak, that is Lorentz covariant: we add a Doppler shift and get

$$(\Phi_\mu{}^\nu)^L = \frac{1}{m_0} \Psi^\dagger e^{-\alpha_I \psi} \beta_\mu{}^\nu \Psi = (\Phi_\mu{}^\nu)\, \cosh(\psi) + (\frac{1}{m_0} \Psi^\dagger \alpha_I \beta_\mu{}^\nu \Psi)\, \sinh(\psi). \tag{201}$$

That allows for the translation of results of quantum measurements on the Dirac level between reference systems connected by a Lorentz boost Λ and a relativistic Doppler shift e^ψ. The result that relativistic quantum fields behave relativistically as EM radiation/photon fields should be highly satisfying.

The operator $S\Lambda^{-1}S^{-1}$ for the Lorentz transformation of the Dirac spinor Ψ exactly matches the one in [21]. The structure of these transformations look familiar. If we define $\gamma' = \cosh(\frac{\psi}{2})$ and $\gamma'\beta' = \sinh(\frac{\psi}{2})$, we get the Lorentz transformation of Ψ as

$$\Psi^L = \begin{bmatrix} \gamma' \hat{\mathbf{1}} & \gamma'\beta'\sigma_I \\ \gamma'\beta'\sigma_I & \gamma'\hat{\mathbf{1}} \end{bmatrix} \begin{bmatrix} \Psi^1 \\ \Psi^2 \end{bmatrix} = \begin{bmatrix} \gamma'\hat{\mathbf{1}}\Psi^1 + \gamma'\beta'\sigma_I \Psi^2 \\ \gamma'\hat{\mathbf{1}}\Psi^2 + \gamma'\beta'\sigma_I \Psi^1 \end{bmatrix}. \tag{202}$$

In the hyperbolic formulation, the details of the Lorentz transformation of Ψ gives

$$\Psi^L = \begin{bmatrix} (\Psi^1)^L \\ (\Psi^2)^L \end{bmatrix} = \begin{bmatrix} \cosh(\frac{\psi}{2})\hat{\mathbf{1}} & \sinh(\frac{\psi}{2})\sigma_I \\ \sinh(\frac{\psi}{2})\sigma_I & \cosh(\frac{\psi}{2}\hat{\mathbf{1}} \end{bmatrix} \begin{bmatrix} \Psi^1 \\ \Psi^2 \end{bmatrix} = \begin{bmatrix} \cosh(\frac{\psi}{2})\hat{\mathbf{1}}\Psi^1 + \sinh(\frac{\psi}{2})\sigma_I \Psi^2 \\ \cosh(\frac{\psi}{2})\hat{\mathbf{1}}\Psi^2 + \sinh(\frac{\psi}{2})\sigma_I \Psi^1 \end{bmatrix}. \tag{203}$$

What we see here is that the Lorentz transformation of the Dirac spinor mixes the two twin Pauli spinors Ψ^1 and Ψ^2. As a consequence, one cannot Lorentz transform a single Pauli spinor in the Dirac representation, so a Lorentz transformation of the Pauli equation without the full Dirac twin is impossible. The Pauli equation on its own cannot possibly be relativistic, not because of the Pauli spin matrices, as is usually thought [22], but due to the spinors involved. At the end of the first Pauli-level part of this paper, I showed that the Pauli-spin energy terms can be derived at the Pauli-level of two by two matrices in a fully relativistic approach, except for the spinors. The spinor representing the Pauli electron with spin up or down can on its own only represent stationary states because in isolation it cannot be boosted. Where the Pauli equation describes an electron in either spin up or spin down situation, its Dirac twin does the same with the positron in either spin up or spin down. In the Dirac representation giving an electron as a spinor a relativistic boost, thus mixing the two spinors after the boost, necessarily involves the positron. Giving an electron a boost can be done by letting it absorb a photon, thus realizing a quantum jump. So the quantum jump of the electron necessarily involves its antiparticle, the positron. As a consequence, in the Schrödinger and the Pauli environment quantum jumps must remain a mystery. In other words, it is rather a waist of time to try to fully understand and analyze the intrinsic aspects of quantum jumps in the Schrödinger and the Pauli theories. This can only be achieved on the Dirac level, by including both Ψ^1 and Ψ^2 (and A_ν, as for example in the form of a Feynman vertex).

3.5. *The Klein Gordon spinor equation in its Dirac level full potential*

Given the general Lagrangian density $\mathscr{L} = \frac{1}{m_0}\Psi^\dagger(\hat{\not{P}}\hat{\not{P}} - \not{E}\not{E})\Psi$ in the space-time Dirac representation one gets the Klein Gordon equation from

$$\frac{\partial \mathscr{L}}{\partial \Psi^\dagger} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial^\mu \Psi^\dagger)} \right) = 0 \tag{204}$$

resulting in

$$\frac{1}{m_0}(\hat{\not{P}}\hat{\not{P}} - \not{E}\not{E})\Psi = 0. \tag{205}$$

In the canonical version we get

$$\frac{1}{m_0}(\hat{\not{P}} + q\not{A})(\hat{\not{P}} + q\not{A})\Psi = \frac{1}{m_0}\not{E}\not{E}\Psi. \tag{206}$$

leading to

$$\frac{1}{m_0}\hat{\not{P}}\hat{\not{P}}\Psi + \frac{q}{m_0}\hat{\not{P}}\not{A}\Psi + \frac{q}{m_0}\not{A}\hat{\not{P}}\Psi + \frac{q^2}{m_0}\not{A}\not{A}\Psi = -U_0\not{1}\Psi \tag{207}$$

and using $\hat{\not{P}} = -\mathbf{i}\hbar\not{\partial}$ we get

$$-\frac{\hbar^2}{m_0}\not{\partial}\not{\partial}\Psi - \frac{\mathbf{i}q\hbar}{m_0}\not{\partial}\not{A}\Psi - \frac{\mathbf{i}q\hbar}{m_0}\not{A}\not{\partial}\Psi + \frac{q^2}{m_0}\not{A}\not{A}\Psi = -U_0\not{1}\Psi \tag{208}$$

and, including a multiplication by a factor $\frac{1}{2}$,

$$-\frac{\hbar^2}{2m_0}\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_t^2\right)\not{1}\Psi - \frac{\mathbf{i}q\hbar}{2m_0}\left(\not{\partial}\not{A}\Psi + \not{A}\not{\partial}\Psi\right) + \frac{q^2}{2m_0}\left(\boldsymbol{A}^2 - \frac{1}{c^2}\phi^2\right)\not{1}\Psi = -\frac{1}{2}U_0\not{1}\Psi. \tag{209}$$

The $\left(\not{\partial}\not{A}\Psi + \not{A}\not{\partial}\Psi\right)$ part of the equation needs detailed examining. Using the chain rule for the derivation, we get

$$\left(\not{\partial}\not{A}\Psi + \not{A}\not{\partial}\Psi\right) = \left((\not{\partial}\not{A})\Psi + \vec{\not{\partial}}\not{A}\Psi + \not{A}\not{\partial}\Psi\right) \tag{210}$$

in which the arrow in $\vec{\not{\partial}}$ means that the derivation skips $\not{A}$ and only applies to Ψ. Due to the non-commutative character of the math, this is the best way to encode the chain rule.

The term $\not{\partial}\not{A}$ produces the electromagnetic field leading to

$$\not{\partial}\not{A} = \begin{bmatrix} -\frac{1}{c}\partial_t\hat{\mathbf{T}} & \boldsymbol{\nabla}\cdot\mathbf{K} \\ -\boldsymbol{\nabla}\cdot\mathbf{K} & \frac{1}{c}\partial_t\hat{\mathbf{T}} \end{bmatrix} \begin{bmatrix} \frac{1}{c}\phi\hat{\mathbf{T}} & \boldsymbol{A}\cdot\mathbf{K} \\ -\boldsymbol{A}\cdot\mathbf{K} & -\frac{1}{c}\phi\hat{\mathbf{T}} \end{bmatrix} = \tag{211}$$

$$\begin{bmatrix} (\frac{1}{c^2}\partial_t\phi + \boldsymbol{\nabla}\cdot\boldsymbol{A})\hat{\mathbf{1}} - (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\mathbf{K} & -\frac{1}{c}\partial_t\boldsymbol{A}\cdot\hat{\mathbf{T}}\mathbf{K} - \frac{1}{c}\boldsymbol{\nabla}\phi\cdot\hat{\mathbf{T}}\mathbf{K} \\ -\frac{1}{c}\partial_t\boldsymbol{A}\cdot\hat{\mathbf{T}}\mathbf{K} - \frac{1}{c}\boldsymbol{\nabla}\phi\cdot\hat{\mathbf{T}}\mathbf{K} & (\frac{1}{c^2}\partial_t\phi + \boldsymbol{\nabla}\cdot\boldsymbol{A})\hat{\mathbf{1}} - (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\mathbf{K} \end{bmatrix} = \tag{212}$$

$$\begin{bmatrix} -\boldsymbol{B}\cdot\mathbf{K} & \frac{1}{c}\boldsymbol{E}\cdot\hat{\mathbf{T}}\mathbf{K} \\ \frac{1}{c}\boldsymbol{E}\cdot\hat{\mathbf{T}}\mathbf{K} & -\boldsymbol{B}\cdot\mathbf{K} \end{bmatrix} = -\boldsymbol{B}\cdot\begin{bmatrix} \mathbf{i}\boldsymbol{\sigma} & 0 \\ 0 & \mathbf{i}\boldsymbol{\sigma} \end{bmatrix} + \frac{1}{c}\boldsymbol{E}\cdot\begin{bmatrix} 0 & -\boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} = -\mathbf{i}\boldsymbol{B}\cdot\boldsymbol{\Sigma} - \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\alpha}. \tag{213}$$

The terms $\vec{\not{\partial}}\not{A}\Psi + \not{A}\not{\partial}\Psi$ leads to a cancellation of all the parts that are anti-commutative and a doubling of the commutative parts. In the above, that would mean cancellation of the

EM field and retaining the Lorenz gauge part. It result in the survival of the norm $\hat{\mathbf{1}}$ parts, so to

$$\not{A}\not{\partial} = \begin{bmatrix} \frac{1}{c}\phi\hat{\mathbf{T}} & \boldsymbol{A}\cdot\mathbf{K} \\ -\boldsymbol{A}\cdot\mathbf{K} & -\frac{1}{c}\phi\hat{\mathbf{T}} \end{bmatrix} \begin{bmatrix} -\frac{1}{c}\partial_t\hat{\mathbf{T}} & \boldsymbol{\nabla}\cdot\mathbf{K} \\ -\boldsymbol{\nabla}\cdot\mathbf{K} & \frac{1}{c}\partial_t\hat{\mathbf{T}} \end{bmatrix} = \tag{214}$$

$$\begin{bmatrix} (\frac{1}{c^2}\phi\partial_t + \boldsymbol{A}\cdot\boldsymbol{\nabla})\hat{\mathbf{1}} & 0 \\ 0 & (\frac{1}{c^2}\phi\partial_t + \boldsymbol{A}\cdot\boldsymbol{\nabla})\hat{\mathbf{1}} \end{bmatrix} = \tag{215}$$

and so

$$\vec{\not{\partial}}\not{A}\Psi + \not{A}\not{\partial}\Psi = 2\frac{1}{c^2}\phi\not{1}\partial_t\Psi + 2\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi. \tag{216}$$

The result of the closer analysis is that we have

$$-\frac{\mathbf{i}q\hbar}{2m_0}\left(\not{\partial}\not{A}\Psi + \not{A}\not{\partial}\Psi\right) = -\frac{\mathbf{i}q\hbar}{2m_0}\left(-\mathbf{i}\boldsymbol{B}\cdot\boldsymbol{\Sigma}\Psi - \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\alpha}\Psi + 2\frac{1}{c^2}\phi\not{1}\partial_t\Psi + 2\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi\right) = \tag{217}$$

$$-\frac{q\hbar}{2m_0}\boldsymbol{B}\cdot\boldsymbol{\Sigma}\Psi + \frac{\mathbf{i}q\hbar}{2m_0c}\boldsymbol{E}\cdot\boldsymbol{\alpha}\Psi - \frac{\mathbf{i}q\hbar}{m_0c^2}\phi\not{1}\partial_t\Psi - \frac{\mathbf{i}q\hbar}{m_0}\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi = \tag{218}$$

$$-\boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi + \mathbf{i}\boldsymbol{E}\cdot\boldsymbol{\pi}_s\Psi - \frac{\mathbf{i}q\hbar}{m_0c^2}\phi\not{1}\partial_t\Psi - \frac{\mathbf{i}q\hbar}{m_0}\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi \tag{219}$$

The complete Klein Gordon equation then results in

$$-\frac{\hbar^2}{2m_0}\not{1}\boldsymbol{\nabla}^2\Psi + \frac{\hbar^2}{2m_0c^2}\not{1}\partial_t^2\Psi - \frac{\mathbf{i}q\hbar}{m_0c^2}\phi\not{1}\partial_t\Psi - \frac{\mathbf{i}q\hbar}{m_0}\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi \tag{220}$$

$$+\frac{q^2}{2m_0}\boldsymbol{A}^2\not{1}\Psi - \frac{q^2\phi^2}{2m_0c^2}\not{1}\Psi = -\frac{1}{2}U_0\not{1}\Psi + \boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi - \mathbf{i}\boldsymbol{E}\cdot\boldsymbol{\pi}_s\Psi. \tag{221}$$

This can be rearranged into

$$-\frac{\hbar^2}{2m_0}\not{1}\boldsymbol{\nabla}^2\Psi + \frac{q^2}{2m_0}\boldsymbol{A}^2\not{1}\Psi - \frac{\mathbf{i}q\hbar}{m_0}\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi - \boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi = \tag{222}$$

$$\frac{q^2\phi^2}{2m_0c^2}\not{1}\Psi + \frac{\mathbf{i}q\hbar}{m_0c^2}\phi\not{1}\partial_t\Psi - \frac{\hbar^2}{2m_0c^2}\not{1}\partial_t^2\Psi - \frac{1}{2}U_0\not{1}\Psi - \mathbf{i}\boldsymbol{E}\cdot\boldsymbol{\pi}_s\Psi. \tag{223}$$

This wave equation has a probability tensor for which the closed system condition is met, one that includes the Dirac current continuity equation. It has a linear in time derivative damping term, it has a quadratic in time derivative harmonic term and it has a Hooke's law term. In the above arrangement one has the familiar terms on the left and the terms that are ignored, misrepresented or that have never been derived on the right. In case of a stationary state, we get

$$\frac{\mathbf{i}q\hbar}{m_0c^2}\phi\not{1}\partial_t\Psi = -\frac{Uq\phi}{m_0c^2}\not{1}\Psi \simeq -q\phi\not{1}\Psi = -V\not{1}\Psi \tag{224}$$

Using this we can now rearrange the equation into

$$-\frac{\hbar^2}{2m_0}\not{1}\boldsymbol{\nabla}^2\Psi + \frac{q^2}{2m_0}\boldsymbol{A}^2\not{1}\Psi - \frac{\mathbf{i}q\hbar}{m_0}\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi - \boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi + V\not{1}\Psi = \tag{225}$$

$$\frac{q^2\phi^2}{2m_0c^2}\not{1}\Psi - \frac{\hbar^2}{2m_0c^2}\not{1}\partial_t^2\Psi - \frac{1}{2}U_0\not{1}\Psi - \mathbf{i}\boldsymbol{E}\cdot\boldsymbol{\pi}_s\Psi. \tag{226}$$

In the classic interpretation, all the terms on the right hand side are reduced to $E\Psi$, giving

$$-\frac{\hbar^2}{2m_0}\not{1}\boldsymbol{\nabla}^2\Psi+\frac{q^2}{2m_0}\boldsymbol{A}^2\not{1}\Psi-\frac{\mathbf{i}q\hbar}{m_0}\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi-\boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi+V\not{1}\Psi=E\Psi. \tag{227}$$

The left hand side is then dubbed the Hamiltonian of the system and that abbreviates the equation to $\hat{H}\Psi = E\Psi$.

For stationary states it is also possible to reduce another term as

$$-\frac{\mathbf{i}q\hbar}{m_0}\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi=-\frac{q\boldsymbol{p}}{m_0}\cdot\boldsymbol{A}\not{1}\Psi\simeq -q\boldsymbol{v}\cdot\boldsymbol{A}\not{1}\Psi=-\boldsymbol{J}\cdot\boldsymbol{A}\not{1}\Psi \tag{228}$$

In a stationary magnetic field for which $\boldsymbol{A}=-\frac{1}{2}\boldsymbol{r}\times\boldsymbol{B}$, this term can also be rewritten as

$$-\frac{\mathbf{i}q\hbar}{m_0}\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi=-\frac{q}{2m_0}\not{1}\boldsymbol{L}\cdot\boldsymbol{B}\Psi, \tag{229}$$

see [23, p. 144].

The two first order derivative terms can be combined into

$$-\frac{\mathbf{i}q\hbar}{m_0c^2}\phi\not{1}\partial_t\Psi-\frac{\mathbf{i}q\hbar}{m_0}\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi=q\phi\not{1}\Psi-\boldsymbol{J}\cdot\boldsymbol{A}\not{1}\Psi=J_\mu A^\mu\not{1}\Psi. \tag{230}$$

These terms are the particle field interaction terms. Together with $\Psi^\dagger$ we get an interaction probability term as

$$\Psi^\dagger J_\mu A^\mu\not{1}\Psi. \tag{231}$$

Together with the $\boldsymbol{B}$ and $\boldsymbol{E}$ terms we have the charge-EM-field interaction terms

$$-\frac{\mathbf{i}q\hbar}{2m_0}\left(\not{\partial}\not{A}\Psi+\not{A}\not{\partial}\Psi\right)=-\boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi+\mathbf{i}\boldsymbol{E}\cdot\boldsymbol{\pi}_s\Psi+q\phi\not{1}\Psi-\boldsymbol{J}\cdot\boldsymbol{A}\not{1}\Psi \tag{232}$$

$$=-\boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi+\mathbf{i}\boldsymbol{E}\cdot\boldsymbol{\pi}_s\Psi+J_\mu A^\mu\not{1}\Psi \tag{233}$$

We have the two terms

$$-\frac{\mathbf{i}q\hbar}{m_0}\not{1}\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi-\boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi=-\frac{q}{2m_0}\not{1}\boldsymbol{L}\cdot\boldsymbol{B}\Psi-\frac{q}{m_0}\boldsymbol{B}\cdot\boldsymbol{S}\Psi= \tag{234}$$

$$-\not{1}\boldsymbol{\mu}_L\cdot\boldsymbol{B}\Psi-\boldsymbol{B}\cdot\boldsymbol{\mu}_s\Psi=-\boldsymbol{B}\cdot(\boldsymbol{\mu}_L\not{1}+\boldsymbol{\mu}_s)\Psi \tag{235}$$

with orbital magnetic momentum $\boldsymbol{\mu}_L$ and spin magnetic momentum $\boldsymbol{\mu}_s$, so with total magnetic momentum $\boldsymbol{\mu}_J=\boldsymbol{\mu}_L\not{1}+\boldsymbol{\mu}_s$ [23, p. 188]. These are the known terms. But parallel to these we have

$$q\phi\not{1}\Psi+\mathbf{i}\boldsymbol{E}\cdot\boldsymbol{\pi}_s\Psi \tag{236}$$

as integral part of the relativistic $\left(\not{\partial}\not{A}\Psi+\not{A}\not{\partial}\Psi\right)$. In the Hydrogen atom, the first term determines the main quantum number n, and the second term should be that radius plus or minus the reduced Compton wavelength. As a two valued zitter variation on the main quantum number. If a constant external electric field is applied, these two terms should be observable as the linear Stark effect. In the same line of reasoning, the diamagnetic term containing $\boldsymbol{A}^2$ should have its quadratic Stark effect term containing $q^2\phi^2$ as its relativistic companion.

Interestingly, the main quantum number term $V\Psi = q\phi\mathbb{1}\Psi$ was derived from the original damping term $-\frac{\mathbf{i}q\hbar}{m_0c^2}\phi\mathbb{1}\partial_t\Psi$. Quantum jumps might then be connected to $-\Psi^\dagger\frac{\mathbf{i}q\hbar}{m_0c^2}\phi\mathbb{1}\partial_t\Psi$, when interpreted in a Feynman manner. A quantum jump as a relativistic boost of spinors and enclosed four vectors should be analyzed using the complete Klein Gordon Lagrangian at the Dirac level, as $\mathscr{L} = \frac{1}{m_0}\Psi^\dagger(\hat{\not{P}}\hat{\not{P}} - \not{E}\not{E})\Psi$. Such a quantum jump of an electron, even inside the Hydrogen atom, should include the positron at some level. A quantum jump should always be fast enough to allow virtual positrons to participate in the process of emitting or absorbing a photon. It is my opinion that a fusion of relativistic QFT and the usual Schrödinger-Pauli analysis of atomic physics should be realized in order to get a grip on the internal dynamics of quantum jumps. On the Schrödinger-Pauli level of two by two spin matrices and two valued spinors, the intrinsics of quantum jumps will remain a mystery.

Quantum jumps in the Hydrogen atom should be analyzed intrinsically on a relativistic quantum field level using the Lagrangian $\mathscr{L} = \frac{1}{m_0}\Psi^\dagger(\hat{\not{P}}\hat{\not{P}} - \not{E}\not{E})\Psi$ and the related inertial probability/field tensor ${\Phi_\mu}^\nu = \Psi^\dagger\gamma_\mu\gamma^\nu\Psi$ with inertial probability/field closed system condition

$$\partial_\nu{\Phi_\mu}^\nu = \partial_\nu\Psi^\dagger\gamma_\mu\gamma^\nu\Psi = 0. \tag{237}$$

Of course, a quantum jump implies an open system, due to its photon exchange and its inevitable momentary virtual positron appearance and disappearance, a consideration that should temper expectation. A system with a primary electron that includes the photon that is being emitted or absorbed during a time interval in which a positron appears on the scene as well might again be considered closed.

What should be avoided at all times in the fermion domain is to reduce the Klein Gordon equation as derived in this section to a Pauli level equation or a scalar equation on the Schrödinger level. On the Pauli level, the spinors cannot be properly boosted, only stationary states are allowed and the intrinsics of the quantum jumps will be lost.

4. Conclusion regarding the proposed generalization of the Dirac current into a probability tensor with a closed system condition

As for the Lorentz transformation of the usual Lagrangian current density element $\mathscr{L} = \overline{\Psi}\not{P}\Psi = \Psi^\dagger\gamma_0\not{P}\Psi$, this is part of the general Lagrangian density element $\mathscr{L} = \frac{1}{m_0}\Psi^\dagger\not{P}\not{P}\Psi$, the Lorentz transformation properties have been demonstrated. Given the Lorentz covariance of the general $\mathscr{L}$, the Lorentz covariance of the closed system condition for the general Lagrangian density $\partial_\nu\mathscr{L} = 0$ is obvious, because ∂_ν is a four-vector in all reference systems. This includes the Lorentz covariance of the continuity equation for the Dirac current as $m_0c\partial_\nu\overline{\Psi}\gamma^\nu\Psi = 0$, as its time-like part.

The gamma tensor $\gamma_\mu\gamma^\nu$ is given by

$$\gamma_\mu\gamma^\nu = \begin{bmatrix}\gamma_0 & \gamma_1 & \gamma_2 & \gamma_3\end{bmatrix}\begin{bmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\\ \gamma_3\end{bmatrix} = \begin{bmatrix}\gamma_0\gamma_0 & \gamma_1\gamma_0 & \gamma_2\gamma_0 & \gamma_3\gamma_0\\ \gamma_0\gamma_1 & \gamma_1\gamma_1 & \gamma_2\gamma_1 & \gamma_3\gamma_1\\ \gamma_0\gamma_2 & \gamma_1\gamma_2 & \gamma_2\gamma_2 & \gamma_3\gamma_2\\ \gamma_0\gamma_3 & \gamma_1\gamma_3 & \gamma_2\gamma_3 & \gamma_3\gamma_3\end{bmatrix} = \begin{bmatrix}\mathbb{1} & -\alpha_1 & -\alpha_2 & -\alpha_3\\ \alpha_1 & -\mathbb{1} & -\mathbf{i}\Sigma_3 & \mathbf{i}\Sigma_2\\ \alpha_2 & \mathbf{i}\Sigma_3 & -\mathbb{1} & -\mathbf{i}\Sigma_1\\ \alpha_3 & -\mathbf{i}\Sigma_2 & \mathbf{i}\Sigma_1 & -\mathbb{1}\end{bmatrix} \tag{238}$$

The probability density tensor is given by

$$\Phi_\mu{}^\nu = \Psi^\dagger\gamma_\mu\gamma^\nu\Psi = \begin{bmatrix} \Psi^\dagger\mathbb{1}\Psi & -\Psi^\dagger\alpha_1\Psi & -\Psi^\dagger\alpha_2\Psi & -\Psi^\dagger\alpha_3\Psi \\ \Psi^\dagger\alpha_1\Psi & -\Psi^\dagger\mathbb{1}\Psi & -\Psi^\dagger\mathbf{i}\Sigma_3\Psi & \Psi^\dagger\mathbf{i}\Sigma_2\Psi \\ \Psi^\dagger\alpha_2\Psi & \Psi^\dagger\mathbf{i}\Sigma_3\Psi & -\Psi^\dagger\mathbb{1}\Psi & -\Psi^\dagger\mathbf{i}\Sigma_1\Psi \\ \Psi^\dagger\alpha_3\Psi & -\Psi^\dagger\mathbf{i}\Sigma_2\Psi & \Psi^\dagger\mathbf{i}\Sigma_1\Psi & -\Psi^\dagger\mathbb{1}\Psi \end{bmatrix}. \tag{239}$$

In the space-time bèta matrices representation, we have $\beta_\mu = \mathbf{i}\gamma_\mu$, so $\beta_\mu\beta^\nu = \mathbf{i}\gamma_\mu\mathbf{i}\gamma^\nu = -\gamma_\mu\gamma^\nu$. So in the space-time representation, we have

$$\Phi_\mu{}^\nu = \Psi^\dagger\beta_\mu\beta^\nu\Psi = -\Psi^\dagger\gamma_\mu\gamma^\nu\Psi \tag{240}$$

For the proper velocity, we know that $\not{U}\not{U} = -c^2\mathbb{1}$. Using $\not{U} = U_\mu\beta^\mu$ we can write this as

$$\not{U}\not{U} = U_\mu\beta^\mu U_\nu\beta^\nu = U_\mu U^\nu\beta_\mu\beta^\nu = -U_\mu U^\nu\gamma_\mu\gamma^\nu = -c^2\mathbb{1} \tag{241}$$

So we have

$$\Psi^\dagger\not{U}\not{U}\Psi = -U_\mu U^\nu\Psi^\dagger\gamma_\mu\gamma^\nu\Psi = -U_\mu U^\nu\Phi_\mu{}^\nu = -c^2\Psi^\dagger\Psi \tag{242}$$

The Dirac current can be arrived at by using the coordinate velocity's rest system coordinates as V^ν to get

$$J^\nu = \Phi_\mu{}^\nu V^\mu = \begin{bmatrix} \Psi^\dagger\mathbb{1}\Psi & -\Psi^\dagger\alpha_1\Psi & -\Psi^\dagger\alpha_2\Psi & -\Psi^\dagger\alpha_3\Psi \\ \Psi^\dagger\alpha_1\Psi & -\Psi^\dagger\mathbb{1}\Psi & -\Psi^\dagger\mathbf{i}\Sigma_3\Psi & \Psi^\dagger\mathbf{i}\Sigma_2\Psi \\ \Psi^\dagger\alpha_2\Psi & \Psi^\dagger\mathbf{i}\Sigma_3\Psi & -\Psi^\dagger\mathbb{1}\Psi & -\Psi^\dagger\mathbf{i}\Sigma_1\Psi \\ \Psi^\dagger\alpha_3\Psi & -\Psi^\dagger\mathbf{i}\Sigma_2\Psi & \Psi^\dagger\mathbf{i}\Sigma_1\Psi & -\Psi^\dagger\mathbb{1}\Psi \end{bmatrix} \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c\Psi^\dagger\mathbb{1}\Psi \\ c\Psi^\dagger\alpha_1\Psi \\ c\Psi^\dagger\alpha_2\Psi \\ c\Psi^\dagger\alpha_3\Psi \end{bmatrix}. \tag{243}$$

The generalized Lagrangian probability density element $\mathscr{L} = \frac{1}{m_0}\Psi^\dagger\not{P}\not{P}\Psi$ can be written as

$$\mathscr{L} = \frac{1}{m_0}\Psi^\dagger\not{P}\not{P}\Psi = \Psi^\dagger\not{U}\not{P}\Psi = m_0\Psi^\dagger\not{U}\not{U}\Psi = -m_0c^2\Psi^\dagger\Psi \tag{244}$$

but then we also have the stress-energy probability density Lagrangian product

$$\mathscr{L} = \frac{1}{m_i}\Psi^\dagger\not{P}\not{P}\Psi = \Psi^\dagger\not{V}\not{P}\Psi = -\frac{1}{\gamma}m_0c^2\Psi^\dagger\Psi = \Psi^\dagger L\Psi = V_\mu P^\nu\Phi_\mu{}^\nu = T_\mu{}^\nu\Phi_\mu{}^\nu. \tag{245}$$

Because I already proved the Lorentz covariance of this Lagrangian density, the Lorentz covariance of $\Phi_\mu{}^\nu$ is now proven too, and thus also the Lorentz covariance of the closed system condition. For completeness however, the Lorentz transformation of the probability/field tensor in the Dirac bèta representation is given by

$$\left(\Phi_\mu{}^\nu\right)^L = \left(\Psi^L\right)^\dagger\left(\beta_\mu\right)^L\left(\beta^\nu\right)^L\left(\Psi\right)^L = \tag{246}$$

$$\left(\Psi^\dagger S\Lambda^{-1}S^{-1}\right)\left(S\Lambda^{-1}S^{-1}\beta_\mu S\Lambda S^{-1}\right)^L\left(S\Lambda^{-1}S^{-1}\beta^\nu S\Lambda S^{-1}\right)^L\left(S\Lambda^{-1}S^{-1}\Psi\right) = \tag{247}$$

$$\Psi^\dagger\Lambda^{-1}\Lambda^{-1}\beta_\mu{}^\nu\Psi = \Psi^\dagger e^{\alpha_l\psi}\beta_\mu{}^\nu\Psi = \Phi_\mu{}^\nu\cosh(\psi) + \Psi^\dagger\alpha_l\beta_\mu{}^\nu\Psi\ \sinh(\psi), \tag{248}$$

thus demonstrating that the probability/field density undergoes a full Doppler shift under a Lorentz boost. And because in the product $\mathscr{L} = T_\mu{}^\nu\Phi_\mu{}^\nu$ one either Lorentz transforms the coordinates in $T_\mu{}^\nu$, leaving the matrices in $\Phi_\mu{}^\nu$ invariant or the matrices in $\Phi_\mu{}^\nu$, leaving the coordinates in $T_\mu{}^\nu$ invariant, the appearance of a full Doppler shift in $\mathscr{L}$ as the result of a Lorentz boost is also proven. The Lorentz covariance of the Doppler shifted condition $\partial_\nu \Phi_\mu{}^\nu = 0$ follows.

Back to this Lagrangian, we also get

$$\frac{\partial \mathscr{L}}{\partial (P_\mu)} = V_\nu \Phi_\mu{}^\nu, \tag{249}$$

$$\frac{\partial \mathscr{L}}{\partial (V_\nu \Phi_\mu{}^\nu)} = P_\mu \tag{250}$$

and

$$\frac{\partial \mathscr{L}}{\partial \Phi_\mu{}^\nu} = T_\mu{}^\nu. \tag{251}$$

For a system with external forces applied, the last equation also leads to

$$\partial_\nu \left(\frac{\partial \mathscr{L}}{\partial \Phi_\mu{}^\nu} \right) = \partial_\nu T_\mu{}^\nu = F_\mu. \tag{252}$$

And for closed systems we get

$$\partial_\nu \left(\frac{\partial \mathscr{L}}{\partial \Phi_\mu{}^\nu} \right) = \partial_\nu T_\mu{}^\nu = 0. \tag{253}$$

We can reverse the order for closed systems and get

$$\partial_\nu \left(\frac{\partial \mathscr{L}}{\partial T_\mu{}^\nu} \right) = \partial_\nu \Phi_\mu{}^\nu = 0. \tag{254}$$

The created environment, including the closed system condition for the above product, closes in on General Relativity's concepts and basic elements, as the symmetric stress-energy tensor density and its closed system condition is.

The Klein Gordon probability equation in my space-time bèta matrices environment is

$$\Psi^\dagger (\not{P}\not{P} - \not{E}\not{E})\Psi = \Psi^\dagger (\not{P} - \not{E})(\not{P} + \not{E})\Psi = 0 \tag{255}$$

and can be split into two Dirac equations as

$$\Psi^\dagger (\not{P} - \not{E}) = 0 \tag{256}$$

$$(\not{P} + \not{E})\Psi = 0 \tag{257}$$

These two equations have the same solutions, but, in the Weyl representation, the roles of the twin spinors are reversed. In terms of the Dirac fields, the role of particle and anti-particle are reversed. If one then goes from the Weyl representation to the Dirac representation using the S operator, the particle and anti-particle fields will be mixed in both Dirac spinor twins. In terms of the space-time basis, the S operator adds a time-reversal to one half of the dual space-time basis. The lower version is the standard Dirac equation. Its Lagrangian then is usually given as

$$\mathscr{L} = \Psi^\dagger \gamma_0 (\not{P} + \not{E})\Psi = \overline{\Psi}(\not{P} + \not{E})\Psi \tag{258}$$

with the Dirac adjoint. In the standard approach, the γ_0 is added in order to produce a Dirac probability current and a Dirac probability current continuity equation with realistic properties for its probability and its Lorentz covariance. Due to the peculiar role of γ_0 in the Dirac adjoint and in all those proves and demonstrations, it has the appearance of being somewhat forced. After all, the γ_0 is the supposed to be the time-like part of a supposedly space-time four vector γ_μ, so why and how should their product $\gamma_0\gamma_\mu$ have Lorentz invariant or covariant properties? In my environment, the γ_0 and the γ_μ are part of the pauliquat dual spin-norm sphere, which complicates thing even more. In the perspective developed in this paper, the γ_0 in the Dirac adjoint has the positive property to connect to the first column of the probability tensor and the negative property to hide the true Lorentz transformation properties of the absolute value of spinors, that is that $\Psi^\dagger\Psi$ Doppler shifts under a Lorentz boost as $\Psi^\dagger e^{\alpha_I \psi}\Psi$.

The Lagrangian $\mathscr{L} = \overline{\Psi}(\not{P} + \not{E})\Psi$ is like the primary hub of the Standard Model. By going backwards in this section, this primary hub can be generalized into a Lagrangian that closes in on gravity. In the process, the Dirac current is generalized into a probability/field density tensor and the Dirac current continuity condition is encapsulated in the closed system condition for this tensor. The use of the Dirac adjoint is in need of a critical assessment, due to the questionable role given to the time-like γ_0 in this adjoint.

References

[1] Synge, J. L., Quaternions, Lorentz Transformations and the Conway-Dirac-Eddington Matrices *Commun. Dublin Inst. Ser. A* **21** (1972), 1–67.

[2] de Haas, E. P. J., The geodetic precession as a 3D Schouten precession and a gravitational Thomas precessionł, *Canadian Journal of Physics* **92** (2014), 1082–1093

[3] Pauli, W., *Theory of Relativity* (Dover, New York, 1958).

[4] Minkowski, H., Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern, *Math. Ann.* **68** (1910), 472–525.; *Nachr. Ges. Wiss. Göttingen* (1908), 53–111.

[5] Dirac, P. A. M., Is there an Æther? *Nature* **168** (1951), 906–907.

[6] Frenkel, J., Die Elektrodynamik des rotierenden Elektrons *Zeit. f. Phys.* **37** (1926), 243–262.

[7] Kramers, H. A., *Quantum Mechanics* (N-H Publishing Company, Amsterdam, 1958) p. 225-232.

[8] de Broglie, L., *La Théorie des Particules de Spin 1/2 (Électrons de Dirac)* (Gauthier-Villars, Paris, 1952) p. 40-42.

[9] Dirac, P. A. M., Forms of Relativistical Dynamics, *Rev. Mod. Phys.* **21** (1949), 392–399.

[10] von Laue, M., Zur Dynamik der Relativitätstheorie *Ann. Phys.* **35** (1911), 524–542.

[11] von Laue, M., *Die Relativitätstheorie*, 3rd ed. (Friedrich Vieweg und Sohn, Braunschweig, 1919).

[12] Meyers, N. H., A poynting theorem for moving bodies and the relativistic mechanics of extended objects, *Journal of The Franklin Institute-engineering and Applied Mathematics* **266** (1958) 439-464.

[13] Kholmetskii, A., Missevitch, O., Yarman, T., Poynting Theorem, Relativistic Transformation of Total EnergyMomentum and Electromagnetic EnergyMomentum Tensor *Foundations of Physics* **46** (2015) 236–261.

[14] Breit, G., An Interpretation of Diracs Theory of the Electron *PNAS***14** (1928), 553–559.

[15] Nikolić, H., How (not) to teach Lorentz covariance of the Dirac equation, *Eur. J. Phys.* **35** (2014) 035003.

[16] Gross, F., *Relativistic Quantum Mechanics and Field Theory*, (John Wiley & Sons, Weinheim,

1993).
[17] Greiner, W. *Relativistic Quantum Mechanics. Wave Equations* (Springer-Verlag, Berlin, 1990).
[18] Stone, M. *The Physics of Quantum Fields* (Springer-Verlag, New York, 2000).
[19] Peskin, M. E., Schroeder, D. V. *An Introduction to Quantum Field Theory* (Perseus Books, Reading Ma., 1995).
[20] Zee, A. *Quantum field theory in a nutshell* (Princeton University Press, Princeton, 2000).
[21] Darwin, C. G., The wave equations of the electron. *Proc. R. Soc. London A***118** (1928), 654–680.
[22] Kragh, H., The Genesis of Dirac's Relativistic Theory of Electrons *Archive for History of Exact Sciences* **24** (1981), 31–67.
[23] Schwabl, F., *Quantum Mechanics* (Springer-Verlag, Berlin, 2007).
[24] Dirac, P. A. M., The quantum theory of the electron *Proc. R. Soc. Lond. A* **117** 610-624 (1928)
[25] Dirac, P. A. M., The quantum theory of the electron. Part II *Proc. R. Soc. Lond. A* **118** 351-361 (1928)