ELEMENTARY SET THEORY CAN BE USED TO PROVE FERMAT'S LAST THEOREM (FLT) V.5

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Abstract. An open problem is proving FLT simply for each $n \in \mathbb{N}, n > 2$. Our proof of FLT is based on our algebraic identity, denoted, for convenience, as $r^n + s^n = t^n$ with $r, s, t > 0$ as functions of variables. For $n \in \mathbb{N}, n > 0$: We relate r, s, t for which $r^n + s^n = t^n$ holds with $x, y, z > 0$ for which $x^n + y^n = z^n$ holds. We infer as true by *direct argument* (not BWOC), for any given $n > 2$, that $\{(x, y, z)|x, y, z \in \mathbb{N}, x^n + y^n = z^n\} \implies \{(r, s, t)|r, s, t \in \mathbb{N}, r^n + s^n = t^n\}$. In addition, we show, for $n > 2$, that $\{(r, s, t)|r, s, t \in \mathbb{N}, r^n + s^n = t^n\} = \varnothing$. Thus, for $n \in \mathbb{N}, n > 2$, it is true that $\{(x, y, z)|x, y, z \in \mathbb{N}, x^n + y^n = z^n\} = \varnothing$.

1. Introduction

FLT states, for $n \in \mathbb{N}, n > 2$, $x, y, z \in \mathbb{N}, x, y, z > 0$ that $x^n + y^n = z^n$ *does not hold*. It is well known that a *simple* proof of FLT for *every* $n \in \mathbb{N}, n > 2$ is lacking.

For $n \in \mathbb{N}$: We use *basics* to devise a *direct proof*, not the *expected* BWOC.

Per Sect. 3, an *identity* with very restricted integral triples for $n \in \mathbb{N}, n > 2$ is :

$$\left((4q^n)^{\frac{1}{n}}\right)^n + \left((p - 2q^n)^{\frac{1}{n}}\right)^n = \left((p + 2q^n)^{\frac{1}{n}}\right)^n . \tag{1}$$

Basic conditions : $n \in \mathbb{N}, n > 0$, $p \in \mathbb{R}$, $p > 0$, $q \in \mathbb{Q}$, $q > 0$ such that $p > 2q^n$.

Denote r for $(4q^n)^{\frac{1}{n}}$; s for $(p-2q^n)^{\frac{1}{n}}$, and t for $(p+2q^n)^{\frac{1}{n}}$ *throughout the paper.*

Therefore, $r, s, t \in \mathbb{N}, r, s, t > 0, r \neq s$ for which $r^n + s^n = t^n$ holds, is similar to, thus comparable to $x, y, z \in \mathbb{N}, x, y, z > 0, x \neq y$ for which $x^n + y^n = z^n$ holds.

We begin, in Sect 2, below, with $r, s, t, x, y, z \in \mathbb{R}$ to subsequently infer a relation between included $r, s, t \in \mathbb{N}, r, s, t > 0$ and included $x, y, z \in \mathbb{N}, x, y, z > 0$.

We argue from an equality of *two sets* to an equality of the two *respective subsets* since an equality of two sets, with both sets nonempty or both sets empty, implies that the respective two subsets are equal, with both nonempty or both empty.

A consistent argument in Sect. 2 requires, for $n = 1, 2$, with $r, s, t, x, y, z > 0$, that $\{(r, s, t)|r, s, t \in \mathbb{N}, r^n + s^n = t^n\} = \{(x, y, z)|x, y, z \in \mathbb{N}, x^n + y^n = z^n\}$ be true; it is clearly true for $n = 1, 2$, but *solely* with $q \in \mathbb{Q}$, $q = \frac{r}{4}, \frac{r}{2}$, respectively; so, $\{(r, s, t)|r, s, t \in \mathbb{N}, r^n + s^n = t^n\} = \{(x, y, z)|x, y, z \in \mathbb{N}, x^n + y^n = z^n\}$ would be false should, instead, $q \in \mathbb{R} - \mathbb{Q}$. So, we must exclude $q \in \mathbb{R} - \mathbb{Q}$ from our proof.

That $\{(r, s, t)|r, s, t \in \mathbb{N}, r^n + s^n = t^n \implies \{(x, y, z)|x, y, z \in \mathbb{N}, x^n + y^n = z^n\}$ is true is shown, in section 2, below, for $n \in \mathbb{N}, n > 2, x, y, z, r, s, t > 0, p \in \mathbb{R}, q \in \mathbb{Q}$; therefore, equation $\{(x, y, z)|x, y, z \in \mathbb{N}, x^n + y^n = z^n\} = \varnothing$ (*which is FLT*) is true since we show in section 3, below, that $\{(r, s, t)|r, s, t \in \mathbb{N}, r^n + s^n = t^n\} = \varnothing$.

Date: November 25, 2018.

For any given $n \in \mathbb{N}, n > 0$: Letting r, s, t respectively denote $(4q^n)^{\frac{1}{n}}, (p-2q^n)^{\frac{1}{n}}$, and $(p+2q^n)^{\frac{1}{n}}$, with $q \in \mathbb{Q}, p \in \mathbb{R}$, $p, q, > 0$ for which $r^n + s^n = t^n$ holds, and with x, y, z such that $x^n + y^n = z^n$ holds, all the sets that we use in Sect. 2, below, are :

Let A be $\{(r, s, t)|r, s, t \in \mathbb{R}, r, s, t > 0\}$: An ∞ of elements for $n > 0$; for $n = 1$, for example $\{3\pi, 4\pi, 7\pi\}, n = 2$, e.g., $\{3\pi, 4\pi, 5\pi\}$, for $n = 3$, e.g. $\{1, 2, 9^{\frac{1}{3}}\}$.

Let B be $\{(r, s, t)|r \cdot s, t \in \mathbb{N}, r, s, t > 0$: An ∞ of elements for $n > 0$; for $n = 1$, e.g. $\{12, 7\}$, and, for $n = 2$, e.g. $\{12, 5\}$, for $n = 3$ e.g. $\{(4-2\sqrt{2})^{\frac{1}{3}} \cdot (4+2\sqrt{2})^{\frac{1}{3}}, 2\}$.

Let C be $\{(r, s, t)|r, s, t \in \mathbb{N}, r, s, t > 0\}$: An ∞ of elements for $n = 1, 2$; for $n = 1$, e.g. $\{3, 4, 7\}$, and, for $n = 2$, e.g. $\{3, 4, 5\}$; likely no elements for $n \geq 3$.

Let D be $\{(x, y, z)|x, y, z \in \mathbb{R}, x, y, z > 0\}$: An ∞ of elements for $n > 0$; for $n = 1$, e.g. $\{6\pi, 8\pi, 14\pi\}$, for $n = 2$, e.g. $\{6\pi, 8\pi, 10\pi\}$, for $n = 3$, e.g. $\{2, 4, 264^{\frac{1}{3}}\}$.

Let E be $\{(x, y, z)|x \cdot y, z \in \mathbb{N}$: An ∞ of elements for $n > 0$; for $n = 1$, e.g. $\{48, 14\}$; for $n = 2$, e.g. $\{48, 10\}$; for $n = 3$, e.g. $\{(32-16\sqrt{2})^{\frac{1}{3}} \cdot (32+16\sqrt{2})^{\frac{1}{3}}, 2\}$.

Let F be $\{(x, y, z)|x, y, z \in \mathbb{N}, x, y, z > 0\}$: An ∞ of elements for $n = 1, 2$; for $n = 1$, e.g. $\{6, 8, 14\}$, and, for $n = 2$, e.g. $\{6, 8, 10\}$; likely no elements for $n \geq 3$.

Let G be $\{\frac{r \cdot s}{t}|(r, s, t) \in A\}$: An ∞ of elements for $n > 0$; for $n = 1$, for example $\{(12\pi^2)/7\pi\}$, and, for $n = 2$, for example $\{(12\pi^2)/5\pi\}$, for $n = 3$, e.g. $\{(2 \cdot 1)/9^{\frac{1}{3}}\}$.

Let H be $\{\frac{r \cdot s}{t}|(r, s, t) \in B\}$: An ∞ of elements for $n > 0$; for $n = 1$, for example $\{12/7\}$, and, for $n = 2$, e.g. $\{12/5\}$, for $n = 3$, e.g. $\{((4-2\sqrt{2})^{\frac{1}{3}} \cdot (4+2\sqrt{2})^{\frac{1}{3}})/2\}$.

Let J be $\{\frac{r \cdot s}{t}|(r, s, t) \in C\}$: An ∞ of elements for $n = 1, 2$; for $n = 1$, for example $\{(3 \cdot 4)/7\}$, and, for $n = 2$, e.g. $\{(3 \cdot 4)/5\}$; likely no elements for $n \geq 3$.

Let K be $\{\frac{x \cdot y}{z}|(x, y, z) \in D\}$: An ∞ of elements for $n > 0$; for $n = 1$, for example $\{(48\pi^2)/14\pi\}$, and, for $n = 2$, e.g. $\{(48\pi^2)/10\pi\}$, for $n = 3$, e.g. $\{(4 \cdot 2)/72^{\frac{1}{3}}\}$.

Let L be $\{\frac{x \cdot y}{z}|(x, y, z) \in E\}$: An ∞ of elements for $n > 0$; for $n = 1$, e.g. $\{48/14\}$ and, for $n = 2$, e.g. $\{48/10\}$, for $n = 3$, e.g. $\{((32-16\sqrt{2})^{\frac{1}{3}} \cdot (32+16\sqrt{2})^{\frac{1}{3}})/2\}$.

Let M be $\{\frac{x \cdot y}{z}|(x, y, z) \in F\}$: An ∞ of elements for $n = 1, 2$; for $n = 1$, for example $\{(6 \cdot 8)/14\}$ and, for $n = 2$, e.g $\{(6 \cdot 8)/10\}$; likely no elements for $n \geq 3$.

2. Our Direct Proof Using Sets and Respective Subsets

Proposition 2.1. *For any given $n \in \mathbb{N}, n > 0$: $H = L$, with $H, L \neq \varnothing$.*

Proof. For *any given* $n \in \mathbb{N}, n > 0$: $\frac{(4q^n)^{\frac{1}{n}}(p-2q^n)^{\frac{1}{n}}}{(p+2q^n)^{\frac{1}{n}}} \in G$, so, $\frac{r \cdot s}{t} \in G$, and $\frac{x \cdot y}{z} \in K$ are equally restricted, as follows : With any given $q \in \mathbb{Q}, q > 0$, unrestricted $p \in \mathbb{R}, p > 0$ *varies such that* $\frac{r \cdot s}{t} \in G$ takes *any given* $\frac{x \cdot y}{z} \in K$; also, $rs/t < r$, $xy/z < x$. Clearly, K includes G. Thus, for any given $n > 0$ it is true that $\{\frac{r \cdot s}{t} \in G\} = \{\frac{x \cdot y}{z} \in K\}$. Since we focus on $x, y, z, r, s, t \in \mathbb{N}$: *Let* $z, t \in \mathbb{N}$, which exist for each $n > 0$, implying $\{\frac{r \cdot s}{t} \in H \subset G\} = \{\frac{x \cdot y}{z} \in L \subset K\}$ with $H, L \neq \varnothing$. □

Proposition 2.2. *Existing $x, y, z \in F$ are rational multiples of existing $r, s, t \in C$.*

Proof. For any given $n \in \mathbb{Z}, n > 0$: Define constant $\frac{\alpha \cdot \alpha}{\alpha}$ with $\alpha \in \mathbb{Q}, \alpha > 0$. Equation $\frac{r \cdot s}{t} \in H = \frac{x \cdot y}{z} \in L$ yields $\frac{r \cdot \alpha \cdot s \cdot \alpha}{((r \cdot \alpha)^n + (s \cdot \alpha)^n)^{\frac{1}{n}}} \in H = \frac{x \cdot y}{(x^n + y^n)^{\frac{1}{n}}} \in L$ for which $\{r \cdot \alpha \cdot s \cdot \alpha | \frac{r \cdot s}{t} \in H\} = \{x \cdot y | \frac{x \cdot y}{z} \in L\}$; $\{t \cdot \alpha | \frac{r \cdot s}{t} \in H\} = \{z | \frac{x \cdot y}{z} \in L\}$.

Hence, $\{s \cdot \alpha | \frac{r \cdot s}{t} \in H\} = \{y | \frac{x \cdot y}{z} \in L\}$, and $\{r \cdot \alpha | \frac{r \cdot s}{t} \in H\} = \{x | \frac{x \cdot y}{z} \in L\}$.

So, $\{r \cdot \alpha, s \cdot \alpha | \frac{r \cdot s}{t} \in J \subset H\} = \{x, y || \frac{x \cdot y}{z} \in M \subset L\}$ with $J, M \neq \varnothing$ or $J, M = \varnothing$.

Consequently, $\{(r \cdot \alpha, s \cdot \alpha, t \cdot \alpha | r, s, t \in C\} = \{(x, y, z \in F\}$: The Fermat triple $(x, y, z) \in F$ is a rational multiple of $(r, s, t) \in C$ with $F, C \neq \varnothing$, or $F, C = \varnothing$. □

Thus, for $n \in \mathbb{N}, n > 0$ we prove Props. 2.1- 2.2 with $p \in \mathbb{R}$, and $q \in \mathbb{Q}$.

3. Results and Conclusion

With $(4q^n)^{\frac{1}{n}}, (p - 2q^n)^{\frac{1}{n}}, (p + 2q^n)^{\frac{1}{n}} \in \mathbb{Q}$, or $r, s, t \in \mathbb{Q}$, respectively, of Sect. 1 :
Term $(4q^n)^{\frac{1}{n}} \in \mathbb{Q}$ reduces to $2^{\frac{2}{n}} q \in \mathbb{Q}$. So, such $2^{\frac{2}{n}} q \in \mathbb{Q}$ and $r \in \mathbb{Q}$ are identical.

Thus, for $n \in \mathbb{N}, n > 2$: There are no values, with $q \in \mathbb{Q}$, for $2^{\frac{2}{n}} q \in \mathbb{Q}$.
Hence, for $n \in \mathbb{N}, n > 2$: There are no values, with $q \in \mathbb{Q}$, for $2^{\frac{2}{n}} q \in \mathbb{N} \subset \mathbb{Q}$.

For $n \in \mathbb{N}, n > 2$, the fact that such $r \in \mathbb{N}$ is impossible shows that $C = \varnothing$.
[For $n \in \mathbb{N}, n > 2$, the fact that such $r \in \mathbb{N}$ is impossible shows also that $\{r \cdot \alpha | r, s, t \in A\} \neq x | x, y, z \in D\}$: Term x can be integral, e.g., $2^3 + 3^3 = (35)^{\frac{1}{3}}$. However, we show $\{r \cdot \alpha | r, s, t \in C\} = \{x | x, y, z \in F\}$ to be true, in Sect 2, above.]

Per our proof of proposition 2.2, above, it is true that $C \implies F$.
Consequently, $F = \varnothing$. In other words, for $n \in \mathbb{N}, n > 2$, the following is true :
Equation $x^n + y^n = z^n$ does not hold for (x, y, z) with $x, y, z \in \mathbb{N}, x, y, z > 0$.

QED.