Refutation of first-order continuous induction on real closed fields

Abstract: Bu mapping definitions, theorems, and propositions, we refute the first-order continuous induction principle on real closed fields.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated *proof* value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). Results are a 16-valued truth table in row-major and horizontal, or repeating fragments of 128-tables for more variables. (See ersatz-systems.com.)

LET p, q, r, s, t: *x*, *y*, ε *or z*, ϕ, ψ;
~ Not; & And, ∧; > Imply, →; < Not Imply, less than;
= Equivalent, <==>; @ Not Equivalent;
% possibility, for one or some; # necessity, for every or all;
(p@p) zero; ~(y<x) (x≤y).

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"Continuous Induction", "Induction over the Continuum", "Real Induction", "Non-Discrete Induction", or the like, are some terms used by authors for referring to some statements about the continuum R. These statements are as strong as the Completeness Axiom of R and a motivation for their introduction into the literature of mathematics is the easy and sometimes unified ways they provide for proving some basic theorems of mathematical analysis. ... The continuous induction principle introduced in [5] (see also [6]) is equivalent to the following:

(IND"$_R$): $\exists x\forall y{\le}x\phi(y) \wedge \exists\varepsilon{>}0\forall x\phi(x) \rightarrow \forall y[x{\le}y{\le}x{+}\varepsilon \rightarrow \phi(y)] \rightarrow \forall x\phi(x)$ (Def. 2.2.1)

((~(((p&s)&q)<(%p&#q))&(%r>((p@p)&(#p&s))))
>(~((p+r)<~(#q<p))>(s&#q))) > (#p&s) ;
NNNN **FFFF** NN**F**N **F**N**F**N (Def. 2.2.2)

(IND$_R$): $\exists x\forall y{<}x\phi(y) \wedge \forall x\forall y{<}x\phi(y) \rightarrow \exists z{>}x\forall y{<}z\phi(y) \rightarrow \forall x\phi(x)$ (Def. 2.4.1a)

((((%p&#q)<((p&s)&q))&((#p&#q)<((p&s)&q)))>
(%r>((p&#q) <(p+((r&s)&q)))))>((#p&s)&p) ;
FFFF **FFF**N **F**N**F**N **F**N**F**N (Def. 2.4.2a)

$\exists x\forall y{<}x\phi(y) \wedge \forall x\forall y{<}x\phi(y) \rightarrow \exists\varepsilon{>}0\forall y{<}x{+}\varepsilon\ \phi(y) \rightarrow \forall x\phi(x)$ (Def. 2.4.1b)

((((%p&#q)<((p&s)&q))&((#p&#q)<((p&s)&q)))>
(%r>(((p@p)&#q)<(p+((r&s)&q)))))>((#p&s)&p) ;
FFFF **FFF**N **F**N**F**N **F**N**F**N (Def. 2.4.2b)

Remark 2.4: Defs. 2.4.2a and 2.4.2b are equivalent by truth table result.

(IND'$_R$): $\exists x \forall y \le x \phi(y) \wedge \forall x \forall y \le x \phi(y) \rightarrow \exists z > x \forall y < z \phi(y) \wedge \forall x[\forall y < x \phi(y) \rightarrow \phi(x)]$
$\rightarrow \forall x \phi(x)$ (Def. 2.6.1)

(((~((p&s)&q)<(%p&#q))&~(((p&s)&q)<(#p&#q)))>
((%r>((p&#q)<((r&s)&q)))&((#q<((#p&s)&q))>(s&#p))))>((#p&s)&p) ;
CCTC TTTC CTTN TTTN (Def. 2.6.2)

In any dense linear order without endpoints ... , the scheme IND$_R$ holds,
if and only if IND'$_R$ holds.
(IND$_R$ <==> IND'$_R$) (Def. 2.4.1a) = (Def. 2.6.1) (Thrm. 2.7.1)

(((((%p&#q)<((p&s)&q))&((#p&#q)<((p&s)&q)))>
(%r>((p&#q) <(p+((r&s)&q)))))>((#p&s)&p))=
((((~((p&s)&q)<(%p&#q))&~(((p&s)&q)<(#p&#q)))>
((%r>((p&#q)<((r&s)&q)))&((#q<((#p&s)&q))>
(s&#p))))>((#p&s)&p)) ; NN**F**N **FFFF** NN**FF** **F**N**F**T (Thrm. 2.7.2)

In any ordered Abelian group whose linear order is dense without endpoints,
if IND$_R$ holds then IND"$_R$ holds too. But not vice versa: IND"$_R$ holds in the
rational numbers but IND$_R$ does not.
(IND$_R$ </=> IND"$_R$): (Def. 2.4.1b) > (Def. 2.2.1) (Thrm. 2.8.1)

(((((%p&#q)<((p&s)&q))&((#p&#q)<((p&s)&q)))>
(%r>((p&#q) <(p+((r&s)&q)))))>((#p&s)&p))>
(((~(((p&s)&q)<(%p&#q))&(%r>((p@p)&(#p&s))))
>(~((p+r)<~(#q<p))>(s&#q))) > (#p&s)) ;
TTTT TTTC TTTT TTTT (Thrm. 2.8.2)

(SUP): $\exists x \phi(x) \wedge \exists y \forall x[\phi(x) \rightarrow x \le y] \rightarrow \exists z \forall y \forall x[\phi(x) \rightarrow x \le y] \leftrightarrow z \le y$ (Def. 3.1.1)

((s&%p)&((s&#p)>~(%q<#p)))>(((s&#p)>~(#q<#p))=~(q<%r)) ;
TTTT TTTT TTTC TTTT (Def. 3.1.2)

(INF): $\exists x \phi(x) \wedge \exists y \forall x[\phi(x) \rightarrow y \le x] \rightarrow \exists z \forall y \forall x[\phi(x) \rightarrow y \le x] \leftrightarrow y \le z$ (Def. 3.2.1)

((s&%p)&((s&#p)>~(#p<%q)))>(((s&#p)>~(#p<#q))=~(%r<q)) ;
TTTT TTTT NNTT NNTT (Def. 3.2.2)

In any dense linear order without endpoints ... , the scheme SUP holds,
if and only if INF holds.
(SUP $\Longleftrightarrow$INF): (Def. 3.1.1) = (Def. 3.2.1) (Prop. 3.3.1)

(((s&%p)&((s&#p)>~(%q<#p)))>(((s&#p)>~(#q<#p))=~(q<%r)))=
(((s&%p)&((s&#p)>~(#p<%q)))>(((s&#p)>~(#p<#q))=~(%r<q))) ;
TTTT TTTT NNTC NNTT (Prop. 3.3.2)

(CUT): $\exists x \exists y[\phi(x) \wedge \psi(y)] \wedge \forall x \forall y[\phi(x) \wedge \psi(y) \rightarrow x<y] \rightarrow \exists$
$z \forall x \forall y[\phi(x) \wedge \psi(y) \rightarrow x \leq z \leq y]$ (Def. 3.4.1)

((((s&%p)&(t&%q))&(s&#p))&((t&#q)>(#p<#q)))>
(((s&#p)&(t&#q))>~(q<~(t<p))) ; `TTTT TTTT TTTT TTTT` (Def. 3.4.2)

In any dense linear order without endpoints ... , the scheme CUT holds,
if and only if SUP holds.
(CUT <==> SUP): (Def. 3.4.1) = (Def. 3.1.1) (Prop. 3.5.1)

(((((s&%p)&(t&%q))&(s&#p))&((t&#q)>(#p<#q)))>
(((s&#p)&(t&#q))>~(q<~(t<p))))=
(((s&%p)&((s&#p)>~(%q<#p)))>(((s&#p)>~(#q<#p))=~(q<%r))) ;
`TTTT TTTT TTTC TTTT` (Prop. 3.5.2)

In any dense linear order without endpoints ... , the scheme INF holds,
if and only if INDR holds.
(INF <==> IND_R): (Def. 3.2.1) = (Def. 2.4.1a) (Prop. 3.6.1)

(((s&%p)&((s&#p)>~(#p<%q)))>(((s&#p)>~(#p<#q))=~(%r<q)))=
(((((%p&#q)<((p&s)&q))&((#p&#q)<((p&s)&q)))>
(%r>((p&#q) <(p+((r&s)&q)))))>((#p&s)&p)) ;
FFFF **FFF**N CT**F**N CT**F**N (Prop. 3.6.2)

As rendered, Defs. 2.2, 2.4, 2.6, 3.1-3 are *not* tautologous. Def. 3.4 is tautologous, but as expected from Tarski. Thrms. 2.7 and 2.8 are *not* tautologous, and Props.3.5 and 3.6 are *not* tautologous.

Remark 3.5: Thrm. 2.8 and Prop. 3.5 produce truth tables which diverge from tautology by one value of falsity, as `C` for contingency. We terminate our evaluation after Section 3.

The Defs., Thrms., and Props. refute the first-order continuous induction principle on real closed fields.