(pk mk qk) or an Unexpected Inconsistency

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Abstract. This note proves the inconsistency of the Peano arithmetic (PA) by deriving both a strengthened form of the strong Goldbach conjecture and its negation.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Theorem. *The Peano arithmetic (PA) is inconsistent.*

Proof. We define S_g := { (pk, mk, qk) | k, m $\in \mathbb{N}$; p, q $\in \mathbb{P}_3$, p < q; m = (p + q) / 2 } and we consider the following two statements on S_g :

(G) For each k ≥ 1, there is no nk, n ≥ 4, different from all the mk.
¬(G) For each k ≥ 1, there is an nk, n ≥ 4, different from all the mk.

The case (G) means that the numbers m take all integer values x ≥ 4 and the case ¬(G) means that the numbers m do not take all integer values x ≥ 4. So, we have

(1) There is a difference in S_g under the two conditions (G) and ¬(G).

On the other hand, for each k ≥ 1 such an nk from ¬(G) can be written as some pk when n is prime, as some pk' when n is composite and not a power of 2, or as 4k' when n is a power of 2; p $\in \mathbb{P}_3$; k, k' $\in \mathbb{N}$. The expressions pk, pk' for nk are first components of S_g triples and 4k' is component of (3k', 4k', 5k'). Since these are the same triples when there is no such nk, i.e. in case (G), we obtain that the triples of S_g are the same in both cases. So,

(2) There is no difference in S_g under the two conditions (G) and ¬(G).

As (2) is the negation of (1), this is a contradiction in PA.

Note: If nk from ¬(G) could be a number that cannot be expressed by a S_g triple component or if nk, k = 1, could be the arithmetic mean of a pair of primes not used in S_g, then S_g would be different under the conditions (G) and ¬(G), without causing any contradiction. However, these two options are excluded by the characteristics of S_g.

□

Actually, the above argument uses a strengthened form of the strong Goldbach conjecture and its negation:

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

¬SSGB: *There is an even integer greater than 6 that cannot be expressed as the sum of two different primes*.

SSGB is equivalent to saying that all integers x ≥ 4 appear as m in a component mk of S_g. Therefore, SSGB is equivalent to the case (G) and the negation ¬SSGB is equivalent to the case ¬(G). We have seen above that the S_g triples are the same in these two cases. This means that both SSGB and ¬SSGB hold.