$\mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity, $[\mathbf{SU(4)}]^4$ Unification, Hermitian Matrix Geometry and Nonsymmetric Kaluza-Klein Theory

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Abstract

We review briefly how $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity (real-complex-quaterno-octonionic Gravity) naturally can describe a grand unified field theory of Einstein's gravity with a Yang-Mills theory containing the Standard Model group $SU(3) \times SU(2) \times U(1)$. In particular, the $C \otimes H \otimes O$ algebra is explored deeper. It is found that it can furnish the gauge group $[\mathbf{SU(4)}]^4$ revealing the possibility of extending the Standard Model by introducing additional gauge bosons, heavy quarks and leptons, and a $fourth$ family of fermions with profound physical implications. An analysis of $C \otimes H \otimes O$-valued gravity reveals that it bears a connection to Nonsymmetric Kaluza-Klein theories and complex Hermitian Matrix Geometry. The key behind these connections is in finding the relation between $C \otimes H \otimes O$-valued metrics in *two complex* dimensions with metrics in *higher* dimensional *real* manifolds ($D = 32$ real dimensions in particular). It is desirable to extend these results to hypercomplex, quaternionic manifolds and Exceptional Jordan Matrix Models.

Keywords: Nonassociative Geometry, Clifford algebras, Quaternions, Octonionic Gravity, Unification, Strings.

1 Introduction

This introduction is a review of our recent work [1] and may be skipped by those readers familiar with it. Recently we have argued how $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity (real-complex-quaterno-octonionic Gravity) naturally can describe a grand unified field theory of Einstein's gravity with a Yang-Mills theory containing the Standard Model group $SU(3) \times SU(2) \times U(1)$ [1]. It was based on an extension of the work by [2],[3],[4]. The quaternion algebra is defined by $q_i q_j = -\delta_{ij} q_o + \epsilon_{ijk} q_k; i, j, k = 1, 2, 3$, and q_o is the identity element. Given an octonion $\mathbf{X}$ it can be expanded in a basis (e_o, e_a) as

$$\mathbf{X} = x^o \ e_o \ + \ x^a \ e_a, \ a = 1, 2, \cdots, 7. \tag{1.1}$$

where e_o is the identity element. The Noncommutative and Nonassociative algebra of octonions is determined from the relations

$$e_o^2 = e_o, \ e_o e_a = e_a e_o = e_a, \ e_a e_b = -\delta_{ab} e_o \ + C_{abc} e_c, \ a, b, c = 1, 2, 3, ...7. \tag{1.2}$$

The non-vanishing values of the fully antisymmetric structure constants C_{abc} is chosen to be $\mathbf{1}$ for the following 7 sets of index triplets (cycles) [4]

$$(124), \ (235), \ (346), \ (457), \ (561), \ (672), \ (713) \tag{1.3}$$

Each cycle represents a quaternionic subalgebra. The values of C_{abc} for the other combinations are zero. The latter 7 sets of index triplets (cycles) correspond to the 7 lines of the Fano plane.

The octonion conjugate is defined

$$\bar{\mathbf{X}} = x^o \ e_o \ - \ x^m \ e_m. \tag{1.4}$$

and the norm

$$N(\mathbf{X}) \ = \ < \ \mathbf{X} \ \mathbf{X} \ > \ = \ Real \ (\bar{\mathbf{X}} \ \mathbf{X}) \ = (x_o \ x_o \ + x_k \ x_k). \tag{1.5}$$

The inverse

$$\mathbf{X}^{-1} \ = \ \frac{\bar{\mathbf{X}}}{N(\mathbf{X})}, \quad \mathbf{X}^{-1}\mathbf{X} = \mathbf{X}\mathbf{X}^{-1} = 1. \tag{1.6}$$

The non-vanishing associator is defined by

$$\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\} = (\mathbf{X}\mathbf{Y})\mathbf{Z} - \mathbf{X}(\mathbf{Y}\mathbf{Z}) \tag{1.7}$$

In particular, the associator

$$\{e_i, e_j, e_k\} \ = \ d_{ijkl} \ e_l, \quad d_{ijkl} \ = \ \epsilon_{ijklmnp} \ c^{mnp}, \ i, j, k.... = 1, 2, 3, ...7 \tag{1.8}$$

There are **no** matrix *representations* of the Octonions due to the non-associativity, however Dixon has shown how many Lie algebras can be obtained from the left/right action of the octonion algebra on itself [4]. $\mathbf{O}_L$ and $\mathbf{O}_R$ are identical, isomorphic to the matrix algebra $R(8)$ of 8×8 real matrices. The 64-dimensional bases are of the form $\mathbf{1}, e_{La}, e_{Lab}, e_{Labc}$, or $\mathbf{1}, e_{Ra}, e_{Rab}, e_{Rabc}$, where, for example, if $\mathbf{x} \in \mathbf{O}$, then $e_{Lab}[\mathbf{x}] = e_a(e_b\mathbf{x})$, and $e_{Rab}[\mathbf{x}] = (\mathbf{x}e_a)e_b$.

From the structure constants of the Octonion algebra one can associate to the left action of e_a on e_o and e_b

$$e_{La}\ [e_o] \ = \ e_a\ e_o \ = \ e_a, \ \ e_{La}\ [e_b] \ = \ e_a\ e_b \ = \ C_{abc}\ e_c \tag{1.9}$$

the following 8×8 *antihermitian* matrix $\mathbf{M}_{La} : e_{La} \leftrightarrow \mathbf{M}_{La}$, and whose entries are given by

$$(M^L_a)_{bc} = C_{abc},\ a, b, c = 1, 2, \cdots, 7;\ (M^L_a)_{00} = 0,\ (M^L_a)_{0c} = \delta_{ac},\ (M^L_a)_{c0} \ = \ -\delta_{ac} \tag{1.10}$$

And similar procedure for the right actions, Due to the non-associativity of the Octonions one has $e_1e_2 = e_4$, but $\mathbf{M}_{L1}\mathbf{M}_{L2} \neq \mathbf{M}_{L4}$!, because there are **no** matrix representations of the non-associative Octonion algebra, and as a result one has that

$$\mathbf{M}_{La}\ \mathbf{M}_{Lb} \ \neq \ C_{abc}\ \mathbf{M}_{Lc} \tag{1.10}$$

Dixon [4] many years ago published a monograph pointing out the key role that the composition algebra (the Dixon algebra) $\mathbf{T} = \mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$ had in the architecture of the Standard Model. More recently, it has been shown by Furey how this algebra acting on itself allows to find the Standard Model particle representations [5]. For this reason we constructed in [1] a gravitational theory based on a $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued metric defined as

$$\mathbf{g}_{\mu\nu}(x^\mu) \ = \ g_{(\mu\nu)}(x^\mu) + g^{IA}_{\mu\nu}(x^\mu)\,(q_I \otimes e_A),\ \ q_I = q_o, q_1, q_2, q_3;\ \ e_A = e_o, e_1, e_2, \cdots, e_7 \tag{1.11}$$

where the ordinary $4D$ spacetime coordinates are $x^\mu, \mu = 0, 1, 2, 3$, and $g_{(\mu\nu)}$ is the standard Riemannian metric. The extra "internal" $C \otimes H \otimes O$-valued metric components are explicitly given by

$$(g_{(\mu\nu)} + i g_{[\mu\nu]})^{oo},\ \ (g_{[\mu\nu]} + i g_{(\mu\nu)})^{ko},\ \ (g_{[\mu\nu]} + i g_{(\mu\nu)})^{oa},\ \ (g_{(\mu\nu)} + i g_{[\mu\nu]})^{ka} \tag{1.12}$$

$k = 1, 2, 3; a = 1, 2, \cdots, 7$. The index o is associated with the real units q_o, e_o. The bar conjugation amounts to $i \to -i;\ q_k \to -q_k;\ e_a \to -e_a$, so that $\bar{\mathbf{g}}_{\mu\nu} = \mathbf{g}_{\nu\mu}$.

The generalization of the line interval considered in [2], [3] based on the metric (3.1) is then given by

$$ds^2 \ = \ < \ \mathbf{g}_{\mu\nu}\ dx^\mu\ dx^\nu \ > \ = \ (\ g_{(\mu\nu)} \ + \ g^{oo}_{(\mu\nu)}\ )\ dx^\mu\ dx^\nu \tag{1.13}$$

where the operation $< \cdots >$ denotes taking the *real* components. From eq-(1.13) one learns that the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued metric leads to a *bimetric* theory of gravity where the two metrics are, respectively, $g_{(\mu\nu)}, g^{oo}_{(\mu\nu)} = h_{(\mu\nu)}$.

The $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued affinity was given by

$$\mathbf{\Upsilon}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}(g_{\mu\nu}) + \mathbf{\Theta}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}(g_{\mu\nu}) + \delta^\rho_\mu \ \mathbf{A}_\nu =$$

$$\Gamma^\rho_{\mu\nu}(g_{\mu\nu}) + \delta^\rho_\mu \ \left(\ A^{oo}_\nu \ (q_o \otimes e_o) + A^{ia}_\nu \ (q_i \otimes e_a) + A^{io}_\nu \ (q_i \otimes e_o) + A^{oa}_\nu \ (q_o \otimes e_a) \ \right) \tag{1.14}$$

Thus we have decomposed the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued affinity $\mathbf{\Upsilon}^\rho_{\mu\nu}$ into a real-valued "external" part Γ plus an "internal" part $\mathbf{\Theta}^\rho_{\mu\nu}$. The base spacetime connection may be chosen to be the torsionless Christoffel connection

$$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} = \frac{1}{2} \ g^{\rho\sigma} \ (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{1.15}$$

but the 'internal" part $\mathbf{\Theta}^\rho_{\mu\nu}$ of the connection is taken to be *independent* of the metric, like in the Palatini formalism.

The $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued curvature tensor $\mathbf{R}^\sigma_{\rho\mu\nu} = \mathcal{R}^\sigma_{\rho\mu\nu} + \mathbf{\Omega}^\rho_{\sigma\mu\nu}$, involving the base spacetime and internal space curvature is defined by

$$\mathbf{R}^\sigma_{\rho\mu\nu} = \mathbf{\Upsilon}^\sigma_{\rho\mu,\nu} - \mathbf{\Upsilon}^\sigma_{\rho\nu,\mu} + \mathbf{\Upsilon}^\sigma_{\tau\nu} \ \mathbf{\Upsilon}^\tau_{\rho\mu} - \mathbf{\Upsilon}^\sigma_{\tau\mu} \ \mathbf{\Upsilon}^\tau_{\rho\nu}. \tag{1.16}$$

$$\mathbf{R}^\sigma_{\rho\mu\nu} = \mathcal{R}^\sigma_{\rho\mu\nu}(\Gamma^\rho_{\mu\nu}) + \delta^\sigma_\rho \ \mathbf{F}_{\mu\nu}. \tag{1.17}$$

where $\mathcal{R}^\sigma_{\rho\mu\nu}(\Gamma^\rho_{\mu\nu})$ is the base spacetime Riemannian curvature associated to the symmetric Christoffel connection $\Gamma^\rho_{\mu\nu}$.

The "internal" space $\mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued curvature is

$$\mathbf{\Omega}^\rho_{\sigma\mu\nu} = \delta^\rho_\sigma \ \mathbf{F}_{\mu\nu} \tag{1.18}$$

with

$$\mathbf{F}_{\mu\nu} = \mathbf{A}_{\mu,\nu} - \mathbf{A}_{\nu,\mu} - [\ \mathbf{A}_\mu \ , \ \mathbf{A}_\nu \ ]. \tag{1.19}$$

and where the field $\mathbf{A}_\mu$ can be read directly in terms of the internal space affinity from the relation

$$\mathbf{\Theta}^\rho_{\mu\nu} = \delta^\rho_\mu \ \mathbf{A}_\nu \tag{1.20}$$

There are 32 complex-valued fields (64-real valued fields)

$$\mathbf{A}_\mu = \{A^{oo}_\mu, A^{io}_\mu, A^{oa}_\mu, A^{ia}_\mu\} \tag{1.21}$$

and the commutators in eq-(1.19) are defined by

$$[q_I \otimes e_A, \ q_J \otimes e_B] = \frac{1}{2} \ \{q_I, q_J\} \otimes [e_A, e_B] + \frac{1}{2} \ [q_I, q_J] \otimes \{e_A, e_B\} \tag{1.22}$$

which lead to the following explicit components for $\mathbf{F}_{\mu\nu}$

$$F^{oo}_{\mu\nu} = \partial_\mu A^{oo}_\nu - \partial_\nu A^{oo}_\mu \tag{1.23}$$

$$F^{oc}_{\mu\nu} \ = \ \partial_\mu A^{oc}_\nu \ - \ \partial_\nu A^{oc}_\mu \ + \ (A^{oa}_\mu \ A^{ob}_\nu \ - \delta_{ij} \ A^{ia}_\mu \ A^{jb}_\nu) \ C^c_{ab} \tag{1.24}$$

$$F^{ko}_{\mu\nu} \ = \ \partial_\mu A^{ko}_\nu \ - \ \partial_\nu A^{ko}_\mu \ + \ (A^{io}_\mu \ A^{jo}_\nu \ - \delta_{ab} \ A^{ia}_\mu \ A^{jb}_\nu) \ f^k_{ij} \tag{1.25}$$

$$F^{kc}_{\mu\nu} \ = \ \partial_\mu A^{kc}_\nu \ - \ \partial_\nu A^{kc}_\mu \ + \ A^{oa}_\mu \ A^{kb}_\nu \ C^c_{ab} \ + \ A^{io}_\mu \ A^{jc}_\nu \ f^k_{ij} \tag{1.26}$$

The next step was to embed the Standard Model Gauge Fields into the Internal Connection $\mathbf{\Theta}^\rho_{\mu\nu}$. Eqs-(1.23-1.26) yield the following 32 complex-valued non-vanishing field strengths

$$F^{oo}_{\mu\nu}, \quad F^{ko}_{\mu\nu}, \quad F^{oc}_{\mu\nu}, \quad F^{kc}_{\mu\nu}, \quad k = 1,2,3; \quad c = 1,2,\cdots,7 \tag{1.27}$$

Given the $U(1)$ Maxwell field

$$\mathcal{F}_{\mu\nu} \ = \ \partial_\mu \mathcal{A}_\nu \ - \ \partial_\nu \mathcal{A}_\mu \tag{1.28}$$

the Maxwell kinetic term in the Standard Model action is embedded as follows

$$\mathcal{F}_{\mu\nu} \ \mathcal{F}^{\mu\nu} \ \subset \ F^{oo}_{\mu\nu} \ (F^{\mu\nu}_{oo})^* \tag{1.29}$$

Given the $SU(2)$ field strength

$$\mathcal{F}^k_{\mu\nu} \ = \ \partial_\mu \mathcal{A}^k_\nu \ - \ \partial_\nu \mathcal{A}^k_\mu \ + \ \mathcal{A}_\mu{}^i \ \mathcal{A}^j_\nu \ \epsilon^k_{ij} \tag{1.30}$$

the $SU(2)$ Yang-Mills term is embedded as

$$\mathcal{F}^i_{\mu\nu} \ \mathcal{F}^{\mu\nu}_i \ (i = 1,2,3) \ \subset \ (F^{ko}_{\mu\nu}) \ (F^{\mu\nu}_{ko})^* \ (k = 1,2,3) \tag{1.31}$$

Since the $SU(2)$ algebra is isomorphic to the algebra of quaternions, the embedding (1.31) is very natural. The chain of subgroups

$$SO(8) \ \supset \ SO(7) \ \supset \ G_2 \ \supset \ SU(3) \tag{1.32}$$

related to the round and squashed seven-spheres : $S^7 \simeq SO(8)/SO(7), S^7_* \simeq SO(7)/G_2$, reflect how the $SU(3)$ group is embedded. The number of generators of $SO(8)$, $SO(7)$ are 28 and 21 respectively. There are $7 + 21 = 28$ complex-valued field strengths, respectively

$$F^{oc}_{\mu\nu}, \quad F^{kc}_{\mu\nu}, \quad k = 1,2,3; \quad c = 1,2,\cdots,7 \tag{1.33}$$

such that the $SU(3)$ Yang-Mills terms can be embedded into the contribution of the above $7 + 21 = 28$ complex-valued fields as follows

$$\mathcal{F}^\alpha_{\mu\nu} \ \mathcal{F}^{\mu\nu}_\alpha \ (\alpha = 1,2,\ldots,7,8) \ \subset \ (F^{oc}_{\mu\nu}) \ (F^{\mu\nu}_{oc})^* \ + \ (F^{kc}_{\mu\nu}) \ (F^{\mu\nu}_{kc})^* \ (c = 1,2,\ldots,7) \tag{1.34}$$

and where the $SU(3)$ field strength is given by

$$\mathcal{F}^{\gamma}_{\mu\nu} \ = \ \partial_\mu \mathcal{A}^{\gamma}_{\nu} \ - \ \partial_\nu \mathcal{A}^{\gamma}_{\mu} \ + \ \mathcal{A}_{\mu}{}^{\alpha} \ \mathcal{A}^{\beta}_{\nu} \ f^{\gamma}_{\alpha\beta} \tag{1.35}$$

Having reviewed some of the results in [1] we shall proceed in the next section to show how the matrix realization of the $C \otimes H \otimes O_L$ algebra naturally leads to a rank-16 $u(4) \oplus (4) \oplus u(4) \oplus u(4)$ algebra. This, in turn, suggests to extend the Standard Model based on the $SU(3) \times SU(2) \times U(1)$ group to one based on $[SU(4)]^4$. In the final section we show how to establish the correspondence among $C \otimes H \otimes O$-valued gravity, generalized Hermitian geometry and Nonsymmetric Kaluza-Klein Theory. The construction in section **3** must *not* be confused with the model of $R \otimes C \otimes H \otimes O$-valued gravity discussed above.

2 $SU(4)_C \times SU(4)_F \times SU(4)_L \times SU(4)_R$ Unification

Given that the complex quaternionic algebra $C \otimes H$ is isomorphic to the Pauli spin algebra with the 2×2 matrices $q_0 = \mathbf{1}_{2\times 2}$, $q_k = i\sigma_k, k = 1, 2, 3$, and the left action of the octonionic algebra on itself is represented by the 8×8 matrices $e_{LA} = \mathbf{M}^L_A, A = 0, 1, \cdots, 7$, then the $4 \times 8 = 32$ generators $q_I \otimes e_{LA}$ of the $C \otimes H \otimes O_L$ algebra can be represented by 32 complex 16×16 matrices, which is tantamount to 64 real 16×16 matrices, and which is compatible with the fact that 64 $(2 \times 4 \times 8)$ is the dimension of the $C \otimes H \otimes O_L$ algebra.

Each complex 16×16 matrix, above, can be expanded in terms of the basis elements of the complex Clifford algebra $Cl(8, C)$ comprised of $2^8 = 256$ complex 16×16 matrices. However this is far too cumbersome. It is easier if we expand each of the above 32 complex 16×16 matrices in terms of the tensor products $\Gamma_M \otimes \mathbf{1}_{4\times 4}$, where $\Gamma_M (M = 1, 2, \cdots, 32 = 2^5)$ is the basis of the complex Clifford algebra $Cl(5, C)$ comprised of 32 complex 4×4 matrices, and $\mathbf{1}_{4\times 4}$ is the unit 4×4 matrix.

Therefore we end up having that the 32 complex 16×16 matrix generators $q_I \otimes e_{LA}$ of the $C \otimes H \otimes O_L$ algebra can be expanded in terms of a linear combination of the 32 $Cl(5, C)$ algebra generators Γ_M as follows

$$q_I \ \otimes \ e_{LA} \ = \ (\mathbf{M}^L_{IA})_{16\times 16} \ = \ \sum_{M=1}^{32} \ C^M_{IA} \ (\Gamma_M)_{4\times 4} \ \otimes \ \mathbf{1}_{4\times 4}, \tag{2.1}$$

where $I = 0, 1, 2, 3; A = 0, 1, 2, \cdots, 7$, and C^M_{IA} are complex numerical coefficients.

Let us recall the following isomorphisms among real and complex Clifford algebras [6]

$$Cl(2m + 1, C) = Cl(2m, C) \oplus Cl(2m, C) \sim M(2^m, C) \oplus M(2^m, C) \ \Rightarrow$$

$$Cl(5,C) \ = \ Cl(4,C) \oplus Cl(4,C) \tag{2.2}$$

where $M(2^m, C)$ is the $2^m \times 2^m$ matrix algebra over the complex numbers (some authors [4] use the different notation $\mathbf{C}(2^m)$).

Also one has

$$Cl(4,C) \sim M(4,C) \sim Cl(4,1,R) \sim Cl(2,3,R) \sim Cl(0,5,R) \tag{2.3}$$

$$Cl(4,C) \sim M(4,C) \sim Cl(3,1,R) \oplus \mathbf{i}\ Cl(3,1,R) \sim M(4,R) \oplus \mathbf{i}\ M(4,R) \tag{2.4}$$

$$Cl(4,C) \sim M(4,C) \sim Cl(2,2,R) \oplus \mathbf{i}\ Cl(2,2,R) \sim M(4,R) \oplus \mathbf{i}\ M(4,R) \tag{2.5}$$

where $M(4,R), M(4,C)$ is the 4×4 matrix algebra over the reals and complex numbers, respectively.

In [6] we showed, by recurring to the Weyl unitary "trick", how from each one of the $Cl(3,1,R)$ commuting sub-algebras inside the $Cl(4,C)$ algebra one can also obtain the $u(p,q)$ algebras with the provision $p+q=4$. Namely, the $u(p,q)$ algebra generators are given by suitable linear combinations of the $Cl(3,1,R)$ generators. In particular, the $u(2,2) = su(2,2) \oplus u(1)$ algebra contains the conformal algebra in four dimensions $su(2,2) \sim so(4,2)$. When $p=4, q=0$, the algebra is $u(4) = u(1) \oplus su(4) \sim u(1) \oplus so(6)$.

To sum up, given that the algebra $M(4,C) \sim gl(4,C)$ is also the complexification of $u(4)$ ($sl(4,C)$ is the complexification of $su(4)$), and by virtue of eqs-(2.2), the $Cl(5,C)$ algebra can be decomposed into $four$ copies of $u(4)$

$$Cl(5,C) \ = \ Cl(4,C) \oplus Cl(4,C) \ \sim \ u(4) \oplus u(4) \oplus u(4) \oplus u(4) \tag{2.6}$$

The dimension of the four copies of $u(4)$ is $4 \times 16 = 64$ which matches the dimension of the $C \otimes H \otimes O_L$ algebra, as expected (64 is also the dimension of the real $Cl(6)$ algebra). Consequently, the $C \otimes H \otimes O_L$ algebra, by virtue of the decomposition in eq-(2.1), can accommodate a grand unified group given by

$$SU(4)_C \times SU(4)_F \times SU(4)_L \times SU(4)_R \ \subset \ U(4) \times U(4) \times U(4) \times U(4) \tag{2.7}$$

The gauge group $SU(3)_C \times SU(3)_F \times SU(3)_L \times SU(3)_R$ can naturally be embedded into the above $[SU(4)]^4$ group. The former group involving a unification of left-right $SU(3)_L \times SU(3)_R$ chiral symmetry, color $SU(3)_C$ and family $SU(3)_F$ symmetries in a maximal rank-8 subgroup of E_8 was proposed by [7] as a landmark for future explorations beyond the Standard Model (SM). This model is called the $SU(3)$-family extended SUSY trinification model [7]. Among the key properties of this model are the unification of SM Higgs and lepton sectors, a common Yukawa coupling for chiral fermions, the absence of the μ-problem, gauge couplings unification and proton stability to all orders in perturbation theory.

The standard model (SM) fermions (quarks, leptons) can be embedded into the fermionic matter belonging to the following $SU(4)_C \times SU(4)_F \times SU(4)_L \times SU(4)_R$ representations as follows

$$Q_{SM} \subset \mathbf{Q} = (4, 4, \bar{4}, 1), \quad Q^c_{SM} \subset \mathbf{Q}^c = (\bar{4}, \bar{4}, 1, 4), \tag{2.8}$$

$$L_{SM} \subset \mathbf{L} = (1, 4, \bar{4}, 1), \quad \mathbf{L}^c = (1, \bar{4}, 1, 4) \tag{2.9}$$

where the $\mathbf{Q}, \mathbf{Q}^c, \mathbf{L}, \mathbf{L}^c$ multiplets include the addition of heavy quarks (anti-quarks); leptons (anti-leptons), and an extra $fourth$ family of fermions (and their anti-particles). The first (left handed) quark family is

$$\mathbf{Q}_1 \equiv \begin{pmatrix} u_r & d_r & U_r & D_r \\ u_b & d_b & U_b & D_b \\ u_g & d_g & U_g & D_g \\ Q_u & Q_d & Q_U & Q_D \end{pmatrix} \tag{2.10}$$

where Q_u, Q_d, Q_U, Q_D, and $U_{r,b,g}, D_{r,b,g}$ are the additional quarks . As usual r, b, g stand for red, blue, green color. The charge conjugate multiplet containing the (right-handed) anti-quarks of the first family is

$$\mathbf{Q}^c_1 \equiv \begin{pmatrix} \overline{u_r} & \overline{u_b} & \overline{u_g} & \overline{Q_u} \\ \overline{d_r} & \overline{d_b} & \overline{d_g} & \overline{Q_d} \\ \overline{U_r} & \overline{U_b} & \overline{U_g} & \overline{Q_U} \\ \overline{D_r} & \overline{D_b} & \overline{D_g} & \overline{Q_D} \end{pmatrix} \tag{2.11}$$

By $\overline{u_r}$ one means $u^c_{\bar{r}}$, the up anti-quark with anti-red color, etc $\cdots$. Whereas $\overline{Q_u} = Q^c_u, \cdots$. And similar assignments for the remaining quark families.

The lepton multiplet will include the ordinary leptons (neutrino, electron, $\cdots$), plus the addition of charged $E_-, E_+, \cdots$, and neutral leptons $N_E, N^c_E, \cdots$. The first (left handed) lepton multiplet is comprised of $\{\nu_e, e_-, N_E, E_-\}$, and its (right handed) anti-multiplet is comprised of $\{\nu^c_e, e_+, N^c_E, E_+\}$. If necessary, one may also have to add extra fermions to cancel anomalies.

An analysis of the models based on $SU(4)_C \times SU(3)_L \times SU(3)_R$, and a preliminary discussion of $SU(4)_C \times SU(4)_L \times SU(4)_R$ can be found in [8]. Their lepton assignment differs from ours. An early $SU(4)_C \times SU(4)_F$ model, and based on an extension of the Pati-Salam group $SU(4)_C \times SU(2)_L \times SU(2)_R$, was proposed by [9]. Examples of a fourth family extension of the Standard Model can be found in [10].

Concluding this section, the algebraic structure of $C \otimes H \otimes O_L$ led to the group $[SU(4)]^4$ and reveals the possibility of extending the standard model by introducing additional gauge bosons, heavy quarks and leptons, and a $fourth$ family of fermions. The physical implications are enormous.

3 $C \otimes H \otimes O$-valued gravity, Matrix geometry and Nonsymmetric Kaluza-Klein Theory

In the final section we show how to establish the correspondence among $C \otimes H \otimes O$-valued gravity, generalized Hermitian Matrix geometry and Nonsymmetric

Kaluza-Klein Theory. It must *not* be confused with the model of $R \otimes C \otimes H \otimes O$-valued gravity discussed previously in section **1**.

We begin by recalling that the standard Hermitian metric on a complex D-dim manifold whose complex coordinates are $z^\mu, \bar{z}^\mu, \mu = 1, 2, \cdots, D; \bar{\mu} = \bar{1}, \bar{2}, \cdots, \bar{D}$, satisfies the properties [11]

$$g_{\mu\nu} \ = \ g_{\bar{\mu}\bar{\nu}} \ = \ 0, \ \ g_{\mu\bar{\nu}} = g_{\bar{\nu}\mu} \neq 0, \ \ (g_{\mu\bar{\nu}})^* \ = \ g_{\bar{\mu}\nu} = g_{\nu\bar{\mu}} \neq 0 \tag{3.1}$$

The real infinitesimal line interval ds^2 is given by

$$ds^2 \ = \ g_{\mu\bar{\nu}} \ dz^\mu \ d\bar{z}^\nu \ + \ g_{\bar{\mu}\nu} \ d\bar{z}^\mu \ dz^\nu \tag{3.2}$$

The $H \otimes O$-valued extension of the above Hermitian metric leads to a real infinitesimal line interval of the form

$$ds^2 \ = \ \frac{1}{16} \ Trace \ (\ \mathbf{g}_{\mu\bar{\nu}} \ dz^\mu \ d\bar{z}^\nu \ + \ \mathbf{g}_{\bar{\mu}\nu} \ d\bar{z}^\mu \ dz^\nu \ ) \tag{3.3}$$

and provided in terms of the trace of the 16×16 matrix-valued $\mathbf{g}_{\mu\bar{\nu}}, \mathbf{g}_{\bar{\mu}\nu}$ components as we shall explain next.

Given that the $2 \times 4 \times 8 = 64$ generators of the $C \otimes H \otimes O_L$ algebra can be represented by 32 *complex* 16×16 matrices $(\mathbf{M}^L_{IA})_{16\times16}$ (or 64 *real* 16×16 matrices), the $C \otimes H \otimes O$-valued metric components appearing in (3.3) can be expanded in a quaterno-octonionic basis, and rewritten in a 16×16-matrix form, in the following fashion

$$\mathbf{g}_{\mu\bar{\nu}}(z^\mu, \bar{z}^\mu) \ = \ \sum_{I,A} \mathbf{g}^{IA}_{\mu\bar{\nu}}(z^\mu, \bar{z}^\mu) \ (q_I \otimes e_{LA})^{JK} \ = \ g^{JK}_{\mu\bar{\nu}}(z^\mu, \bar{z}^\mu) \tag{3.4}$$

$$\mathbf{g}_{\bar{\mu}\nu}(z^\mu, \bar{z}^\mu) \ = \ \sum_{I,A} \mathbf{g}^{IA}_{\bar{\mu}\nu}(z^\mu, \bar{z}^\mu) \ (q_I \otimes e_{LA})^{JK} \ = \ g^{JK}_{\bar{\mu}\nu}(z^\mu, \bar{z}^\mu) \tag{3.5}$$

The coordinates are $z^\mu, \bar{z}^\mu \in C^2$. The matrix indices' range is $J, K = 1, 2, \cdots, 16$. The quaternion indices are $I = 0, 1, 2, 3$, and the octonion indices $A = 0, 1, 2, \cdots, 7$, respectively, and such that the components $g^{JK}_{\mu\bar{\nu}}(z^\mu, \bar{z}^\mu), g^{JK}_{\bar{\mu}\nu}(z^\mu, \bar{z}^\mu)$ are *complex-conjugates* of each other ensuring that the interval $(ds)^2$ in eq-(3.3) is real.

The non-vanishing connection coefficients of a Hermitian complex manifold are given by [11]

$$\Gamma^\rho_{\mu\nu} \ = \ g^{\rho\bar{\lambda}} \ \partial_\mu g_{\bar{\lambda}\nu} \ = \ g^{\rho\bar{\lambda}} \ \frac{\partial g_{\bar{\lambda}\nu}}{\partial z^\mu}; \quad \Gamma^{\bar{\rho}}_{\bar{\mu}\bar{\nu}} \ = \ g^{\bar{\rho}\lambda} \ \partial_{\bar{\mu}} g_{\lambda\bar{\nu}} \ = \ g^{\bar{\rho}\lambda} \ \frac{\partial g_{\lambda\bar{\nu}}}{\partial \bar{z}^\mu} \tag{3.6}$$

The non-vanishing curvature components are

$$R^\rho_{\sigma\bar{\mu}\nu} \ = \ \partial_{\bar{\mu}} \Gamma^\rho_{\nu\sigma}, \ \ R^{\bar{\rho}}_{\bar{\sigma}\mu\bar{\nu}} \ = \ \partial_\mu \Gamma^{\bar{\rho}}_{\bar{\nu}\bar{\sigma}} \tag{3.7}$$

The Ricci tensor components are

$$R_{\bar\mu\nu} \ = \ R^{\rho}_{\rho\bar\mu\nu}, \quad R_{\mu\bar\nu} \ = \ R^{\bar\rho}_{\bar\rho\mu\bar\nu} \tag{3.8}$$

and the Ricci scalar is

$$R \ = \ g^{\mu\bar\nu} \ R_{\mu\bar\nu} \ + \ g^{\bar\mu\nu} \ R_{\bar\mu\nu} \tag{3.9}$$

Under (anti) holomorphic coordinate transformations

$$z'^{\mu} \ = \ z'^{\mu}(z^{\rho}), \quad \bar z'^{\mu} \ = \ \bar z'^{\mu}(\bar z^{\rho}) \tag{3.10}$$

the metric components transform as

$$g'_{\rho\bar\sigma} \ = \ \frac{\partial z^{\mu}}{\partial z'^{\rho}} \ \frac{\partial \bar z^{\nu}}{\partial \bar z'^{\sigma}} \ g_{\mu\bar\nu}, \quad g'_{\bar\rho\sigma} \ = \ \frac{\partial \bar z^{\mu}}{\partial \bar z'^{\rho}} \ \frac{\partial z^{\nu}}{\partial z'^{\sigma}} \ g_{\bar\mu\nu} \tag{3.11}$$

$$g'_{\bar\rho\bar\sigma} \ = \ g'_{\rho\sigma} \ = \ 0 \tag{3.12}$$

Let us take the ordinary Hermitian metric in $D = 2$ complex dimensions case as an example ($D = 4$ real dimensions) whose coordinates are $z^{\mu}, \bar z^{\mu}, \ \mu, \nu = 1, 2$ and $\bar\mu, \bar\nu = \bar 1, \bar 2$. The invariant measure of integration under the (anti) holomorphic coordinate transformations (3.10) is

$$d\Omega \ \equiv \ dz^1 \wedge dz^2 \wedge d\bar z^1 \wedge d\bar z^2 \ \sqrt{det(g_{\mu\bar\nu}(z, \bar z))} \ \sqrt{det(g_{\bar\mu\nu}(z, \bar z))} \tag{3.13}$$

and the analog of the Einstein-Hilbert action is

$$S \ = \ \frac{1}{2\kappa^2} \ \int R \ d\Omega \tag{3.14}$$

where R is given by eq-(3.9) and κ^2 is the gravitational coupling, ($8\pi G$ in ordinary Einstein gravity in $4D$).

To extend these definitions to the $C \otimes H \otimes O$-valued metric case is more complicated due to the noncommutativity and nonassociativity. One may begin, firstly, by finding the relation between $C \otimes H \otimes O$-valued metrics in *two complex* dimensions with metrics in *higher* dimensional *real* manifolds.

Focusing on one simple example given by the two-complex dimensional case (four real dimensions) $z^{\mu}, \bar z^{\mu} \in C^2$, so that the $C \otimes H \otimes O$-valued metric components $g^{JK}_{\mu\bar\nu}(z^{\mu}, \bar z^{\mu})$ have a one-to-one correspondence with the components of the 32×32 complex matrix $g_{MN} = g_{(MN)} + i g_{[MN]}$, with $M, N = 1, 2, \cdots, 32$. Similarly, the $C \otimes H \otimes O$-valued metric components $g^{JK}_{\bar\mu\nu}(z^{\mu}, \bar z^{\mu})$ have a one-to-one correspondence with the components of the 32×32 complex matrix $(g_{MN})^* = g_{(MN)} - i g_{[MN]} = g_{NM}$.

Let us decompose the 32×32 complex metric $g_{MN} = g_{(MN)} + i g_{[MN]}$ in the following Kaluza-Klein (KK) form

$$g_{MN}(x^{\alpha}; y^{a}) \ = \ \begin{pmatrix} g_{\alpha\beta} \ + \ h_{ab} \ A^a_{\alpha} \ A^b_{\beta} & A^b_{\alpha} \ h_{ab} \\ A^a_{\beta} \ h_{ab} & h_{ab} \end{pmatrix} \tag{3.15}$$

such that

$$g_{\alpha\beta} = g_{(\alpha\beta)} + ig_{[\alpha\beta]}; \quad h_{ab} = h_{(ab)} + ih_{[ab]} \tag{3.16}$$

The four-dimensional spacetime indices range from $\alpha, \beta = 1, 2, 3, 4$, and the internal space indices range is $a, b = 1, 2, \cdots, 28$. Similar results apply to the complex conjugate $(g_{MN})^*(x^\alpha; y^a)$. Note that the *real* dimensions of the higher dimensional space is $32 = 4 + 28$.

It is important to emphasize that the above Kaluza-Klein decomposition is *not* the standard one associated to *symmetric* metrics but one corresponding to the Nonsymmetric Kaluza-Klein (Jordan-Thiry) Theory and whose structure is far richer than the conventional one. Completely new results in comparison to the standard symmetric Kaluza-Klein theory have been obtained by [12].

The Ricci scalar

$$\mathcal{R} \ = \ g^{MN}\mathcal{R}_{MN} \ + \ (g^{MN}\mathcal{R}_{MN})^* \tag{3.17}$$

allows to construct the higher dimensional gravitational action

$$S \ = \ \frac{1}{2\kappa^2} \int d^{32}X \ [\ ||det(g_{MN})||\ ]^{\frac{1}{2}} \ \mathcal{R}(X) =$$

$$\frac{1}{2\kappa^2} \int d^{32}X \ [det(g_{MN}) \ det(g_{MN})^*]^{\frac{1}{4}} \ \mathcal{R}(X) \tag{3.18}$$

writing the norm of a complex number as $||z|| = \sqrt{(zz^*)}$ is the reason why there is a 4-th root in (3.18). After the Kaluza-Klein reduction from $D = 32$ to $D = 4$: $g_{MN}(x^\alpha; y^a) \to g_{MN}(x^\alpha)$, eq-(3.18) becomes

$$S \ = \ \frac{\Omega_{28}}{2\kappa^2} \int d^4x \ [det(g_{MN}(x)) \ det(g_{MN}(x))^*]^{\frac{1}{4}} \ \mathcal{R}(x) \tag{3.19}$$

where $\int d^{28}y = \Omega_{28}$ is the volume of the 28-dimensional compact internal space.

To sum up, given $\mu, \nu = 1, 2$; $\bar{\mu}, \bar{\nu} = \bar{1}, \bar{2}$, and $M, N = 1, 2, \cdots, 32$; the Nonsymmetric Kaluza-Klein *reduction* from $D = 32$ to $D = 4$: $g_{MN}(x^\alpha; y^a) \to g_{MN}(x^\alpha)$ would allow to establish the following correspondence between $C \otimes H \otimes O$-valued metrics in *two complex* dimensions and complex-valued metrics in *higher* dimensional *real* manifolds

$$g^{JK}_{\mu\bar{\nu}}(z^\mu, \bar{z}^\mu) \leftrightarrow \ g_{MN}(x^\alpha) \ = \ g_{(MN)}(x^\alpha) + ig_{[MN]}(x^\alpha); \quad \alpha = 1, 2, 3, 4 \tag{3.20}$$

and similary

$$g^{JK}_{\bar{\mu}\nu}(z^\mu, \bar{z}^\mu) \leftrightarrow \ (g_{MN})^*(x^\alpha) \ = \ g_{(MN)}(x^\alpha) - ig_{[MN]}(x^\alpha); \quad \alpha = 1, 2, 3, 4 \tag{3.21}$$

Finally, after the correspondence of eqs-(3.20, 3.21) is established we may then propose the action (3.19), after the Kaluza-Klein reduction, to be the one which corresponds to the $H \otimes O$-extension of the prior gravitational action (3.14) associated with the Hermitian metric in a two-dimensional complex manifold.

An interesting coincidence is that the line interval $ds^2 = \eta_{MN} dX^M dX^N$ in a $D = 32$-dim Euclidean space has $SO(32)$ for its isometry group. $SO(32)$ and $E_8 \times E_8$ are the groups associated with the anomaly-free heterotic string in $D = 10$. A KK compactification from $D = 32$ to $D = 4$ on a 14 complex-dimensional internal space $CP^{14} = \frac{SU(15)}{U(14)}$ yields a $SU(15)$ Yang-Mills in $D = 4$. $SU(15)$ can be embedded into $SO(32)$ as $SU(15) \subset SU(16) \subset SO(32)$.

The simplest case is that of a metric in $D = 1$ complex dimension (2 real dimensions) $g^{JK}_{\mu\bar\nu} = g^{JK}_{1\bar 1}(z, \bar z)$ which corresponds to a 16×16 complex metric g_{MN} in 16 real dimensions. A KK compactification from $D = 16$ to $D = 2$ on a 7 complex-dimensional internal space $CP^7 = \frac{SU(8)}{U(7)}$ yields a $SU(8)$ YM in $D = 2$. $SU(8) \subset SO(16)$ which is the isometry group of a 16-dim Euclidean space.

To extend the definitions of the Ricci scalar (3.9) to the $C \otimes H \otimes O$-valued metric $\mathbf{g}$ case is more complicated due to the noncommutativity and nonassociativity. For example, one would have terms of the form $\mathbf{g}\partial(\mathbf{g}\partial\mathbf{g})$, $\mathbf{g}(\mathbf{g}\partial\mathbf{g})(\mathbf{g}\partial\mathbf{g})$, such that their products are *no* longer associative, and due to the noncommutativity, the results also depend on the *ordering* of those products.

To finalize this section we propose the construction of a generalized Hermitian Matrix geometry as follows. After the correspondence in eqs-(3.20, 3.21) is made, one could treat each one of the components of $\mathbf{g}_{\mu\bar\nu}, \mathbf{g}_{\bar\mu\nu}$ as if they were 16×16 matrices, and if one chooses an specific *ordering* of those matrices in the products in $\mathbf{g}\partial(\mathbf{g}\partial\mathbf{g})$, $\mathbf{g}(\mathbf{g}\partial\mathbf{g})(\mathbf{g}\partial\mathbf{g})$, one could then define the $H \otimes O$-valued extension of the Ricci tensor (3.8). Furthermore, due to the cyclic property of the trace operation, the $H \otimes O$ extension of the Ricci scalar of eq-(3.9) is given in terms of the trace of the product of the 16×16 complex matrices as follows

$$\mathbf{R} \ = \ \frac{1}{16}\ Trace\ \left(\ \mathbf{g}^{\mu\bar\nu}\ \mathbf{R}_{\mu\bar\nu}\ +\ \mathbf{g}^{\bar\mu\nu}\ \mathbf{R}_{\bar\mu\nu}\ \right) \tag{3.22}$$

To find the analog of the Einstein-Hilbert action in the $C \otimes H \otimes O$-valued metric requires to construct the proper *measure*. We may define the *block* determinant Det of $g^{JK}_{\mu\bar\nu}(z^\mu, \bar z^\mu)$ in terms of antisymmetrized sums of products of determinants of 16×16 matrices. Namely,

$$Det\ (g^{JK}_{\mu\bar\nu}(z^\mu, \bar z^\mu))\ =\ \frac{1}{(2!)^2}\ \epsilon^{\mu_1\mu_2}\ \epsilon^{\bar\nu_1\bar\nu_2}\ det(g^{JK}_{\mu_1\bar\nu_1})\ det(g^{JK}_{\mu_2\bar\nu_2}) \tag{3.23}$$

where the determinant of the 16×16 matrix block is

$$det(g^{JK}_{\mu_1\bar\nu_1})\ =\ \frac{1}{(16!)^2}\ \epsilon_{J_1J_2\cdots J_{16}}\ \epsilon_{K_1K_2\cdots K_{16}}\ g^{J_1K_1}_{\mu_1\bar\nu_1}\ g^{J_2K_2}_{\mu_1\bar\nu_1}\ \cdots\ g^{J_{16}K_{16}}_{\mu_1\bar\nu_1} \tag{3.24}$$

and

$$det(g^{JK}_{\mu_2\bar\nu_2})\ =\ \frac{1}{(16!)^2}\ \epsilon_{J_1J_2\cdots J_{16}}\ \epsilon_{K_1K_2\cdots K_{16}}\ g^{J_1K_1}_{\mu_2\bar\nu_2}\ g^{J_2K_2}_{\mu_2\bar\nu_2}\ \cdots\ g^{J_{16}K_{16}}_{\mu_2\bar\nu_2} \tag{3.25}$$

Similarly we can define the block determinant $Det\ (g^{JK}_{\bar\mu\nu}(z^\mu, \bar z^\mu))$ and extend these definitions to other complex-dimensions beyond $D = 2$. The measure of integration is a generalization of (3.13) and given by

$$D\Omega \ \equiv \ dz^1 \wedge dz^2 \wedge d\bar z^1 \wedge d\bar z^2 \ \sqrt{Det(g^{JK}_{\mu\bar\nu}(z,\bar z))}\ \sqrt{Det(g^{JK}_{\bar\mu\nu}(z,\bar z))} \tag{3.26}$$

The generalization of the Einstein-Hilbert action in eq-(3.14) is given in terms of $\mathbf{R}$ in eq-(3.22), and the measure (3.26), as follows

$$S \ = \ \frac{1}{32\kappa^2} \int D\Omega\ Trace_{16\times 16}\ \left(\ \mathbf{g}^{\mu\bar\nu}\ \mathbf{R}_{\mu\bar\nu} \ + \ \mathbf{g}^{\bar\mu\nu}\ \mathbf{R}_{\bar\mu\nu} \ \right) \tag{3.27}$$

Therefore, the gravitational action (3.27) based on "coloring" the graviton by attaching internal indices $\mathbf{g}_{\mu\bar\nu} \rightarrow g^{JK}_{\mu\bar\nu}, \cdots$ and associated to the 16×16 matrices, is the one corresponding to a $C \otimes H \otimes O$-valued metric, and defined over a complex Hermitian manifold in two complex-dimensions. We propose that this matrix approach could be an example of a generalized Hermitian Matrix geometry, and which must *not* be confused with the current work on generalized geometry, double field theories, exceptional field theories in M-theory, see [13] and references therein.

Going back to the line interval of eq-(3.3), under unitary $U(16)$ symmetry transformations $\mathbf{U}^\dagger = \mathbf{U}^{-1}$ acting on the 16×16 matrix indices only

$$\mathbf{g}_{\mu\bar\nu} \ \rightarrow \ \mathbf{U}\ \mathbf{g}_{\mu\bar\nu}\ \mathbf{U}^{-1}, \ \ \mathbf{g}_{\bar\mu\nu} \ \rightarrow \ \mathbf{U}\ \mathbf{g}_{\bar\mu\nu}\ \mathbf{U}^{-1} \tag{3.28}$$

the interval ds^2 (3.3) will remain invariant due to the cyclic property of the Trace

$$Trace\ \left(\ \mathbf{U}\ \mathbf{g}_{\mu\bar\nu}\ \mathbf{U}^{-1} \ \right) \ = \ Trace\ \left(\ \mathbf{U}^{-1}\ \mathbf{U}\ \mathbf{g}_{\mu\bar\nu} \right) \ = \ Trace\ (\ \mathbf{g}_{\mu\bar\nu}\ ) \tag{3.29a}$$

$$Trace\ \left(\ \mathbf{U}\ \mathbf{g}_{\bar\mu\nu}\ \mathbf{U}^{-1} \ \right) \ = \ Trace\ \left(\ \mathbf{U}^{-1}\ \mathbf{U}\ \mathbf{g}_{\bar\mu\nu} \right) \ = \ Trace\ (\ \mathbf{g}_{\mu\bar\nu}\ ) \ \Rightarrow \tag{3.29a}$$

$$Trace\ \left(\ \mathbf{U}\ \mathbf{g}_{\mu\bar\nu}\ \mathbf{U}^{-1}\ dz^\mu\ d\bar z^\nu \ + \ \mathbf{U}\ \mathbf{g}_{\bar\mu\nu}\ \mathbf{U}^{-1}\ d\bar z^\mu\ dz^\nu \ \right) \ =$$

$$Trace\ (\ \mathbf{g}_{\mu\bar\nu}\ dz^\mu\ d\bar z^\nu \ + \ \mathbf{g}_{\bar\mu\nu}\ d\bar z^\mu\ dz^\nu \ ) \tag{3.30}$$

Therefore, the unitary group $U(16)$ acts as an *isometry* group. In ordinary KK theory the gauge symmetries in lower dimensions emerge from the isometry group of the compactified internal space. In the previous section one had $C \otimes H \otimes O_L$ algebra $\leftrightarrow$ 32 complex 16×16 matrices $\leftrightarrow$ 64 real 16×16 matrices $\leftrightarrow$ 64 generators of the rank-16 $u(4) \oplus u(4) \oplus u(4) \oplus u(4)$ algebra. The $u(16)$ has also rank 16, like the $so(32)$ and $e_8 \oplus e_8$ algebras, but in this case the isometry group $U(16)$ is larger than $[U(4)]^4$.

To conclude, we have explored the $C \otimes H \otimes O$ algebra deeper and led us to the gauge group $[SU(4)]^4$ (suggesting the plausible existence of a fourth family). Whereas $C \otimes H \otimes O$-valued gravity bear connections to Nonsymmetric

Kaluza-Klein theories and complex Hermitian Matrix Geometry. It is desirable to extend these results to hypercomplex, quaternionic manifolds and Exceptional Jordan Matrix Models.

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References

[1] C. Castro, "R ⊗ C ⊗ H ⊗ O-valued Gravity as a Grand Unified Field Theory" submitted to Adv. in Appl. Clifford Algebras, 2018.

[2] S. Marques and C. Oliveira, J. Math. Phys **26** (1985) 3131. Phys. Rev **D 36** (1987) 1716.

[3] Carlos Castro, "The Noncommutative and Nonassociative Geometry of Octonionic Spacetime, Modified Dispersion Relations and Grand Unification" J. Math. Phys, **48**, no. 7 (2007) 073517.

[4] G. M. Dixon, *"Division Algebras, Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics"* (Kluwer, Dordrecht, 1994).

G. M. Dixon, *Division Algebras, Lattices, Physics and Windmill Tilting* (ICG 2010)

G.M. Dixon, "(1,9)-Spacetime → (1,3)-Spacetime : Reduction ⇒ $U(1) \times SU(2) \times SU(3)$"; arXiv : hep-th/9902050.

G.M. Dixon, J. Math. Phys **45** , no 10 (2004) 3678.

[5] C. Furey, "Standard Model from an Algebra ? " (Ph.D thesis) arXiv : 1611.09182.

C. Furey, "$SU(3)_C \times SU(2)_L \times U(1)_Y (\times U(1)_X)$ as a symmetry of division algebraic ladder operators" Eur. Phys. J. C (2018) 78:375.

[6] Carlos Castro, "A Clifford $Cl(5, C)$ Unified Gauge Field Theory of Conformal Gravity, Maxwell and $U(4) \times U(4)$ Yang-Mills in $4D$" Advances in Applied Clifford Algebras **22**, no. 1 (2012), page 1-21

[7] J. E. Camargo-Molina, A. P. Morais, A. Ordell, R. Pasechnik, and J. Wessen, "Scale hierarchies, symmetry breaking and particle spectra in SU(3)-family extended SUSY trinification", arXiv : 1711.05199.

[8] P. Filiviez Perez and S. Ohmer, "Unification and Local Baryon Number" arXiv : 1612.07165

[9] S. Rajpoot and M. Singer, J. Phys. **G 5** : Nuc. Phys. no. 7 (1979) 871.

S. Rajpoot, Phys. Rev. **D 22**, no. 9 (1980) 2244.

J. Pati and A. Salam, Phys. Rev. Lett **31** (1973) 661; Phys. Rev. **D 8** (1973) 1240; Phys. Rev. **D 10** (1974) 275.

[10] Chiu Man Ho, Pham Q. Hung and T. W. Kephart, "Conformal Completion of the Standard Model with a Fourth Generation", JHEP 1206 (2012) 045.

A. Jourjine, "The Spectrum of the 4-Generation Dirac-Kaehler Extension of the SM" Phys. Lett. **B 695** (2011) 482

[11] M. Nakahara, *Geometry, Topology and Physics* (IOP Publishing, 1998).

[12] M. W. Kalinowski, "Fermion Fields in the (Non)Symmetric Kaluza-Klein Theory" , arXiv : 1512.04365.

M. W. Kalinowski, The Nonsymmetric Kaluza-Klein Theory and Modern Physics. A novel approach, Fortschritte der Physik 64, p. 190 (2016).

M. W. Kalinowski, Nonsymmetric Kaluza-Klein (Jordan-Thiry) Theory in a general nonabelian case, Int. J. Theor. Phys. 30, p. 281 (1991).

M. W. Kalinowski, Scalar fields in the Nonsymmetric Kaluza-Klein (Jordan-Thiry) Theory, arXiv: hep-th/0307242v10, 7 Jul 2015.

J. W. Moffat, Generalized theory of gravitation and its physical consequences, in: Proceeding of the VII International School of Gravitation and Cosmology. Erice, Sicilly, ed. by V. de Sabbata, World Scientific, Singapore, p. 127, 1982.

[13] C. Hull and U. Lindstrom, " The Generalized Complex Geometry of (p, q) Hermitian Geometries" arXiv: 1810.06489.

V. Marotta and R. Szabo, "Para-Hermitian Geometry, Dualities and Generalized Flux Backgrounds" arXiv : 1810. 03953.