

On the logical foundation of physics (part 1) - draft

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The aim of this work is, first, to set the basis for a formal model of science, then, it is to show that the laws of physics are its theorems. Necessarily, all theories that are the product of science are theorems of our model. Therefore, as we will argue, our model is the logical foundation of physics.

Within the context of prior work our model is best understood as an axiomatic realization of the participatory universe envisioned by John Archibald Wheeler³, in which the observer's practice of science brings about the reality he attempts to understand. As our model is constructive of a mathematical structure isomorphic to nature, it deprecates Karl Popper's definition of science based on falsifiability.

In our model, nature is understood as the "basket" which holds the proofs that all observations are formally verifiable (and thus reproducible). For the universal case, the "basket" is a mathematical structure isomorphic to the physical universe. In this case, the theory is epistemologically complete, fundamental, predictive, free of physical baggage (in the sense of Max Tegmark⁴), logically necessary, guaranteed to survive falsifiability, and provably cannot be improved by any scientific means.

³ John A Wheeler. Information, physics, quantum: The search for links. *Complexity, entropy, and the physics of information*, 8, 1990

⁴ Max Tegmark. *Our mathematical universe: My quest for the ultimate nature of reality*. Vintage, 2014

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1 Introduction

1.1 *Experimental facts, not axioms, have priority*

We intuitively understand scientific inquiry as a methodology to improve our understanding of the objective world. According to falsifiability, the evidence is to be collected with the intent to falsify hypotheses. Through an iterative process, ever more validated physical theories are produced, tested, and falsified. Confidence in a scientific theory is increased by actively attempting to falsify it (and failing to do so). The end goal of scientific inquiry is a final theory which would presumably explain all known and future experimental facts.

In practice, the inquiry process is usually divided into an experimental part and a theoretical part. Experimentalists gather experimental data, patterns are noticed within this data, and theoreticians formalize these patterns within the model of mathematics.

Whilst those involved understand the world through the practice of science, the theories so produced are, however, unaware of the process which created them. Indeed, each such formal theory is defined first and foremost as a set of axioms. Then, its theorems are the indubitable consequence of its axioms. Although falsifying a theory ought to be a scientist's primary motivation, this possibility of future falsification is, however, not derivable from the axioms of the theory alleged to be a correct description of reality.

This typical type of formal construction does not correspond to how the world is scientifically understood to be. First and foremost, scientists understand that experimental facts have priority and thus they overwrite any hypothesized set of axioms. Consequently, in

the framework of scientific inquiry and in practice, axioms are disposable, mutable, and interchangeable. Therefore, to be formally scientific, physical theories ought to be constructed from a "facts-implies-theory" basis rather than an "theory-implies-facts" basis.

Here, we present a framework able to produce formal physical theories in such a manner. The primary step will be to reverse the usual formalization of a typical physical theory. Instead of describing the theory (with axioms) and solving for a description of the facts (listing the theorems), we will first describe the facts, then solve for the theories that explain them. Reversing the problem in this way greatly reduces the difficulty and dissolves away a plurality philosophical problems in regards to physics, and even metaphysics.

1.2 *The facts imply the theory*

This strategy is key as it reverses the usual implication of a theory T with respect to set of experimental facts F . To better see the difference, let's compare it to a typical physical theory.

Typically in theoretical physics, T is hypothesized from experimental data and then tested. The axioms of T are presented as the primary actors of the theory, and they command the most attention. In most cases, the axioms directly represent the laws or symmetries of nature and are inspired from empirical data.

Using mathematics, the axioms of T can be unpacked into theorems. If all theorems of the theory are found to be an element of F , then the theory is effective, and it has not been falsified. However, if a mismatch is found, T is falsified and must be replaced with an alternative.

Due to this formulation, physical theories so produced will erroneously proclaim, on paper, that the facts are a consequence of the theory. Indeed, formulated as such, the facts are obtained by unpacking T into its theorems. This typical construction will semantically claim the following:

$$T \implies F \quad (\text{The theory implies the facts}) \quad (1)$$

The knowledge that the origin of T actually lies within F is understood in the minds of those who hypothesized T (as the consequence of scientific inquiry) but is absent from the formal description of T . Constructed as such, T is fundamental, and F is a mere consequence of it. Thus, T is unaware of its scientific origin.

This implication is reversed for the model presented in this work. Indeed, in this new model, the fundamental actors are now the elements of F , and the formal theory T that explains F is implied by

F . Thus, the relation is reversed, and the model semantically claims, correctly, that it is instead the facts that implies the theory:

$$F \implies T \quad (\text{The facts imply the theory}) \quad (2)$$

Backed by the results presented throughout this work, we will argue that the incorrect implication (i.e., the theory implies the facts) is the primary error in the way T is typically constructed. Once the relationship is reversed, solving for T is surprisingly simple. In our model, rather than guessing T via iterative falsifiability, T and its properties will be obtained as theorems of the model.

2 *Desiderata for a formal model of science*

To provide historical context, we review a number of desiderata that have either been suggested in the literature, and new ones that we would like to have in a formal model of science. This section introduces the key concepts used to produce our model and it eases entry to the next section where we list the axioms of the model.

Desideratum 1. *A formal model of science is necessarily more fundamental than any theory produced by science. Such a model should therefore serve as the logical foundation of physics.*

Let's explain why this is the case. First, we recall that a theory that is able to prove (or disprove) the axioms of another theory is considered more fundamental than the latter. In the case where the axioms are proven, the latter theory is said to be a theorem of the former.

In the case of scientific theories, the existence of a formal model of science (such as the one presented here) precludes their status as axiomatic theories and instead necessitate that they be theorems of our model. Indeed, this is unavoidable; a formal model of science implies that all theories that are the product of science be a theorem of it. Failure to be the case, this would mean that the formal model of science is simply insufficiently expressive.

Desideratum 2. *Since a formal model of science is more fundamental than any alternative physical theories, its starting points (axioms) should be elementary and, more importantly, free of physical baggage.*

Physical baggage, in the sense of (Tegmark [2014]) is an explicit reference to a physical entity within the axioms of the theory. For example, references to the mass, to time or to space, in classical mechanics, are its physical baggage. As these are referenced in the axioms, the theory is unable to explain the origin of such. Thus, by

virtue of being the most general scientific theory, the formal model of science must be completely free of physical baggage.

It is through the practice of science generally that we are lead to believe in an objective reality, where multiple observers appear to share the same information about the world. Thus, we would like, using the model, to prove the emergence of objective reality.

Desideratum 3. *A formal model of science should explain how the practice of science is able to bring about a provable objective reality, strictly using the tools of science. Concepts such as well-defined and reproducible experiments will be useful in that regards.*

This is aiming high, but if anything is up to the task, it will be science - as objective reality is its domain. It goes without saying that there are subtleties with such an approach. We will now introduce a series of hints to guide us towards the construction of the appropriate model. The first hint connects the practice of science to the emergence of an objective reality, in the sense given by John Archibald Wheeler.

2.1 *Hint 1: John Archibald Wheeler*

Here, we will directly quote John Archibald Wheeler and give an interpretation of his participatory universe hypothesis within the context of our work. We will argue that an axiomatic realization of John Archibald Wheeler's participatory universe hypothesis is a formal model of science. Suppose there exists an observer capable of practicing science. Such observer may, under experimental freedom, select which experiments to carry out. And thus this observer is able to have an impact in the World by the choice of experiments carried out.

We summarize John Archibald Wheeler's participatory universe hypothesis as follows. First, for any experiments, regardless of its simplicity or complexity, the registration of counts (in the form binary yes-or-no alternatives, the bit) is taken as a common book-keeping tool unifying the practice of science. Further to that, John Archibald Wheeler suggests (in the aphorism "it from bit") that what we consider to be the "it" is simply one out of many possible mixture of theoretical glue that binds the "bits" together. The "it" is, well, hypotheticalal, thus what is truly important is the bit. He states;

"It from bit symbolizes the idea that every item of the physical world has at bottom — at a very deep bottom, in most instances — an immaterial source and explanation; that what we call reality arises in the last analysis from the posing of yes-no questions and the registering of equipment-evoked responses; in short, that all things physical are information-theoretic in origin and this is a participatory universe"

Here, John Archibald Wheeler explicitly implies that the bit is the anchor. The bit comes into being in the final act, so to speak, and then constraints the possible "it"s, whose theoretical formulation must be consistent with all bits generated thus far. Furthermore, he mentions that the bit is registered following an equipment-evoked response.

To further illustrate, John A. Wheeler gives example of the theme of it from bit. One of which is the photon.

"With polarizer over the distant source and analyzer of polarization over the photodetector under watch, we ask the yes or no question, "Did the counter register a click during the specified second?" If yes, we often say, "A photon did it." We know perfectly well that the photon existed neither before the emission nor after the detection. However, we also have to recognize that any talk of the photon "existing" during the intermediate period is only a blown-up version of the raw fact, a count."

With this example John A. Wheeler restricts ontology to epistemology. For him, it makes little sense to speak of the photon existing (or not existing) until a detector registers a count. But he goes further and suggests that even after the registration of a count, deducing that the photon existed in between the counts is a "blown-up version of the raw fact, a count". Here, John A. Wheeler implies that the counts are what is real, not the theory that explains the counts. The theory is one hypothesis among many alternative and is, at best, a mathematical tool to make some sense of the counts, which by themselves define the world irrespective of the theory.

John A. Wheeler did not produce a formal theory of the participatory universe, because, I believe, of the following missing part:

Let's say that I were to provide you with a list of bits. Let's say I give you 111010110001001110101010101. How valuable is this information? Probably not much —can you guess why not? As a hint, imagine if I were to tell you that these bits represent the winning numbers of the next lottery draw. Then, all of sudden and although the sequence of bits stays the same, the bits are much more valuable.

John A. Wheeler does mention that the bits are the result of the registering of equipment-evoked responses. Although he spent considerable time developing the idea of the bits and their significance, he however, and to my knowledge, did not further developed the idea of the equipment-evoked response and how this ties, precisely, with the bits. To formalize his hypothesis one has to recognize that the bits only have meaning if they are associated to a well-specified experiment. Bits without context are meaningless.

Desideratum 4. *A formal model of science should be able to give context to each bits of information that can be acquired about the World. Consistent*

with the practice of science, this should be done by defining a well-specific experiment for each bit which ascribe a meaning to it.

Our model thus describe a World in which its substance is emergent from the contextualization of the bits of information that it contains. To make the idea precise, we must now find the proper way to contextualize this information while understanding that the main difficulty will be to do so in a manner free of physical baggage. I believe this was the primary roadblock encountered by John A. Wheeler: formalizing equipment-evoked response seems to require a physical description of said equipment, and as this would contain physical baggage, then the fundamentality of the theory would be compromised. The next hint will tell us how to do it without physical baggage.

2.2 Hint 2: Gregory Chaitin

As stated, John A. Wheeler's participatory universe hypothesis requires two parts to be made precise. The first part relates to the bits which counts the clicks on the detector as yes-no alternatives, and the second part relates to fleshing out a precise description of the experimental setups under which counts can be registered. Indeed, it is useless to register counts if one does not understand the setup under which these counts occur. An experiment can only be reproduced, and understood, given a full description of both parts: counts and setup.

In the case where we want our model to be fundamental, we must find a way to describe arbitrary experiments using no physical baggage. This is one of the main difficulty and it is at this point that Gregory Chaitin's ideas of a method of "describing mathematics as an assortment of halting experiments" enters the picture.

Gregory Chaitin summarizes his work on the halting probability (Chaitin [1975]) in the book *Meta Math!* (Chaitin [2004]). In it, he argues that the existence of a proof that Ω is algorithmically random implies that the set of all mathematical truths is not reducible to a finitely axiomatic system, and further reinforces the result greatly. Since, almost all practical work in mathematics is done using finitely axiomatic systems, this cause one's eyebrow to be raised. First let us quickly introduce what omega is:

$$\Omega = \sum_{p \text{ halts}} 2^{-|p|} \quad (3)$$

where $|p|$ denotes the length of p , an element of a prefix-free code. The Chaitin's construction (Ω , halting probability, Chaitin's constant)

is defined for a universal Turing machine as a sum over its domain (the set of programs that halts for it) where the term $2^{-|p|}$ acts as a special probability distribution which guarantees that the value of the sum, Ω , is between 0 and 1 (Kraft inequality). Knowing Ω is enough to know the programs that halt and those that do not for the universal Turing machine it is defined for. Since the halting problem is unsolvable, Ω must therefore be non-computable. In fact, Ω 's connection to the halting problem guarantees that it is algorithmically random, normal and incompressible.

The algorithmic randomness of Ω not only connects to the well-known incompleteness theorems of Gödel and Alan Turing, but also improves upon them. Further in the book, Gregory presents the strongest incompleteness theorem he claims to have produced in his career. He states:

"A finitely axiomatic system (FAS) can only determine as many bits of Ω as its complexity.

As we showed in Chapter V, there is (another) constant c such that a formal axiomatic system FAS with program-size complexity $H(FAS)$ can never determine more than $H(FAS) + c$ bits of the value for Ω .

These are theorems of the form "The 39th bit of Ω is 0" or "The 64th bit of Ω is 1". (This assumes that the FAS only enables you to prove such theorems if they are true.)"

To understand what Gregory is saying, we must understand Kolmogorov complexity. The Kolmogorov complexity of a binary string s is the length of the shortest program p which outputs s for some universal Turing machine. We say that the shortest program p is an elegant program for s .

With this, Gregory claims the following. 1. Any finitely axiomatic system FAS can be encoded as a string s . 2. The shortest program p which outputs this string s is the elegant representation of the theory. 3. Any finitely axiomatic system cannot prove more bits of Ω than the length of its elegant representation (plus a constant c dependant upon the choice of programming language). In relation to proving the bits of Ω , this places an intrinsic limit on the size the proof landscape of a theory can be based on how short its elegant representation is.

From this last theorem, Gregory concludes the following:

"I therefore believe that we cannot stick with a single finitely axiomatic system, as Hilbert wanted, we've got to keep adding new axioms, new rules of inference, or some other kind of new mathematical information to the foundations of our theory. And where can we get new stuff that cannot be deduced from what we already know? Well, I'm not sure, but I think that it may come from the same place that physicists get

their new equations: based on inspiration, imagination and on — in the case of math, computer, not laboratory-experiments."

Finally, Gregory Chaitin further suggests:

"So this is a "quasi-empirical" view of how to do mathematics, which is a term coined by Lakatos in an article in Thomas Tymoczko's interesting collection *New Directions in the Philosophy of Mathematics*. And this is closely connected with the idea of so-called "experimental mathematics", which uses computational evidence rather than conventional proof to "establish" new truths. This research methodology, whose benefits are argued for in a two-volume work by Borwein, Bailey and Girgensohn, may not only sometimes be extremely convenient, as they argue, but in fact it may sometimes even be absolutely necessary in order for mathematics to be able to progress in spite of the incompleteness phenomenon..."

At this point Gregory focuses his attention away from axioms and instead to focus on the bits of Ω as a general measuring device for the creative content of arbitrary finitely axiomatic systems. Gregory asks:

[Is] Ω concentrated creativity? [Is] each bit of Ω [equal to] one bit of creativity? Can human intellectual progress be measured by the number of bits of Ω that we know, or are currently capable of knowing, as a function of time?

Then Gregory intuits that a more creative mathematical theory ought to expand its axiomatic basis. He further states:

To develop a model of mathematics that is biological, that is, that evolves and develops, that's dynamic, not static. Perhaps a time-dependent formal axiomatic system?

The intuition of Gregory appears to be that a sufficiently complex system (mathematics, nature, life, etc) might require a progressively richer algorithmic landscape to be properly described. However, since all finitely axiomatic system can only prove n bits of Ω (finitely many), they all hit an algorithmic plateau at some point and are thus unsuitable to complete the task.

Here, we will burrow Gregory's intuition but we will flip it. If Gregory is right, and that the properties of Ω are conducive to a scientific approach to mathematics, perhaps the appropriate insight for our purposes is in the reverse; we instead seek to use Ω , its properties and insights, to formalize the practice of science using mathematics.

Mathematics is traditionally understood in the logical direction that axioms implies theorems. We suggest that science ought to be defined, formally, in the reverse direction: theorems implies axioms,

and that this can be made precise with the halting probability Ω . In this framework, we assume that we are provided with a universal Turing machine, that we are allowed to run programs on it, and that we use the halting behaviour of said programs to increase our knowledge in regards to the system (i.e., to formulate a theory T of halting experiments).

Using the halting probability makes it is therefore possible to construct a mathematical structure which is defined by an assortment of halting experiments whose results are encoded (in a maximally compressed manner) as the bits of Ω . The axioms of this structure would no longer be the primary focus of the theory, instead, the assortment of halting-experiments and its properties becomes the focus. Indeed, in this construction, the axioms would refer to a choice of Turing-complete programming language, and the theorems of interest would be provable up to a constant c which does depend upon the axioms, but otherwise have little to no impact.

Reprising John A. Wheeler's aphorism of "it from bit", and of "counts from equipment-evoked responses"; we can identify a pattern connecting the two ideas. The "it" becomes the choice of Turing complete programming language and it glues the bit together. The counts are produced when a program halts. Finally, the equipment-evoke responses are the programs p that are run on the universal Turing machine. As the programs are well-specific, reproducible and Turing complete, they act as the experiments of the system.

This reversal should be reminiscent of the introduction to this paper in which we argued against the axiom-to-fact formulation of physics, in favor of a fact-to-axioms approach, consistent with the actual practice of science.

Desideratum 5. *The formal model of science should be heavily inspired by Chaitin's construction. Each concept required for the definition and estimation of Ω , should be mapped to an analogous scientific concept. The model therefore burrows the mathematical rigor of the halting probability, and its properties, to define the practice of science.*

In this context, the practice of science consists of estimating Ω by executing arbitrary programs, within some priors, on a universal Turing machine and to note those that halt. At some point we might stop and we would obtain a falsifiable (but not provable) estimation of Ω .

We might now wonder what connection to the physical world this purely mathematical structure has. For this, we need our next hint.

2.3 Hint 3: Alan Turing

A function u is universal iff for all functions f , there exists an input e such that for all inputs x , $u(e, x) = f(x)$. Or, as a first order logic sentence:

$$\forall f \exists e [\forall x [u(e, x) = f(x)]] \quad (4)$$

This definition of universality was given by (Rogers and Rogers [1967]) and it is a generalization to the concept of the universal Turing machine introduced by (Turing [1937]). A theory which can constructively prove the existence of such u is Turing complete.

Desideratum 6. *To convince us that the practice of science is closed on the informational content of the World, experiments must be Turing complete.*

The concept of universality is epistemologically interesting. It is the reason why halting experiments and physical experiments are equivalent in some informational sense. Indeed, because of universality, physical experiments can prove any halting experiments and vice-versa. Our ability to construct a physical computer, for instance, allows us to verify the proof of many theorems (ex., the four color theorem), but it also allows us to simulate a physical system. In practice the computer is memory limited, but in principle, there are no memory thresholds beyond which we expect universality to fail in nature. Thus, universality is assumed.

Essentially, the argument we put forward is that two universal structures, whether they are constructed using language, a physical substance, or are purely mathematical, etc., will share a common epistemological core. However, because universality is closed on epistemology, and infinitely many constructions are universal, ontology will be censored. Let us clarify what we mean by this censorship with our next hint.

2.4 Hint 4: Ludwig Wittgenstein

By virtue of universality, universal structures share a common epistemological core but their plurality censors the ontology. To clarify what we mean by this, it helps to introduce Ludwig Wittgenstein's work on the limits of language.

In the *Tractatus Logico-Philosophicus* (1927), Ludwig Wittgenstein lists in a total of seven somewhat enigmatic propositions regarding what he considers are the proper definitions of the world, its properties and the consequences of such. The aim of the *Tractatus* is, in a nutshell, to imply that there is a limit to sensible thoughts (e.g., a limit to the subject of formal language) and that this limit is, in

some sense, the limits of the world. His proof relies on constructing an isomorphism between language, thought and the world. Then as stated by (Biletzki and Matar [2018]), "since language, thought and the world, are all isomorphic, any attempt to say in logic (i.e., in language) "this and this there is in the world, that there is not" is doomed to be a failure, since it would mean that logic has got outside the limits of the world, i.e. of itself."

We will study Ludwig Wittgenstein's work in the narrow context in which it is applicable for a paper primarily concerned with physics. Our first interest will be on how he constructed the isomorphism that he used to support his thesis. Indeed, we would advise that anyone who attempts to produce a mathematical structure isomorphic to the physical world ought to pay close attention to Wittgenstein's work. Finally, we will explain how this leads to ontological censorship.

The propositions listed by Wittgenstein are sub-divided into hundreds of sub-propositions with the intend of clarifying or adding to the main proposition. It is within these sub-propositions that we find the appropriate definitions to construct said isomorphism.

Before we begin, I should warn the reader to expect a permissive description of Wittgenstein's work. Specifically, I should state that I have derived the definitions necessary for the isomorphism independently of the Tractatus, and only post-facto realized the similarities. Thus my goal here is to show that my definitions have been anticipated, to provide a analogous take on the subject and to use the Tractatus as a clarification tool (humor intended) for my own definitions. As a result, we will employ a modernized terminology to discuss Wittgenstein's work which is more appropriate in the context of physics.

To scope the discussion, let us propose two hypothesis roughly summarizing Wittgenstein's work and conclusions, using more modern terminology, and discuss them in more detail. For the first hypothesis, we present a strong and a weak version.

1. An epistemological closure hypothesis:

- (a) (weak) Two universal structures are epistemologically equivalent. This includes, the World, mathematics, science (as defined in this work), Turing-complete languages, etc.
- (b) (strong) The practice of science is closed on the informational content of the World.

For the weak version of the hypothesis, the practice of science defined by Karl Popper (i.e., falsifiability) is appropriate. In a nutshell; inspired by experimentation, we guess a mathematical structure that

we believe to be isomorphic to the physical world, then we attempt to falsify it. If the structure is falsified, we find a new one. Rinse and repeat until cosmological heat death.

For the strong version of the hypothesis, we are provided with a methodology to construct the appropriate mathematical structure. Said methodology is simply the practice of science formalized as a model of mathematics. This latter version is suggestive of a participatory universe in which science proves the real world as it is practiced. The completion of this practice produces a universal mathematical structure isomorphic to the World. The concept of falsifiability is not applicable in this case as the method is constructive of the appropriate isomorphic mathematical structure (i.e., there is no longer a need to guess it).

We now give the second hypothesis. It clarifies the limits of the first one (both versions).

2. An ontological censorship hypothesis: The ontology of the World cannot be revealed by language, or mathematics (or even science).

The practice of science, formalized in our model, imposes only that whatever the "substrate of reality" is, it be Turing complete. This imposition requires the practice to be universal, and since universality is epidemiologically complete, we are confronted with ontological ambiguity. For instance, one could argue that reality is language, or that reality is a computer simulation, or a universal mathematical structure. Indeed, there are infinitely many candidate Turing-complete structures which would be up to the task. As the practice of science is unable to decide which of these structure is the actual one, the ontology of the World is thus censored, ergo, the ontological censorship hypothesis. The choice of ontological theory, provided that the theory is Turing complete, is thus an aesthetic and we will call them *interpretations* (or as Wittgenstein would no doubt eloquently put it; non-sense).

For the purposes of introducing additional desiderata, let us now recall the relevant sub-propositions of the Tractatus that Wittgenstein used to construct the isomorphism between the World and language. Please note that as the terminology of the Tractatus is somewhat dated and inappropriate for physics we will be adopt a very permissive interpretation of the sub-propositions. Our intention here is to show that the definitions we will soon adopt for our formal model of science have been anticipated before, and to increase the clarity by introducing them from more than one perspective.

Specifically, in proposition 1, Wittgenstein states:

"

1. The world is all that is the case.
 - 1.1. The world is the totality of facts, not of things.
 - 1.1.1. The world is determined by the facts, and by their being all the facts.
 - 1.1.2. For the totality of facts determines what is the case, and also whatever is not the case.
 - 1.1.3. The facts in logical space are the world.
 - 1.2. The world divides into facts.
 - 1.2.1. Each item can be the case or not the case while everything else remains the same.
- [...]

"

According to Wittgenstein, the atomic component of the world is not atoms, nor quarks, nor even superstrings, but facts. According to Wittgenstein, we can understand reality with four main ideas: (1) facts, (2) what is the case, (3) the state of affairs and (4) substance.

1. First, we consider the set of all possible facts. But not all facts are actual. For instance, it is a fact that "if a unicorn falls off a cliff and gravity exists, it will splatter on the ground". However, this fact is not actual because one or more of the conditions of the proposition are not met in real life (i.e., gravity exists, but for whatever reasons, there are no unicorns). If one fact is actual, then possibly infinitely many proposition which refers to this fact are also implied. Thus, a complicated web of propositions can emerge immediately from a single fact becoming actual.
 - (a) Because all facts are conditional, then some facts are actual (they are the case) and other facts are not.
 - (b) The state of affairs is defined as the set of all facts that are actual.
2. Secondly, Wittgenstein observes that nothing in logic would dictate which fact is actual (a proposition is a fact iff it is conditional, but what sets the conditions?) and which fact isn't, thus he postulates that the actuality of a fact is random (i.e. not dictated by logic).
3. Thirdly, we reason that the world appears somewhat stable—in the sense that it is not just white noise over the random permutation of actual facts. The sun will hypothetically rise tomorrow with high degree of confidence despite the randomness of the actuality of facts. If the actuality of facts is random, why is there this appearance of stability? To resolve this, Wittgenstein introduces the notion of substance. For Wittgenstein, substance is what persist independently of a change of actual facts.

4. Finally, Wittgenstein concludes that the substance is what nature is, and facts are how we picture nature as.

Wittgenstein dedicates over two hundred sub-propositions for the purposes of explicitly defining a proposition that can be considered a proper fact, and to discuss their properties. His main concern is to show that propositions are sufficiently general to account for all possible states of the world. His secondary concern is to guarantee that each proposition only includes references to a state of the world (and no subjective references for instance). In modern terminology, his concept of a proposition is simply a formal computer program which halts on certain input (e.g., the conditions of the proposition are met), or otherwise does not halt (e.g., the conditions are not met). Recall that the *Tractatus* was published in 1927, ten years before Alan Turing and four years before Godel's incompleteness theorems.

To illustrate, let us recall the example of the unicorn that we have just used. We have taken many shortcuts with this example. For this proposition to truly be a fact in the sense of Wittgenstein, it would require a lot more precision. To imply splattering, the material strength, the structure of whatever the unicorn is made of and the dynamics of such matter has to be embedded with the proposition. The proposition must necessarily follow from the conditions should they be true. It must be infallible. This is why defining such propositions as a computer program is so convenient. Indeed, computer programs must be readable by some universal Turing machine and thus, the behavior of the program is fully specified and only depends upon the input, which in this context is taken as the conditions of the proposition.

In our model, a program that halts, or a physical experiment that holds, is what we consider a fact to be. Since this is universal, it is appropriate to call it a fact in the Wittgenstein's sense. Thus, consistent with Wittgenstein's ideas,

Desideratum 7. *A formal model of science should define the World as the set of all facts.*

and

Desideratum 8. *A formal model of science should define a fact as an (1) experiment holding or terminating (in the physical interpretation) and as a (2) program halting (in the mathematical interpretation).*

In regards to nature (called substance in the *Tractatus*),

Desideratum 9. *A formal model of science should define nature using a terminology (and properties) applicable to all Turing complete structures.*

With proposition 2.24, Wittgenstein defines nature as what subsists independently of the facts of the world. A better way to formulate this (in the sense that it is more conducive to formalization) is to claim that facts implies a certain nature. For instance, we say that experiments implies a nature (with certain resources) to verify the experiment. Whatever resources must be consumed to cause an experiment to terminate are added to the nature in which the experiment is verified. For instance, a large set of difficult experiments, thus imply a correspondingly large nature.

Desideratum 10. *Each fact implies a nature required for its verification.*

Thus, the nature is the "substance" that verifies the contextualization of each bit of information about the World as an assortment for well-specific and reproducible protocols. Now that we have a modest idea of how the model we will recover the idea of a persistent substance, we will now specifically discuss how the laws of physics are emergent from it. Is the random statistics of actual facts sufficient to bound this substance to the laws of physics? For this, we need our next hint.

2.5 Hint 5: Claude Elwood Shannon

The Shannon entropy (Shannon [1948]) is interpreted in the context of receiving a message x (usually a binary string) out of a set of possible messages M . If the message is expected to be selected under some probability distribution $p(x)$, then the entropy of the statistical ensemble corresponds to the information gained by receiving one message from the set. This entropy is defined as

$$H = - \sum_{x \in M} p(x) \ln p(x) \quad (5)$$

The Shannon entropy provides us a way to make Wittgenstein's world mathematically precise. Indeed, to start off:

Desideratum 11. *The set of facts that are actual can be interpreted, in the Shannon sense, as an information-bearing "message/string" about the actual World whose facts are randomly selected from the set of all possible facts.*

Let us expand on this more with two examples.

1. For the first example, let us define the set of all possible World as containing a single element. Then there is only one possible world; the one which is actual. We can perhaps image a model in which a creator wills one and only one world into existence, such that nothing is left to chance.

Intuitively, we understand that this interpretation is minimally informative. But why is that the case? According to Shannon's entropy, the actual world, interpreted as a message selected from the set of all possible facts, is in fact the only possible world. Thus, the entropy of the ensemble is 0, and knowing it versus what is possible provides exactly zero intellectual insight.

2. At the other end of the spectrum, we instead suppose that the set of possible facts is epistemologically complete in the sense given in the previous hint. As the facts of the actual world are randomly selected from this set, the "message" is thus maximally informative. Indeed, to make the "message" more informative, one would have to make the set more epistemologically complete than it already is—an impossible task.

It is in this scenario that we will find the laws of nature emergent from the model. Thus, these laws are, in our model, provably maximally informative in regards to the World. As we will find, the laws of nature are in fact those that no possible World can violate, and that this is the reason our World follows them.

The final part necessary for the laws of nature to emerge is to recall Wittgenstein's definition of substance; as that which persists independently of what is the case. In our model, to characterize this substance, we first use the set of actual facts to set the size of nature required to verify it. Then, taking nature as the prior we maximize the entropy over the possible facts. Finally, we obtain a maximally entropic ensemble over the facts verifiable within a persistent substance. Lets give a more concrete example.

Suppose that the actual world is given by a set of facts \bar{P} which is a proper subset of the set of possible facts. We define the substance \bar{N} as that which verifies \bar{P} . One question we could ask would be how many possible Worlds can be constructed (and verified) using this substance, and that would be an interesting question. But a better way is to reformulate the question is, what is the maximum amount of information we can gain by knowing which world is actual within this substance? The answer is the same as the second example above, except now augmented with a persistent substance in the sense of Wittgenstein.

Then, the reader who recalls his or her introductory classes to statistical physics will no doubt recognize this description as a simple system of statistical physics (in disguise). The nature required to verify the context of the message (i.e., by running each associated programs to termination), is taken as the priors. Finally, maximizing the entropy within these priors yields the laws of nature as an equation of state. We are now ready to enter the more technical parts of

the paper and make the idea precise.

Part I

Formal model of science

3 *Technical introduction*

3.1 *Statistical physics*

We will provide a brief recap of statistical physics. In statistical physics, we are interested in the distribution that maximizes entropy,

$$S = -k_B \sum_{x \in X} p(x) \ln p(x) \quad (6)$$

subject to the fixed macroscopic quantities. The solution for this is the Gibbs ensemble. Typical thermodynamic quantities are:

<i>quantity</i>	<i>name</i>	<i>units</i>	<i>type</i>	
$T = 1/(k_B\beta)$	temperature	K	intensive	(7)

E	energy	J	extensive	(8)
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$p = \gamma/\beta$	pressure	J/m ³	intensive	(9)
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V	volume	m ³	extensive	(10)
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$\mu = \delta/\beta$	chemical potential	J/kg	intensive	(11)
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N	number of particles	kg	extensive	(12)
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Taking these quantities as examples, the partition function becomes:

$$\begin{aligned} & \textit{Gibbs ensemble} \\ Z &= \sum_{x \in X} e^{-\beta E(x) - \gamma V(x) - \delta N(x)} \end{aligned} \quad (13)$$

The probability of occupation of a micro-state is:

$$\begin{aligned} & \textit{Gibbs measure} \\ p(x) &= \frac{1}{Z} e^{-\beta E(x) - \gamma V(x) - \delta N(x)} \end{aligned} \quad (14)$$

The average values and their variance for the quantities are:

$$\bar{E} = \sum_{x \in X} p(x)E(x) \quad \bar{E} = \frac{-\partial \ln Z}{\partial \beta} \quad \overline{(\Delta E)^2} = \frac{\partial^2 \ln Z}{\partial \beta^2} \quad (15)$$

$$\bar{V} = \sum_{x \in X} p(x)V(x) \quad \bar{V} = \frac{-\partial \ln Z}{\partial \gamma} \quad \overline{(\Delta V)^2} = \frac{\partial^2 \ln Z}{\partial \gamma^2} \quad (16)$$

$$\bar{N} = \sum_{x \in X} p(x)N(x) \quad \bar{N} = \frac{-\partial \ln Z}{\partial \delta} \quad \overline{(\Delta N)^2} = \frac{\partial^2 \ln Z}{\partial \delta^2} \quad (17)$$

The laws of thermodynamics can be recovered by taking the following derivatives

$$\left. \frac{\partial S}{\partial \bar{E}} \right|_{V,N} = \frac{1}{T} \quad \left. \frac{\partial S}{\partial \bar{V}} \right|_{E,N} = \frac{p}{T} \quad \left. \frac{\partial S}{\partial \bar{N}} \right|_{E,V} = -\frac{\mu}{T} \quad (18)$$

, which can be summarized as

$$d\bar{E} = TdS - pd\bar{V} + \mu d\bar{N} \quad (19)$$

This is known as the equation of states of the thermodynamic system. The entropy can be recovered from the partition function and is given by:

$$S = k_B (\ln Z + \beta \bar{E} + \gamma \bar{V} + \delta \bar{N}) \quad (20)$$

3.2 Algorithmic thermodynamics

Many authors (Bennett et al. [1998], Chaitin [1975], Fredkin and Toffoli [1982], Kolmogorov [1965], Zvonkin and Levin [1970], Solomonoff [1964], Szilard [1964], Tadaki [2002, 2008]) have discussed the similarity between physical entropy $S = -k_B \sum p_i \ln p_i$ and the entropy in information theory $S = -\sum p_i \log_2 p_i$. Furthermore, the similarity between the halting probability Ω and the Gibbs ensemble of statistical physics has also been studied (Li and Vitanyi [2008], Calude and Stay [2006], Baez and Stay [2012], Tadaki [2002]). Tadaki suggests to augment Ω with a multiplication constant D , which acts as a decompression term on Ω .

$$\text{Chaitin construction} \quad \rightarrow \quad \text{Tadaki ensemble} \quad (21)$$

$$\Omega = \sum_{q \in \text{halts}} 2^{-|q|} \quad \rightarrow \quad \Omega_D = \sum_{q \in \text{halts}} 2^{-D|q|} \quad (22)$$

With this change, the Gibbs ensemble compares to the Tadaki ensemble as follows;

$$\begin{array}{ll}
 \text{Gibbs ensemble} & \text{Tadaki ensemble} \\
 Z = \sum_{x \in X} e^{-\beta E(x)} & \Omega_D = \sum_{q \in \text{halts}} 2^{-D|q|} \quad (23)
 \end{array}$$

Interpreted as a Gibbs ensemble, the Tadaki construction forms a statistical ensemble where each program corresponds to one of its micro-state. The Tadaki ensemble admits a single quantity; the prefix code length $|q|$ conjugated with D . As a result, it describes the partition function of a system, which maximizes the entropy subject to the constraint that the average length of the codes is some constant $\overline{|q|}$;

$$\overline{|q|} = \sum_{q \in \text{halts}} |q| 2^{-|q|} \quad \text{from 15} \quad (24)$$

The entropy of the Tadaki ensemble corresponds to the average length of prefix-free codes available to encode programs.

$$S = k_B \left(\ln \Omega + D \overline{|q|} \ln 2 \right) \quad \text{from 20} \quad (25)$$

The constant $\ln 2$ comes from the base 2 of the halting probability function instead of base e of the Gibbs ensemble.

John C. Baez and Mike Stay (Baez and Stay [2012]) take the analogy further by suggesting an interpretation of algorithmic information theory based on thermodynamics, where the characteristics of programs are considered to be thermodynamic quantities. Starting from Gregory Chaitin's Ω number, the Chaitin construction

$$\Omega = \sum_{q \in \text{halts}} 2^{-|q|} \quad (26)$$

is extended with algorithmic quantities to obtain

$$\begin{array}{ll}
 \text{Gibbs ensemble} & \text{Baez-Stay ensemble} \quad (27)
 \end{array}$$

$$Z = \sum_{x \in X} e^{-\beta E(x) - \gamma V(x) - \mu N(x)} \quad \Omega' = \sum_{q \in \text{halts}} 2^{-\beta E(q) - \gamma V(q) - \delta N(q)} \quad (28)$$

Noting the similarity between the Gibbs ensemble of statistical physics (13) and (28), these authors suggest an interpretation where E is the expected value of the logarithm of the program's runtime, V is the expected value of the length of the program, and N is the expected value of the program's output. Furthermore, they interpret the conjugate variables as (quoted verbatim from their paper);

"

1. $T = 1/\beta$ is the *algorithmic temperature* (analogous to temperature). Roughly speaking, this counts how many times you must double the runtime in order to double the number of programs in the ensemble while holding their mean length and output fixed.
2. $p = \gamma/\beta$ is the *algorithmic pressure* (analogous to pressure). This measures the trade-off between runtime and length. Roughly speaking, it counts how much you need to decrease the mean length to increase the mean log runtime by a specified amount while holding the number of programs in the ensemble and their mean output fixed.
3. $\mu = -\delta/\beta$ is the *algorithmic potential* (analogous to chemical potential). Roughly speaking, this counts how much the mean log runtime increases when you increase the mean output while holding the number of programs in the ensemble and their mean length fixed.

"

–John C. Baez and Mike Stay

From equation (28), they derive analogs of Maxwell’s relations and consider thermodynamic cycles, such as the Carnot cycle or Stoddard cycle. For this, they introduce the concepts of *algorithmic heat* and *algorithmic work*. Other authors have suggested other alternative mappings in other but related contexts (Li and Vitanyi [2008], Tadaki [2008]).

3.3 Feasible Mathematics

In an previous article (Harvey-Tremblay [2017]), I suggested a framework for feasible mathematics based on algorithmic thermodynamics. What is feasible mathematics?

Feasible mathematics is an alternative take on the familiar notions computational complexity theory (CT) such that it better connects to nature. Let us first recall what CT is. CT is the study of the inherent difficulty of computational problems; as such, CT classifies problems by the increase in difficulty associated with an increase in input size. For example, a binary search algorithm will have a difficulty of $O(\log n)$, and thus it has a logarithmic complexity. Indeed, the number of steps required to find an item from n sorted items grows proportionally to the logarithm of n .

Why bother with an alternative take on the subject? CT does not correctly distinguish between all indicators of complexity. As an example, the difficulty between, say, an exponential problem with a small multiplication constant $O(2^n) \times 0.001$ and a polynomial

problem with a large multiplication constant $O(n^2) \times 10^{99999999}$ is bizarrely classified. As far as CT is concerned, the latter problem is much simpler than the first as it only grows in n^2 versus 2^n . However, in practice, the latter might never be solved because there might not be enough resources in the observable universe to do so (even for $n = 1$). Therefore, although this idea is very interesting, something is missing from CT to truly connect it to nature. This is where feasible mathematics comes in.

Some research has been done in the area of feasible numbers (a close cousin of feasible mathematics). Perhaps the most promising is from Vladimir Yu. Sazonov's paper on feasible numbers (Sazonov [1995]). He suggests that feasible numbers are intuitively a set of numbers F which satisfies $0 \in F, F + 1 \subseteq F$ and $2^{1000} \notin F$. Then, he investigates various constructions which he claims allow the consistent treatment of such sets by imposing various restrictions on the expression of proofs. For example; limiting the quantity of symbols allowed in a proof, and/or requiring the encoding for numbers to be in unary, etc. Using these restrictions, he claims to have a system which guarantees that all steps of the proof are feasible (below a certain complexity bound).

The framework for feasible mathematics introduced here takes a different approach. It recognizes that 2^{1000} is a large number but nonetheless, it can be compressed to a short representation. Thus, it accepts that theorems featuring this number can be proven even in the context of limited resources. Feasible mathematics proposes a method to treat feasibility as a limit applicable to proof complexity based on limited available computing resources, as opposed to by axiomatically restricting the language the proof is expressed in.

Feasible mathematics is defined using various "meta-indicators" of complexity that can be associated with any proofs independently from the power of expression of the language of the proof. By bounding proofs based on such indicators, the proof landscape available to a mathematician with limited resources is reduced (made feasible) whilst the expressive power of the language of the proofs remains untouched. As a result, we believe that a representation of mathematical feasibility based on limited computing resources more accurately describe the practical notion of feasibility. As the framework of feasible mathematics is applicable to an arbitrary set of formal axioms, we introduce a distinction between *feasible mathematics* and *universal mathematics*. Universal mathematics is made feasible when, intuitively, the proof landscape of the mathematician is bounded by computational limits. In this sense, all practical work in mathematics is feasible. The main result of the framework is a relation defining the boundary between feasible and universal mathematics for some given

set of computing resources.

The domain of feasible mathematics is defined as an extension to the halting probability Ω of computer science. Using a similar construction, we define a probability Z that represents the probability that a random program will halt within some available computing resources. These resources can be time, memory, clock speed, etc. Z does for feasible mathematics what Ω does for "universal mathematics". Interestingly, when the computing limits are made to vanish, Z will converge to Ω , and thus universal mathematics will be recovered in the regime of unbounded computing resources.

To make this concrete, let us introduce the following scenario as the typical problem of the field. Suppose a research group with access to a supercomputer. Alice has been granted a fixed amount of computing resources to use on the supercomputer. She has further been instructed to run a program q_A . With no prior knowledge of q_A , what is the probability that the program will halt within the allocated resources?

To make feasible mathematics precise, we will consider mathematical proofs as computer programs that are executed on a universal Turing machine. We will then construct a statistical ensemble able to define the boundary between feasible mathematics and universal mathematics. As a first example, we introduce into Ω_D the quantity $t(q)$ (the program-runtime) along with its conjugate P (the halting-power) and we obtain the construction Z .

$$Z = \sum_{q \in \text{halts}} 2^{-Pt(q) - Fx(q)} \quad (29)$$

where

$$Z \in \mathbb{R}_{\geq 0} \quad \text{numerical value of the sum} \quad (30)$$

$$t(q) : q \rightarrow \mathbb{N} \quad \text{number of iterations required for } q \text{ to halt} \quad (31)$$

$$P \in \mathbb{R} \quad \text{conjugate to } t(q) \text{ in units of } (\textit{iterations})^{-1} \quad (32)$$

$$x(q) : q \rightarrow \mathbb{N} \quad \text{number of bits of the size of } q \quad (33)$$

$$F \in \mathbb{R} \quad \text{conjugate to } x(q) \text{ in units of } (\textit{bits})^{-1} \quad (34)$$

The corresponding probability measure is:

$$p(q, P, F) = \frac{1}{Z} 2^{-Pt(q) - Fx(q)} \quad (35)$$

It maximizes the entropy subject to the following constraints:

$$\bar{t} = \sum_{p \in \text{halts}} p(q, P, F) t(q) \quad \text{average program-runtime } \bar{t} \quad (36)$$

$$\bar{x} = \sum_{q \in \text{halts}} p(q, P, F) x(q) \quad \text{average program-size } \bar{x} \quad (37)$$

Each Lagrange multiplier of the partition function Z is a computing resource (in this case: P and F) that must be provided by the supercomputer to fix their respective conjugated mean quantities (\bar{t} and \bar{x}). The resources are interpreted as follows:

Resource	Variable	Conjugate	Function
halting-power	P	program-runtime	$t(q)$ (38)

halting-force	F	program-size	$x(q)$ (39)
---------------	-----	--------------	-------------

- The halting-power counts how many times the mean program-runtime must be doubled in order to double the entropy of the ensemble while holding the mean program-size fixed.
- The halting-force counts how many times the mean program-size must be doubled in order to double the entropy of the ensemble while holding the mean program-runtime fixed.

In the supercomputer analogy, the halting-power can be understood as fixing to the mean time at which programs are made to terminate, and the halting-force as fixing the density of halting programs versus non-halting programs within the ensemble.

There exists alternative constructions of Z such that other resources are fixed by the supercomputer.

Halting-action to program-frequency formulation:

$$Z' = \sum_{q \in \text{halts}} 2^{-Sf(q) - Fx(q)} \quad (40)$$

To formulate this relation, we must introduce the program-frequency $f(q)$ as the inverse of the runtime $t(q)$, thus $f(q) := 1/t(q)$. This formulation fixes a mean frequency \bar{f} by having the supercomputer provide a constant halting-action to the system:

Resource	Variable	Conjugate	Function
halting-action	S	program-frequency	$f(q)$ (41)

halting-force	F	program-size	$x(q)$ (42)
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- The halting-action counts how many times the mean program-frequency must be doubled in order to double the entropy of the ensemble while holding the mean program-size fixed.

Halting-time to program-power formulation:

$$Z'' = \sum_{q \in \text{halts}} 2^{-tP(q) - Fx(q)} \quad (43)$$

This formulation fixes a mean program-power \bar{P} by having the supercomputer provide a constant halting-time to the system:

Resource	Variable	Conjugate	Function
halting-time	t	program-power	$P(q)$ (44)

halting-force	F	program-size	$x(q)$ (45)
---------------	-----	--------------	-------------

- The halting-time counts how many times the mean program-power must be doubled in order to double the entropy in the ensemble while holding the mean program-size fixed.

This formulation describes a system where all programs halt at the same time. To guarantee this behavior, the supercomputer must adjust the computation power on a per program basis $P(q)$.

Time-cutoff formulation:

$$Z''' = \sum_{q \in \text{halts}; t(q) \leq k} 2^{-Fx(q)} \quad (46)$$

The sum Z''' is done only on programs that halt within a time cutoff k . Thus, Z''' contains no "non-halting information" and is computable. Ω is recovered in the limit when $k \rightarrow \infty$.

Size-cutoff formulation:

$$Z'''' = \sum_{q \in \text{halts}; |q| \leq k} 2^{-Fx(q)} \quad (47)$$

The sum Z'''' is performed only on programs with sizes less or equal to k . Ω is recovered in the limit when $k \rightarrow \infty$. Z'''' represents the first n bits Ω up to the cutoff k .

Computational-complexity formulation:

$$Z'''' = \sum_{q \in \text{halts}; O(q) = \log n} 2^{-Fx(q)} \quad (48)$$

The sum only includes programs which provably halts on $\log n$ complexity. (Well this part is slightly embarrassing: I never said feasible mathematics makes computational complexity theory simpler, just that it can recover it as a special case.)

Interpretation via supercomputing reservoirs:

In the context of feasible mathematics, we interpret the supercomputer as taking a similar role to the role taken by the various baths in thermodynamics (heat bath, particle bath, etc). For example, in thermodynamics we would say that a system which can exchange

energy with its environment is in contact with a heat bath. Its temperature will be constant but its total energy would fluctuate as it is exchanged with the bath. By analogy, in feasible mathematics, we would imagine that a computation occurs in a supercomputer which schedule priority, assigns memory, etc. so has to maintain various mean program quantities fixed during the calculation.

The typical Gibbs ensemble in physics is $Z(\beta) = \sum_i e^{-\beta E_i}$. Its average energy is given by $\bar{E} = -\partial \ln Z / \partial \beta$ whose fluctuations are given by $\overline{(\Delta E)^2} = \partial^2 \ln Z / \partial \beta^2$. From these relations, the ensemble can be informally illustrated as a system in contact with a thermal reservoir. In this case, both the system and the reservoir have the same temperature and they can exchange energy. The reservoir is considered large enough that the fluctuations of the smaller system are negligible to its description —mathematically, it has infinite heat capacity. Thus, the reservoir abstractly represents an infinitely deep pool of energy at a given, constant temperature.

In the context of feasible mathematics, supercomputers are interpretable using a similar analogy. The role of the heat baths are replaced with compute baths. Each exchange with the compute baths will produce resources that are then available for the purposes of advancing the computation. The intensity of the exchanges are calibrated by the Lagrange multipliers of the ensemble. For instance, instead of a heat bath, we have a runtime-bath and a tape-bath (associated with the tape of a UTM). The intensity of the exchanges between the system and the runtime-bath are calibrated by halting-power, and those with the tape-bath are calibrated by the halting-force. The reservoirs have mathematically infinite runtime and tape capacities, and thus acts as infinitely deep pools of computing resources. Computing is made possible by the interaction of the reservoirs with the system, and the boundary of the feasibility landscape is calibrated by the halting-power and the halting-force provided to the system by the supercomputer.

4 Axioms of science

4.1 Foundation

Definition 1 (Experiment). *An experiment p is a well-specified protocol TM (formalized as a Turing machine) and an hypothesis h . An experiment holds iff $TM(h)$ is defined. In this case $TM(h)$ produces the result r and $TM(h) = r$. Otherwise, the experiment fails and $TM(h)$ is undefined. Specifically,*

$$TM(h) = \begin{cases} r & [p \text{ holds}] \\ \# & [p \text{ fails}] \end{cases} \quad (49)$$

Definition 2 (World). *The world \overline{W} is the set of all experiments that holds.*

$$\forall p [p \text{ holds} \implies p \in \overline{W}] \quad (50)$$

A formal theory is usually defined as set of first order sentences and its domain as the set of all sentences it is able to prove. However, this definition is not sufficiently reproducible for our purposes. Indeed, given a formal theory, there usually exists arbitrarily many different ways of proving a theorem. To eliminate this ambiguous behavior, we thus instead define a formal theory as follows.

Definition 3 (Theory). *Suppose a set T of sentences $\{s_1, s_2, \dots\}$. Then \overline{T} is a theory for T iff it is a Turing machine that halts on s iff $s \in T$.*

$$\forall s [\overline{T}(s) \text{ halts} \iff s \in T] \quad (51)$$

As the Turing machine is deterministic, each sentence of T admits a single proof, realized by causing \overline{T} to halt on it when taken as input. In this case, we say that \overline{T} proves p iff $\overline{T}(p)$ halts.

4.2 Properties of scientific theories

From these initial definitions, we can list properties of theories (where p is an experiment).

Definition 4 (Effective). A theory $\overline{\mathbb{T}}$ is effective in $\overline{\mathbb{W}}$ iff:

$$\exists p[(\overline{\mathbb{T}} \vdash p) \wedge (p \in \overline{\mathbb{W}})] \quad (52)$$

Definition 5 (Falsified). A theory $\overline{\mathbb{T}}$ is falsified in $\overline{\mathbb{W}}$ iff

$$\exists p[(\overline{\mathbb{T}} \vdash p) \wedge \neg(p \in \overline{\mathbb{W}})] \quad (53)$$

Definition 6 (Complete). A theory $\overline{\mathbb{T}}$ is complete in $\overline{\mathbb{W}}$ iff

$$\forall p[(p \in \overline{\mathbb{W}}) \implies (\overline{\mathbb{T}} \vdash p)] \quad (54)$$

Definition 7 (Sound). A theory $\overline{\mathbb{T}}$ is sound in $\overline{\mathbb{W}}$ iff

$$\forall p[(\overline{\mathbb{T}} \vdash p) \implies (p \in \overline{\mathbb{W}})] \quad (55)$$

4.3 Nature

Definition 8 (Nature). The nature $\overline{\mathbb{N}}$ implied by a theory $\overline{\mathbb{T}}$ is defined for an experiment p as the set of all computing resources required to prove p in $\overline{\mathbb{T}}$. We suppose a function g , defined as

$$g : p, \overline{\mathbb{T}} \rightarrow \overline{\mathbb{N}} \quad (56)$$

The function g takes p and $\overline{\mathbb{T}}$ as inputs then returns the computing resources required for $\overline{\mathbb{T}}$ to prove p .

In Theorem 1, we prove the existence of such a function.

4.4 Practice of science

Definition 9 (Participation). A set $\overline{\mathbb{P}}$ is a participation set iff all of its elements are experiments and its carnality is less than infinity:

$$\forall p[p \in \overline{\mathbb{P}} \implies p \in \overline{\mathbb{W}}] \quad (57)$$

$$|\overline{\mathbb{P}}| < \infty \quad (58)$$

4.5 Observer

To introduce the notion of an observer, we must first explain what a primitive notion is. A primitive notion is a concept that is not formal but is nonetheless used in a formal theory. A theory has to start somewhere. For instance, let's take set theory. As Mary Tiles says in *The Philosophy of Set Theory*; "[The] 'definition' of 'set' is less a definition than an attempt at explication of something which is being given the status of a primitive, undefined, term". Furthermore, quoting Felix Hausdorff; "A set is formed by the grouping together of single objects into a whole. A set is a plurality thought of as a unit.". Finally, Bertrand Russell considered these notions in general and called them "indefinables of mathematics". Here, our primitive notion is that of an observer. An observer is understood by its relationship with the definitions and laws that we introduce.

Primitive Notion 1 (Observer). *Denoted by the symbol \mathcal{O} .*

Axiom 1 (Participatory-observer). *All observers are associated a participation set. To represent the relationship, we define a predicate P . P is true iff \bar{P} is the participation of \mathcal{O} in \bar{W} . Then,*

$$\forall \mathcal{O} [\exists \bar{P} [P(\mathcal{O}, \bar{P})]] \quad (59)$$

Axiom 2 (Existence of the observer). *We brutally claim the existence of \mathcal{O} .*

$$\exists \mathcal{O} \quad (60)$$

4.6 Laws of nature

Definition 10 (Participatory-invariance). *All properties of nature that remain unchanged by participation are laws of nature: ergo, the laws of nature are participatory-invariant. These laws are interpreted in the sense that no act of participation (e.g., no experiments, or groups of experiment) can violate them.*

Participatory-invariance embodies the idea that the observer can participate (e.g., make changes in nature) but that any well-defined participation imply certain restrictions. In the case where the observer participates via the practice of science, those restrictions are the laws of nature.

Participatory-invariance is obtained by formulating an ensemble of feasible mathematics over the set of all possible experiments. To formulate it, we first take the participatory set $\overline{\mathbb{P}}$ associated with \mathcal{O} , then determine the required nature to verify its contextualization.

Finally, taking nature as the prior, we then adopt the principle of maximum entropy to the participation of the observer within the priors. Under these conditions, the restrictions are valid for all participatory sets and thus, are the laws of nature (Definition 11).

Applying participatory-invariance to nature will be done explicitly in theorem (2).

Definition 11 (Laws of nature). *A function $f(\overline{\mathbb{N}}, \overline{\mathbb{P}}) = 0$, called an equation of the state of nature, is a law of nature iff it is emergent from participatory-invariance over $\overline{\mathbb{N}}$ and for all $\overline{\mathbb{P}}$ whose contextualization is verifiable in $\overline{\mathbb{N}}$.*

4.7 Meta-theory

To discuss the subject matter and formulate proofs, we will adopt a sufficiently expressive, ideally Turing complete, formal language. Naturally, we adopt set theory for the task at hand.

Axiom 3. *We adopt the axioms of first-order logic with equality and the axioms of ZF set theory.*

5 *Theses*

Thesis 1 (Central Thesis). *The nature required to do science is, under participatory-invariance, a mathematical structure isomorphic to the (real) physical nature, and the laws of nature are the laws of physics.*

Thesis 2 (Secondary Thesis). *The definition of nature survives the "Cartesian argument", and is therefore presented as a theory of reality. Nature is that which is required for the practice of science (verifying experiments), and thus its existence cannot be refuted by such.*

6 *Foundational theorems of science*

Theorem 1. *There exists a function g that returns the running time and the input size of p iff p is provable in $\overline{\mathbb{T}}$.*

Proof. Non-constructively, we imagine a function g , which can simulate any Turing machine (i.e. it embeds a universal Turing machine). g takes as input a program p and a theory $\overline{\mathbb{T}}$. It then uses the utm to run p on $\overline{\mathbb{T}}$. For each iteration of the utm, g increments a counter by one. Once and if the execution terminates, the function returns the value of the counter along with the size of the input $|p|$. The resources are then the running time $t(p)$ (the value of the counter) and the input-size $r(p)$ required to bring p to termination. \square

Theorem 2 (Main Result). *Here, we take the participation set $\overline{\mathbb{P}}$ associated with an observer \mathcal{O} , as a "message" in the Shannon sense about the actual World taken from the set of all possible Worlds. The message, along with its contextualization, implies a nature. Finally, taking the nature as the prior and maximizing the entropy over the allowable participatory sets yields the laws of nature as an equation of state. This process will be done explicitly here and yields the main result:*

The participatory-invariant ensemble of experiments for a nature \bar{N} defined for the priors \bar{t} and \bar{r} is:

$$Z = \sum_{p \text{ halts}} \exp(\lambda_t t(p) - \lambda_r r(p)) \quad (61)$$

and its equation of state is:

$$dS = -\lambda_t d\bar{t} + \lambda_r d\bar{r} \quad (62)$$

Preliminaries. For consistency with the units of physics, we use the Boltzmann definition of entropy (similar to Shannon's definition but with an extra constant k_B), and we use the natural unit of information e , instead of the bit 2. In addition, as we are working with a purely isomorphic mathematical structure, we will prefix all units with the word *iso*. For instance, the running time of a program will have the units *iso-seconds* (denoting a quantity of iterations) and the length of program p will have the units of *iso-meters* (denoting a quantity of bits), etc. Adding this prefix to all units makes the isomorphism explicit, whilst also reminding us that we are dealing with an isomorphic structure and not necessarily the physical thing itself. Consistent with the isomorphism, we can then interpret our equations as either generally isomorphic (with the prefix), or as a model of the physical (by removing the prefix). The only thing that ought to change between the isomorphic interpretation and the physical interpretation is the removal of the *iso* prefix to the units. Let us now prove the foundational theorems, then in part (II) we will explain how we intend to support the applicability of the isomorphism as a model of the (real) physical nature.

□

Proof. Under the principle of maximum entropy, we seek the probability distribution $\rho: Q \rightarrow \{p \in \mathbb{R} | 0 \leq p \leq 1\}$ and $\sum_{q \in Q} \rho(q) = 1$ which maximizes the entropy S :

$$S = -k_B \sum_{q \in Q} \rho(q) \ln \rho(q) \quad (63)$$

and subject to the priors \bar{t} and \bar{r} (given by function g)

$$\bar{t} = \sum_{q \in Q} \rho(q) t(q) \quad (64)$$

$$\bar{r} = \sum_{q \in Q} \rho(q) r(q) \quad (65)$$

We maximize the entropy using the well-known method of the Lagrange multipliers.

$$\begin{aligned} \mathcal{L} = & \left(-k_B \sum_{q \in Q} \rho(q) \ln \rho(q) \right) + \lambda_0 \left(\sum_{q \in Q} \rho(q) - 1 \right) \\ & + \lambda_1 \left(\sum_{q \in Q} \rho(q) t(q) - \bar{t} \right) + \lambda_2 \left(\sum_{q \in Q} \rho(q) r(q) - \bar{r} \right) \end{aligned} \quad (66)$$

Maximizing \mathcal{L} with respect to $\rho(q)$ is done by taking its derivative and posing it equal to zero:

$$\frac{\partial \mathcal{L}}{\partial \rho(q)} = -k_B \ln \rho(q) - k_B + \lambda_0 + \lambda_1 t(q) + \lambda_2 r(q) = 0 \quad (67)$$

Solving for $\rho(q)$ we obtain:

$$\rho(q) = \exp \left(\frac{k_B + \lambda_0 + \lambda_1 t(q) + \lambda_2 r(q)}{k_B} \right) \quad (68)$$

From the constraint $1 = \sum_{q \in Q} \rho(q)$, we can find the value for λ_0 :

$$1 = \sum_{q \in Q} \rho(q) \quad (69)$$

$$= \sum_{q \in Q} \exp \left(\frac{-k_B + \lambda_0 + \lambda_1 t(q) + \lambda_2 r(q)}{k_B} \right) \quad (70)$$

$$= \exp \left(\frac{-k_B + \lambda_0}{k_B} \right) \sum_{q \in Q} \exp (\lambda_1 t(q) + \lambda_2 r(q)) \quad (71)$$

We define the partition function Z to be

$$Z := \sum_{q \in Q} \exp (\lambda_1 t(q) + \lambda_2 r(q)) \quad (72)$$

Then, we rewrite $\rho(q)$ using Z , we pose $\lambda_1 := 1/t_0$ and $\lambda_2 := -1/r_0$ and we obtain the probability distribution:

$$\rho(q) = \frac{1}{Z} \exp \left(\frac{1}{t_0} t(q) - \frac{1}{r_0} r(q) \right) \quad (73)$$

where $1/t_0$ with units $[1/(\text{iso-seconds})]$ and $-1/r_0$ with units $[1/(\text{iso-meters})]$ are the Lagrange multipliers (a justification for the choice of signs will be provided after the results in section ??). Finally, we obtain the equation of state:

$$dS = -\frac{k_B}{t_0}d\bar{t} + \frac{k_B}{r_0}d\bar{r} \quad (74)$$

□

Definition 12 (Domain of epistemology). *This work adopts the epistemological theory of Infallibilism. As such we consider philosopher Richard L. Kirkham's suggestion that to qualify as knowledge, the justification of a belief must necessitate its truth (Gettier [1963], Kirkham [1984]). Consistent with this suggestion, we formally define the domain of epistemology as the set of all halting programs (defined up to a universal Turing machine).*

Then,

Theorem 3. *Given enough resources, the practice of science is, in the limit, epistemologically complete. To prove this, we show that the partition function Z converges towards Ω_D , the Tadaki ensemble, when $\bar{t} \rightarrow \infty$.*

$$\lim_{\bar{t} \rightarrow \infty} Z \rightarrow \Omega_D \quad (75)$$

Proof. First, we rewrite Ω_D as:

$$\Omega_D = \sum_{i=1}^{\infty} 2^{-H(q_i) - D|q_i|} \quad \text{where } H(q_i) := \begin{cases} 0 & q_i \text{ halts} \\ \infty & \text{otherwise} \end{cases} \quad (76)$$

Second, we note that the runtime $t(q_i)$ of a program q_i will be finite if it halts and infinite otherwise.

$$t(q_i) = \begin{cases} t_i \in \mathbb{R}_{\geq 0} & q_i \text{ halts} \\ \infty & \text{otherwise} \end{cases} \quad (77)$$

Then taking the limit of Z ,

$$\lim_{\mathcal{P} \rightarrow \infty} \frac{1}{\mathcal{P}} t(q_i) = \begin{cases} 0 & q_i \text{ halts} \\ \infty & \text{otherwise} \end{cases} \quad (78)$$

This is the definition of $H(q_i)$. Therefore,

$$\lim_{\mathcal{P} \rightarrow \infty} \frac{1}{\mathcal{P}} t(q_i) \rightarrow H(q_i) \quad (79)$$

Thus,

$$\lim_{\mathcal{P} \rightarrow \infty} Z \rightarrow \Omega_D \quad (80)$$

□

Theorem 4. *Z monotonically converges towards Ω_D as the available resources are increased.*

Proof. Without loss of generality, let us now expand Z explicitly with an example. Assume a system comprised of three micro-states with prefix code-length $|q_1| = 1$, $|q_2| = 2$ and $|q_3| = 3$ and with the running times $t_1 = 5$, $t_2 = \infty$ and $t_3 = 5$. In this example, q_1 and q_2 halt and q_3 does not. For the purposes of simplicity, we can assume that all other programs do not halt. In this case, the system is not universal but let us nonetheless use it as a simplified numerical example. The sum Z becomes;

$$Z(W) = 2^{-1+5W} + 2^{-2+\infty W} + 2^{-3+5W} \quad (81)$$

We will now produce a series of numerical calculations with progressively smaller values of W and we will look at the evolution of the error rate $\xi(W) = \Omega - Z(W)$. For this system, $\Omega = 0.1010\bar{1}_b$.

$(-W)$	$Z(W)$	$\xi(W)$	error	
∞	0	Ω	max	(82)
1	0.000000101... _b	0.10011011 _b	$\approx 2^{-1}$	(83)
0.1	0.011100010... _b	0.00101110... _b	$\approx 2^{-3}$	(84)
0.01	0.100110101... _b	0.00000010... _b	$\approx 2^{-6}$	(85)
0.001	0.011100010... _b	0.00000000... _b	$\approx 2^{-9}$	(86)
\vdots	\vdots	\vdots	\vdots	
0	Ω	0	none	(87)

As we can see, increasing the halting-power ($\mathcal{P} = -1/W$) causes the value Z to monotonically converges towards Ω . The error rate decreases as more valid bits of Ω are obtained. □

Theorem 5. *An observer knowing n bits of Z will be able to decide at most 2^N programs.*

Proof. We consider a numerical value for Z whose first k bits correspond to the bits of Ω . We look at two cases: 1) For the first k bits, Z (as with Ω) can decide 2^N programs per bit. 2) For the bits after k, the situation is a bit more complex:

To recover the feasible programs beyond k , an observer can execute programs on a universal Turing machine in dovetail. As they halt, the observer adds their contribution to Z . Once the value of Z is recovered, then all programs taking longer to halt are beyond the feasible bound, regardless of whether they ultimately halt or not.

□

With theorem (3), (4) and (5), we conclude that the practice of science increases knowledge as it is practiced, and in the limit recovers all epistemological knowledge (under the epistemological theory of infallibility).

Part II

Central Thesis (a): Emergence of time, space and causality

To be continued in the next paper.

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