Real Numbers in the Neighborhood of Infinity

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Abstract

We demonstrate the existence of a broad class of real numbers which are not elements of any number field: those in the neighborhood of infinity. After considering the reals and the affinely extended reals, we prove that numbers in the neighborhood of infinity are ordinary real numbers. We show that real numbers in the neighborhood of infinity obey the Archimedes property of real numbers. As an application in complex analysis, we show that the Riemann zeta function has infinitely many non-trivial zeros off the critical line.

§1 Introduction

It is the popular theme in modern mathematics to define $\mathbb{R}$ by an algebraic approach but here we will define $\mathbb{R}$ by a geometric approach based on the Euclid magnitude [1]. In Euclid's approach, a real number is the length of a line segment. Presently and following Euclid, we will not define $\mathbb{R}$ through its algebraic operations as in the axioms of a complete ordered field. Motivating the current approach, Pugh writes the following in Reference [2].

> "The current teaching trend treats the real number system $\mathbb{R}$ as a given—it is defined axiomatically. Ten or so of its properties are listed, called axioms of a complete ordered field, and the game becomes: deduce its other properties from the axioms. This is something of a fraud, considering that the entire structure of analysis is built on the real number system. For what if a system satisfying the axioms failed to exist? Then one would be studying the empty set!"

Although the Euclid magnitude was the definition of $\mathbb{R}$ throughout most of the past two millennia, modern mathematicians have chosen to adopt, for the time being at least, the currently trendy field axioms. Through such axioms, it is claimed that $\mathbb{R}$ is such that the set

$$\mathcal{R} = \{\mathbb{R}, +, \times\} \ ,$$

satisfies all of the field axioms. Here we will define $\mathbb{R}$ as a cut in the real number line (a line with the label "real number line" attached), and $\mathbb{R}$ will be such that $\mathcal{R}$ does not universally satisfy the field axioms. Nothing is lost

in the present definition, however, because there shall exist $\mathbb{R}_0 \subset \mathbb{R}$ such that the set

$$\mathcal{R}' = \{\mathbb{R}_0, +, \times\} \quad,$$

does satisfy all of the field axioms. Indeed, $\mathcal{R}'$ shall be exactly identical to what is usually called the real number field. By taking the more general approach through a geometric definition of $\mathbb{R}$, we do not lose any of the power of the field axioms because $\mathbb{R}$ contains a subset $\mathbb{R}_0$ (called real numbers in the neighborhood of the origin) which do 100% of the work done by the real number field when it is defined as in $\mathcal{R}$. Nothing at all is lost in this geometric definition of $\mathbb{R}$ but, as we will show, very much is gained when we define $\mathbb{R}$ geometrically as opposed to through its algebraic operations $+$ and $\times$. That which is gained shall be called real numbers in the neighborhood of infinity: $\mathbb{R} \setminus \mathbb{R}_0$. As we will show, the field axioms preclude the existence of numbers in the neighborhood of infinity but the existence of these numbers is very much implied by the historical a-real-number-is-a-cut-in-an-infinite-line approach to $\mathbb{R}$. As Pugh states in the above excerpt from Reference [2], our only burden is to show that there does exist a non-empty set which satisfies the definition. Since our definition is vastly simpler than the ordered field definition, this task is accomplished easily. The main hurdle will be to show that numbers in the neighborhood of infinity satisfy the Archimedes property of real numbers which we will concede as an axiomatic requirement for any valid definition of $\mathbb{R}$.

In the next section of this paper, we will give the geometric definition of $x \in \mathbb{R}$ as a cut in a number line. In the third section, we will discuss the affinely extended real number line which is the real line together with its endpoints at infinity. This is the so-called two point compactification of $\mathbb{R}$. In the fourth section, we give an axiomatic definition for infinity and we never require at any point that $\infty \in \mathbb{R}$. Infinity is not a real number! After developing infinity, we define real numbers in the neighborhood of infinity. We give their properties and show that such numbers, as presently defined, do not satisfy the axioms of a complete ordered field. It is precisely this property which makes the geometric definition of $\mathbb{R}$ better than the algebraic definition; the geometric definition contains all the numbers admitted by the algebraic definition—numbers in the neighborhood of the origin—but also admits another class of numbers in the neighborhood of infinity whose properties are exciting, interesting, and non-trivial. In the sixth section, we examine the axiomatized arithmetic operations bestowed upon numbers in the neighborhood of infinity throughout the previous sections. In the seventh section, we make a direct extension from $\mathbb{R}$ to $\mathbb{C}$. In the final section, we will discuss the Archimedes property of real numbers and show that the Riemann ζ function has infinitely many non-trivial zeros off the critical line. All of these zeros are in the neighborhood of infinity.

§2 Real Numbers

Definition 2.1 The real numbers are defined in interval notation as

$$\mathbb{R} \quad \equiv \quad (-\infty, \infty) \quad,$$

where the interval $(-\infty, \infty)$ is an infinite line.

Definition 2.2 A cut in a line $x \in \mathbb{R}$ separates one line into two pieces as

$$\mathbb{R} \setminus x = (-\infty, x) \cup (x, \infty) \quad.$$

Definition 2.3 A real number $x \in \mathbb{R}$ is a cut in the real number line.

Remark 2.4 A number is a cut in a line. A line is defined a priori. All lines can be cut so all lines are number lines. A given line is the real line by definition. A real number separates the real number line into a set of "larger" real numbers and a set of "smaller" real numbers.

Definition 2.5 Call real numbers in the neighborhood of the origin $x \in \mathbb{R}_0$. Define them such that

$$\mathbb{R}_0 \quad \equiv \quad \{ \, x \mid (\exists n \in \mathbb{N})[-n < x < n] \, \} \quad.$$

Here we define $\mathbb{R}_0$ as the set of all x such that there exists an $n \in \mathbb{N}$ allowing us to write $-n < x < n$.

Definition 2.6 Call real numbers in the neighborhood of infinity $x \in \mathbb{R}_\infty$. Define them as all real numbers except for real numbers in the neighborhood of the origin:

$$\mathbb{R}_\infty \quad \equiv \quad \mathbb{R} \setminus \mathbb{R}_0 \quad.$$

Remark 2.7 The main result of this paper demonstrates that $\mathbb{R}_\infty$ is not the empty set.

Definition 2.8 For $x \in \mathbb{R}$, we have the property

$$\lim_{x \to 0^\pm} \frac{1}{x} = \text{diverges} \quad, \qquad \text{and} \qquad \lim_{n \to \infty} \sum_{k=1}^{n} k = \text{diverges} \quad.$$

§3   Affinely Extended Real Numbers

Definition 3.1 Define two affinely extended real numbers $\pm\infty$ such that for $x \in \mathbb{R}$

$$\lim_{x \to 0^{\pm}} \frac{1}{x} = \pm\infty \quad , \qquad \text{and} \qquad \lim_{n \to \infty} \sum_{k=1}^{n} k = \infty \quad .$$

Definition 3.2 The set of all affinely extended real numbers is

$$\overline{\mathbb{R}} \quad \equiv \quad \mathbb{R} \cup \{\pm\infty\} \quad .$$

Definition 3.3 The affinely extended real numbers are defined in interval notation as

$$\overline{\mathbb{R}} \quad \equiv \quad [-\infty, \infty] \quad .$$

Definition 3.4 An affinely extended real number $x \in \overline{\mathbb{R}}$ is $\pm\infty$ or it is a cut in the affinely extended real number line.

Definition 3.5 In $\overline{\mathbb{R}}$, $\pm\infty$ are such that the limit of any monotonic sequence of real numbers which diverges in $\mathbb{R}$ is equal to ∞ or $-\infty$.

Theorem 3.6 *If $x \in \overline{\mathbb{R}}$ and $x \neq \pm\infty$, then $x \in \mathbb{R}$.*

Proof. Proof follows from Definition 3.2.   ✐

Axiom 3.7 Infinity is such that

$$\infty - \infty = \text{undefined} \quad , \qquad \text{and} \qquad \frac{\infty}{\infty} = \text{undefined} \quad .$$

Axiom 3.8 Infinity does not have the distributive property of multiplication.

§4   Infinity Hat

Definition 4.1 Additive absorption is a property of ∞ such that non-zero $\mathbb{R}_0$ numbers are additive identities of $\pm\infty$. The additive absorptive property is

$$\pm\infty + x = x \pm \infty = \pm\infty \quad , \qquad \text{for} \qquad x \in \mathbb{R}_0 \setminus 0 \quad .$$

Remark 4.2 The arithmetic operations we will assign to real numbers in the neighborhood of infinity will require the removal of the zero additive identity from infinity.

Definition 4.3 Let the symbol $\widehat{\infty}$ be called "infinity hat" and endow it with every property of ∞ except additive absorption. $\pm\widehat{\infty}$ are explicitly two different infinities such that

$$-\left(\pm\widehat{\infty}\right) = \mp\widehat{\infty} \quad .$$

Axiom 4.4 Infinity and infinity hat both describe the same affinely extended real number, *i.e.*: $\|\widehat{\infty}\| = \infty$.

Remark 4.5 Although the operations of $\widehat{\infty}$ differ from those of ∞, we say they are the same number because the present treatment identifies unique numbers with unique points along the extended real number line. The operations do not contribute to or appear in the definition of the number.

Definition 4.6 If an expression using $\widehat{\infty}$ causes a contradiction via additive non-absorptivity, then the hat must be removed to alleviate the contradiction. This property requiring removal of the hat in certain instances is called the non-contradiction property of $\widehat{\infty}$.

Axiom 4.7 The hat which differentiates infinity hat $\widehat{\infty}$ from canonical infinity ∞ is inserted and removed by choice except in the case where it invokes a contradiction and must be removed by definition.

Example 4.8 An example of a statement in which the hat does not invoke a contradiction and may be left in place is

$$x = \widehat{\infty} - b \quad .$$

Example 4.9 An example of a statement in which the hat invokes a contradiction and may not be left in place is given by two sequences

$$x_n = \sum_{k=1}^{n} k \quad , \qquad \text{and} \qquad y_n = c_0 + \sum_{k=1}^{n} k \quad ,$$

where c_0 is some non-zero real number. Since ∞ and $\widehat{\infty}$ are the same number (Axiom 4.4), we can use Definition 3.1 to write

$$\lim_{n\to\infty} x_n = \infty = \widehat{\infty} \quad , \qquad \text{and} \qquad \lim_{n\to\infty} y_n = \infty = \widehat{\infty} \quad .$$

We may also write, however,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} c_0 + \lim_{n\to\infty} x_n = c_0 + \widehat{\infty} \quad .$$

This delivers an equality

$$\widehat{\infty} = c_0 + \widehat{\infty} \quad ,$$

which contradicts the additive non-absorption of $\widehat{\infty}$. At this point, we must obey the non-contradiction property of $\widehat{\infty}$ (Definition 4.6) by removing the hat. Then

$$\infty = c_0 + \infty \ \ ,$$

demonstrates the usual additive absorptive property of infinity and there is no contradiction.

Axiom 4.10 $\widehat{\infty}$ is such that for any non-zero $b \in \mathbb{R}_0$

$$\pm\widehat{\infty} + b = b \pm \widehat{\infty}$$
$$\pm\widehat{\infty} - b = -b \pm \widehat{\infty}$$
$$\pm\widehat{\infty} + \left(-b\right) = \pm\widehat{\infty} - b$$
$$\pm\widehat{\infty} + b = \pm\widehat{\infty} - \left(-b\right)$$
$$-\left(\pm\widehat{\infty}\right) = \mp\widehat{\infty}$$
$$\pm\widehat{\infty} \cdot b = b \cdot \pm\widehat{\infty} = \begin{cases} \pm\widehat{\infty} & \text{if} \quad b > 0 \\ \mp\widehat{\infty} & \text{if} \quad b < 0 \end{cases}$$
$$\frac{\pm\widehat{\infty}}{b} = \begin{cases} \pm\widehat{\infty} & \text{if} \quad b > 0 \\ \mp\widehat{\infty} & \text{if} \quad b < 0 \end{cases}$$
$$\frac{b}{\pm\widehat{\infty}} = 0 \ \ .$$

Axiom 4.11 $\widehat{\infty}$ is such that

$$\pm\widehat{\infty} + 0 = 0 \pm \widehat{\infty} = \pm\widehat{\infty} - 0 = \text{undefined}$$
$$\pm\widehat{\infty} \cdot 0 = 0 \cdot \pm\widehat{\infty} = \text{undefined}$$
$$\frac{\pm\widehat{\infty}}{0} = \text{undefined}$$
$$\frac{0}{\pm\widehat{\infty}} = 0 \ \ .$$

Remark 4.12 We will revisit the lack of zero as an additive identity element for $\widehat{\infty}$ in Example 5.12. We will show that it is required to remove the zero additive identity from infinity if we want to add the freedom to add and subtract real numbers in the neighborhood of infinity.

Remark 4.13 When the ∞ symbol appears as $\widehat{\infty}$, we consider the hat to be an instruction to delay the additive absorption of ∞ indefinitely or until such a delay causes a contradiction. The instruction to "delay additive absorption" should be understood to mean that additive absorption is not a property of $\widehat{\infty}$

but that the additive absorptive property can be implemented trivially after an *ad hoc* decision to remove the hat by choice, or after its removal is required by the non-contradiction property.

Infinity without the hat is afforded with some freedom to do the algebraic operations of its expressions in different orders, and there is no requirement in mathematics that every expression must be simplified as much as possible. Since there is no *a priori* requirement for us to immediately execute the additive absorptive operation in all cases, we add absolutely nothing to infinity with the hat. Rather, the hat is simply a superior alternative to a declaration, "Remember not to simplify this expression via the additive absorptive operation which does not necessarily have to be completed immediately." Instead, we will put the hat there as a reminder not to do that operation while the hat is in place.

We have not added anything to infinity with the hat; the hat is merely an instruction about how to use the algebraic freedom which already exists in the order of operations. While we have not added anything to infinity, we have added something to mathematics. Peeking ahead to Axiom 5.10, an ideal example of that which is gained through the hat is

$$\left(\widehat{\infty} - b\right) - \left(\widehat{\infty} - a\right) = a - b \ \ .$$

After delaying the absorptive operation on the left, there is no infinity remaining on the right into which we might absorb. Such statements are not possible without the axioms of $\widehat{\mathbb{R}}$ (given in the following section.)

§5   Real Numbers in the Neighborhood of Infinity

Definition 5.1 The set of large real numbers in the neighborhood of infinity is
$$\widehat{\mathbb{R}} \quad \equiv \quad \left\{ \pm\left(\widehat{\infty} - b\right) \mid b \in \mathbb{R}_0, \ b > 0 \right\} \ \ .$$

Remark 5.2 We call $\widehat{\mathbb{R}}$ large numbers in the neighborhood of infinity to distinguish them from all numbers in the neighborhood of infinity: $\mathbb{R}_\infty \equiv \mathbb{R} \setminus \mathbb{R}_0$. As we show in Reference [3], it is not the case that $\mathbb{R}_\infty \setminus \widehat{\mathbb{R}} \equiv \emptyset$.

Axiom 5.3 The ordering of $\widehat{\mathbb{R}}$ numbers is

$$
\begin{aligned}
\pm\left(\widehat{\infty} - b\right) &= \pm\left(\widehat{\infty} - a\right) & &\Longleftrightarrow & &a = b \\
\left(\widehat{\infty} - b\right) &> \left(\widehat{\infty} - a\right) & &\Longleftrightarrow & &a > b \\
-\left(\widehat{\infty} - b\right) &> -\left(\widehat{\infty} - a\right) & &\Longleftrightarrow & &a < b \\
\left(\widehat{\infty} - b\right) &> -\left(\widehat{\infty} - a\right) & &\forall & &b, a \in \mathbb{R}_0 \\
\left(\widehat{\infty} - b\right) &> x & &\forall & &b, x \in \mathbb{R}_0
\end{aligned}
$$

$$-\left(\widehat{\infty} - b\right) < x \qquad\qquad \forall \qquad\qquad b, x \in \mathbb{R}_0$$
$$\left(\widehat{\infty} - b\right) < \infty \qquad\qquad \forall \qquad\qquad b \in \mathbb{R}_0$$
$$-\left(\widehat{\infty} - b\right) > -\infty \qquad\qquad \forall \qquad\qquad b \in \mathbb{R}_0 \quad .$$

Theorem 5.4 $\widehat{\mathbb{R}}$ *is a subset of* $\overline{\mathbb{R}}$*.*

Proof. Definition 3.2 gives $\overline{\mathbb{R}} \equiv [-\infty, \infty]$. By Definition 3.4, an affinely extended real number $x \in \overline{\mathbb{R}}$ is a cut in or endpoint of the affinely extended real number line. Definition 2.2 requires that a cut separates one line into two pieces. Observe that

$$\overline{\mathbb{R}} \setminus \left(\widehat{\infty} - b\right) \quad \equiv \quad [-\infty, \widehat{\infty} - b) \cup (\widehat{\infty} - b, \infty]$$
$$\overline{\mathbb{R}} \setminus \left(-\widehat{\infty} + b\right) \quad \equiv \quad [-\infty, -\widehat{\infty} + b) \cup (-\widehat{\infty} + b, \infty] \quad .$$

All $x \in \widehat{\mathbb{R}}$ conform to the definition of affinely extended real numbers so $\widehat{\mathbb{R}} \subset \overline{\mathbb{R}}$. ✑

Main Theorem 5.5 $\widehat{\mathbb{R}}$ *is a subset of* $\mathbb{R}$*.*

Proof. If a number is an affinely extended real number $x \in \overline{\mathbb{R}}$ and $x \neq \pm\infty$, then, by Theorem 3.6, we have $x \in \mathbb{R}$. Theorem 5.4 proves that

$$x \in \widehat{\mathbb{R}} \quad \implies \quad x \in \overline{\mathbb{R}} \quad .$$

In the absence of additive absorption

$$\pm\left(\widehat{\infty} - b\right) \neq \pm\widehat{\infty} = \pm\infty \quad .$$

(Definition 5.1 requires $b \neq 0$.) This proves the theorem.

Alternatively, by the ordering of Axiom 5.3, we have

$$\mathbb{R} \setminus \left(\widehat{\infty} - b\right) \quad \equiv \quad (-\infty, \widehat{\infty} - b) \cup (\widehat{\infty} - b, \infty) \quad .$$

All numbers $x \in \widehat{\mathbb{R}}$ satisfy the definition of $x \in \mathbb{R}$ through Definitions 2.2 and 2.3. This also proves the theorem. ✑

Theorem 5.6 $\widehat{\mathbb{R}}$ *is a subset of* $\mathbb{R}_\infty$*.*

Proof. We have shown in Main Theorem 5.5 that

$$\widehat{\mathbb{R}} \subset \mathbb{R} \quad ,$$

so we will satisfy the definition of $\mathbb{R}_\infty$ (Definition 2.6) if we show that

$$\widehat{\mathbb{R}} \cap \mathbb{R}_0 \quad \equiv \quad \emptyset \quad .$$

Definition 2.5 requires that elements of $\mathbb{R}_0$ satisfy

$$-n < x < n \quad ,$$

For proof by contradiction, assume $\pm\big(\widehat{\infty} - b\big) \in \mathbb{R}_0$. This requires

$$-n < \pm\big(\widehat{\infty} - b\big) < n \quad .$$

Since $b \in \mathbb{R}_0$, we know it has an additive inverse. Add or subtract b to obtain

$$-n + b < \widehat{\infty} < n + b \quad , \qquad \text{and} \qquad -n - b < -\widehat{\infty} < n - b \quad .$$

We obtain a contradiction as $\widehat{\infty}$ cannot be less than the sum of two $\mathbb{R}_0$ numbers and $-\widehat{\infty}$ cannot be greater than the difference of two $\mathbb{R}_0$ numbers. All $\widehat{\mathbb{R}}$ numbers satisfy the definition of $\mathbb{R}_\infty$ (Definition 2.6.)   ▱

Remark 5.7 The remainder of this section defines and makes remarks on the arithmetic operations for $\widehat{\mathbb{R}}$ numbers, and we will also treat these operations in the following section. The purpose in defining operations for $\widehat{\mathbb{R}}$ is to supplement the canonical operations for $\mathbb{R}_0$ and $\infty \sim \widehat{\infty}$. (The canonical operations of $\mathbb{R}_0$ are those given when $\mathcal{R}' = \{\mathbb{R}_0, +, \times\}$ satisfies the field axioms.) Every $\widehat{\mathbb{R}}$ number can be decomposed and its pieces manipulated separately but the main purpose in defining special operations for $\widehat{\mathbb{R}}$ is to define new operations for expressions which are undefined under the arithmetic operations of $\mathbb{R}_0$ and $\widehat{\infty}$ alone, or whose structure vanishes under additive absorption. Although it is easy to obtain statements requiring us to remove the hat from $\widehat{\infty}$ through the non-contradiction property, there is a very broad class of structures where the hat does not imply any contradiction and **this class of structures should be studied**.

Axiom 5.8 The arithmetic operations of $\widehat{\mathbb{R}}$ numbers with $\mathbb{R}_0$ numbers are

$$-\big(\widehat{\infty} - b\big) = -\widehat{\infty} + b$$

$$-\big(-\widehat{\infty} + b\big) = \widehat{\infty} - b$$

$$\pm\big(\widehat{\infty} - b\big) + x = x \pm \big(\widehat{\infty} - b\big) = \begin{cases} \pm\widehat{\infty} \mp \big(b \mp x\big) & \text{if} \quad b \neq x \\ \pm\widehat{\infty} & \text{if} \quad b = x \end{cases}$$

$$\pm\big(\widehat{\infty} - b\big) \cdot x = x \cdot \pm\big(\widehat{\infty} - b\big) = \begin{cases} \pm\big(\widehat{\infty} - xb\big) & \text{if} \quad x \neq 0 \\ \text{undefined} & \text{if} \quad x = 0 \end{cases}$$

$$\frac{\pm\left(\widehat{\infty}-b\right)}{x} = \begin{cases} \pm\widehat{\infty} \mp \dfrac{b}{x} & \text{if} \quad x \neq 0 \\ \text{undefined} & \text{if} \quad x = 0 \end{cases}$$

$$\frac{x}{\pm\left(\widehat{\infty}-b\right)} = 0 \quad .$$

Remark 5.9 Although the axiom of closure is not always explicitly included in the axioms of a complete ordered field, it is usually taken for granted that number fields are closed under their operations. However, by Axiom 5.8

$$x > b \quad \Longrightarrow \quad \left[\left(\widehat{\infty}-b\right)+x\right] \notin \widehat{\mathbb{R}} \quad .$$

Although we have not given the ordering for numbers of the form $\left(\widehat{\infty}+a\right)$ with $a > 0$, it reasonably follows from Axiom 5.3 that such numbers are greater than infinity. Therefore, with $\mathbb{R}$ as presently defined, the set $\mathcal{R} = \{\mathbb{R}, +, \times\}$ is not a number field because it is not closed under its operations.

Axiom 5.10 The arithmetic operations of $\widehat{\mathbb{R}}$ numbers with $\widehat{\mathbb{R}}$ numbers are

$$\pm\left(\widehat{\infty}-b\right) \pm \left(\widehat{\infty}-a\right) = \pm\widehat{\infty} \mp \left(a+b\right)$$
$$\pm\left(\widehat{\infty}-b\right) \mp \left(\widehat{\infty}-a\right) = \pm\left(a-b\right)$$
$$\pm\left(\widehat{\infty}-b\right)\left(\widehat{\infty}-a\right) = \pm\widehat{\infty}$$
$$\frac{\pm\left(\widehat{\infty}-b\right)}{\widehat{\infty}-a} = \pm 1 \quad .$$

Remark 5.11 Axiom 5.10 states that

$$\left(\widehat{\infty}-b\right) - \left(\widehat{\infty}-a\right) = a - b \quad .$$

Although this implies the existence of an additive inverse for every $\widehat{\mathbb{R}}$ number, it does not imply an additive inverse for $\widehat{\infty}$ because the case of $a = b = 0$ is ruled out by the definition of $\widehat{\mathbb{R}}$ (Definition 5.1.)

Example 5.12 Axiom 4.11 states that infinity does not have an additive identity element. We were not able to demonstrate this requirement in Section 4 because we needed first to define the axioms of $\widehat{\mathbb{R}}$ which make it impossible for ∞ or $\widehat{\infty}$ to have zero as an additive identity element. (The additive identity element of $\pm\widehat{\infty}$ is $\pm\widehat{\infty}$ though the multiplicative absorptive property and the definition of multiplication that $2x = x+x$.) This example gives an illustration of the type of contradictions which are avoided by removing the zero additive identity element of infinity. Consider the limit

$$\lim_{x\to\infty} \left(x^2 - x\right) = \infty$$

$$\lim_{x\to\infty}\left(x^2-x\right)=\lim_{x\to\infty}x^2-\lim_{x\to\infty}x=\infty-\infty\quad.$$

This is a typical example used to demonstrate the lack of an additive inverse for ∞. If infinity is bestowed with an additive inverse, then we obtain a contradiction $\infty=0$. The expression $\infty-\infty$, thus, is undefined. If we added the hats to infinity then we could insert the additive identity on the right side of $\infty=\infty-\infty$ to write

$$\begin{aligned}\widehat{\infty}&=\widehat{\infty}-\widehat{\infty}\\&=\widehat{\infty}-\widehat{\infty}+0\\&=\widehat{\infty}-\widehat{\infty}+1-1\\&=\left(\widehat{\infty}-1\right)-\left(\widehat{\infty}-1\right)=0\quad.\end{aligned}$$

We see that unhatted infinity likewise cannot have zero as an additive identity because we could write

$$\begin{aligned}\infty&=\infty-\infty\\&=\infty-\infty+\left(1-1\right)\\&=\widehat{\infty}-\widehat{\infty}+1-1\\&=\left(\widehat{\infty}-1\right)-\left(\widehat{\infty}-1\right)=0\quad,\end{aligned}$$

where we have simply chosen in the second step not to do the additive absorptive operation within the freedom afforded to the order of algebraic operations. By allowing infinity to have zero as an additive identity element, we induce the same contradiction which forbids an additive inverse for infinity.

Remark 5.13 The expressions ∞ and $\widehat{\infty}$ are perfectly well defined but $\infty+0$ and $\widehat{\infty}+0$ are examples of an undefined composition. Since ∞ is not an $\widehat{\mathbb{R}}$ number, this property cannot create problems for the algebra of $\widehat{\mathbb{R}}$ numbers. Essentially, we have traded the zero additive identity element of infinity for the freedom to add and subtract $\widehat{\mathbb{R}}$ numbers.

Axiom 5.14 The additive operation is not associative for $\widehat{\mathbb{R}}+\widehat{\mathbb{R}}$.

Example 5.15 The example demonstrates why $\widehat{\mathbb{R}}+\widehat{\mathbb{R}}$ cannot have the associative property. Through associativity we may easily derive a contradiction from Axiom 5.10 which gives

$$\left(\widehat{\infty}-b\right)+\left(\widehat{\infty}-a\right)=\widehat{\infty}-\left(b+a\right)\quad.$$

To obtain that contradiction, subtract an $\widehat{\mathbb{R}}$ number from both sides as

$$\left[\left(\widehat{\infty}-b\right)+\left(\widehat{\infty}-a\right)\right]-\left(\widehat{\infty}-c\right)=\widehat{\infty}-\left(b+a\right)-\left(\widehat{\infty}-c\right)\quad.$$

Assuming the associative property of addition, we may arrange brackets to write

$$\left(\widehat{\infty} - b\right) + \left[\left(\widehat{\infty} - a\right) - \left(\widehat{\infty} - c\right)\right] = \widehat{\infty} - \left(b + a\right) - \left(\widehat{\infty} - c\right)$$

$$\widehat{\infty} + \left[c - \left(b + a\right)\right] = c - \left(b + a\right) \ .$$

Subtracting the $\mathbb{R}_0$ part from both sides yields

$$\widehat{\infty} = 0 \ ,$$

which is not allowed. Since associativity is required among elements of a number field, this example further demonstrates that the present definition of $\mathbb{R}$ is not such that $\mathcal{R} = \{\mathbb{R}, +, \times\}$ satisfies the field axioms in all cases.

Theorem 5.16 *It is possible to make cuts in the real line which are numbers greater than any $n \in \mathbb{N}$, i.e.: certain cuts in the real number line are real numbers in the neighborhood of infinity.*

Proof. Suppose there exists a line segment AB. Every line segment can be bisected by a cut at its midpoint C. We say C is a midpoint of AB if and only if

$$\operatorname{len} AC = \operatorname{len} CB \ , \qquad \text{and} \qquad \operatorname{len} AC + \operatorname{len} CB = \operatorname{len} AB \ .$$

Define a chart x on AB such that $x \in [0, \pi/2]$. Then define a conformal chart x' such that

$$x' = \tan(x) \ , \qquad \text{and} \qquad x' \in [0, \infty]$$

These bounds on the x' chart are derived through Definition 3.1 as

$$\lim_{\theta \to \pi/2} \tan(\theta) = \lim_{\theta \to \pi/2} \frac{\sin(\theta)}{\cos(\theta)} = \lim_{\substack{x \to 0 \\ y \to 1}} \frac{y}{x} = \lim_{x \to 0} \frac{1}{x} = \infty \ .$$

We know it is possible to bisect any line segment AB at a midpoint C. If we add a chart to AB, it cannot affect this fundamental geometric property that every line segment can be bisected. Adding a chart such that $x'(B) = \infty$ does not disrupt the fundamental geometric properties of line segment. To prove the present theorem by contradiction, assume $x'(C)$ is less than some $n \in \mathbb{N}$ (where $x'(C)$ refers to the value of x' at the geometric midpoint of AB.) Then

$$\operatorname{len} AC < n \ , \qquad \text{and} \qquad \operatorname{len} CB < n \ ,$$

so it follows that

$$\operatorname{len} AC + \operatorname{len} CB < 2n \ .$$

Here we obtain a contradiction because, by the definition of a midpoint, we have

$$\operatorname{len} AC + \operatorname{len} CB = \operatorname{len} AB = \infty \ , \qquad \text{but} \qquad \infty \not< 2n \ .$$

Therefore, when AB is charted in x', the magnitude of the cut at the midpoint C is greater than any natural number. Because $\text{len}[0, \infty] = \text{len}(0, \infty)$, it follows that we can make a cut in the positive branch of the real line at a magnitude greater than any natural number. ✑

Remark 5.17 In Theorem 5.16, we have derived a requirement for two lengths which are equal, less than infinity, and whose sum is equal to infinity. This requirement is given a dedicated treatment in Reference [3].

Theorem 5.18 *An* $\widehat{\mathbb{R}}$ *number does not have a multiplicative inverse.*

Proof. Axiom 5.10 gives

$$\frac{\widehat{\infty} - b}{\widehat{\infty} - a} = 1 \quad ,$$

which allows us to give a simple proof by contradiction. Assume that the numerator has a multiplicative inverse e_b. Multiplying both sides by e_b gives

$$\frac{e_b\big(\widehat{\infty} - b\big)}{\widehat{\infty} - a} = e_b \quad \Longrightarrow \quad e_b = \frac{1}{\widehat{\infty} - a} \quad .$$

By Axiom 5.8, the expression on the left identically zero. However, if $e_b = 0$, then by Axiom 5.8, the product $e_b\big(\widehat{\infty} - b\big)$ is undefined. Therefore, $x \in \widehat{\mathbb{R}}$ cannot have a multiplicative inverse. ✑

Remark 5.19 In Example 6.4, we will discuss the notion that division cannot be defined as multiplication by the inverse for numbers which do not have a multiplicative inverse.

Remark 5.20 In Axiom 5.8, we have preserved the distributive property of multiplication such that

$$0 \cdot \big(\widehat{\infty} - b\big) = \text{undefined} \quad .$$

That distributive product necessarily depends on $0 \cdot \widehat{\infty}$ which is undefined for a very good reason. However, we might revise Axiom 5.8 such that 0 is a multiplicative inverse for all $x \in \widehat{\mathbb{R}}$. If we did, there would still be some unwanted issues because $\widehat{\mathbb{R}}$ includes negative numbers and it is not possible to give 0 a sign such that its product with a negative number is equal to positive 1. If $x \in \widehat{\mathbb{R}}$ and $0 \cdot x = 1$, then we would expect that $0 \cdot \big(-x\big) = -1$. In that case, 0 is not the multiplicative inverse of $-x$.

§6   Limits

Example 6.1 This example shows we cannot always substitute the limit definition of infinity to directly compute all expressions involving ∞. If we use Definition 3.1 to write

$$\infty - \infty = \left(\lim_{x\to 0}\frac{1}{x}\right) - \left(\lim_{x\to 0}\frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{x}\right) = \lim_{x\to 0} 0 = 0 \ ,$$

then we contradict Axiom 3.7 which gives

$$\infty - \infty = \text{undefined}\ \ .$$

A typical example showing why $\infty - \infty$ must be undefined is the limit

$$l = \lim_{x\to\infty}\left(3n - n\right)\ \ .$$

If we compute the limit of a difference, then

$$\lim_{x\to\infty}\left(3x - x\right) = \lim_{x\to\infty} 2x = 2\infty = \infty\ \ ,$$

but if we compute the difference of limits, which should be equal to the limit of the difference, then we obtain

$$\lim_{x\to\infty}\left(3x - x\right) = \lim_{x\to\infty} 3x - \lim_{x\to\infty} x = 3\infty - \infty = \infty - \infty\ \ .$$

If we defined $\infty - \infty = 0$, then we would obtain a contradiction $\infty = 0$ when we set the limit of a difference equal to the difference of two limits. Similar demonstrations show why ∞/∞ must be undefined. However, when we have given in Axiom 5.10 a definition for the expression

$$\left(\widehat{\infty} - b\right) - \left(\widehat{\infty} - a\right) = a - b\ \ ,$$

we avoid all such possible contradictions. To derive a contradiction of the form usually used to demonstrate the lack of additive and multiplicative inverse for infinity, we would need to find some functions $f(x)$ and $g(x)$ such that

$$\lim_{x\to\infty} f(x) = \infty - b\ \ , \qquad \text{and} \qquad \lim_{x\to\infty} g(x) = \infty - a\ \ .$$

There are no such functions. Even if there were, ∞ has the property of additive absorption which would make a and b vanish. More importantly, even if we wrote the limit as x approaches $\widehat{\infty}$, there are no such functions such that

$$\lim_{x\to\widehat{\infty}} f(x) = \widehat{\infty} - b\ \ , \qquad \text{and} \qquad \lim_{x\to\widehat{\infty}} g(x) = \widehat{\infty} - a\ \ .$$

Conjecture 6.2 Although we are not able to use the limit definition of infinity to compute $\infty - \infty$ and ∞/∞ due to the existence of unfixable contradictions, no such contradictions exist for $x \in \widehat{\mathbb{R}}$ because there are no limits such that $l \in \widehat{\mathbb{R}}$ and $\lim f(x) = l$. Therefore, it is possible to use the limit definition of infinity to compute the operations of $\widehat{\mathbb{R}}$ numbers.

Theorem 6.3 *The quotient of a number $x \in \mathbb{R}_0$ divided by a number $y \in \widehat{\mathbb{R}}$ is identically zero (Axiom 5.8.)*

Proof. Let z be any non-zero real number such that

$$\frac{x}{y} = z \quad .$$

Since $||x|| < ||y||$ (Axiom 5.3), we have $||z|| \leq 1$ which implies $z \in \mathbb{R}_0$. (Usually this would imply $||z|| < 1$ but presently we must account for the special case of $\widehat{\mathbb{R}}/\widehat{\mathbb{R}} = \pm 1$.) All $\mathbb{R}_0$ numbers have a multiplicative inverse. We find, therefore, that

$$\frac{x}{zy} = 1 \qquad \Longleftrightarrow \qquad x = zy \quad .$$

The product zy is given by Axiom 5.8 as

$$zy = z \cdot \pm\big(\widehat{\infty} - b\big) = \pm\big(\widehat{\infty} - zb\big) \quad .$$

This delivers a contradiction because it requires that $x = zy$ is a real number in the neighborhood of infinity while we have already defined it to be a real number in the neighborhood of the origin. Therefore, the only possible numerical value for x/y is 0.

Alternatively, the limit definition of infinity gives the same result. Observe that for some $n, b \in \mathbb{R}_0$ we have

$$\frac{n}{\widehat{\infty} - b} = \lim_{x \to 0} \frac{n}{\frac{1}{x} - b} = \lim_{x \to 0} \frac{nx}{1 - bx} = 0 \quad . \qquad \qquad \text{\reflectbox{$\mathscr{O}$}}$$

Example 6.4 This example treats the $\widehat{\mathbb{R}}/\widehat{\mathbb{R}}$ operation. In Axiom 5.8, we have given

$$0 \cdot \big(\widehat{\infty} - b\big) = \text{undefined} \quad .$$

This follows from

$$0 \cdot \big(\widehat{\infty} - b\big) = 0 \cdot \widehat{\infty} - 0 \cdot b \quad ,$$

with $0 \cdot \widehat{\infty}$ being undefined, as per usual. However, in Axiom 5.10 we have given

$$\frac{\widehat{\infty} - b}{\widehat{\infty} - a} = 1 \quad ,$$

which can be written as

$$\frac{\widehat{\infty} - b}{\widehat{\infty} - a} = \big(\widehat{\infty} - b\big)\frac{1}{\widehat{\infty} - a} = \big(\widehat{\infty} - b\big) \cdot 0 \quad .$$

Apparently, $\widehat{\mathbb{R}}/\widehat{\mathbb{R}} = \pm 1$ contradicts the part of Axiom 5.8 which says such expressions are undefined. For this, we need to give to special attention to the $\div$ operator which is usually defined as multiplication by the inverse. Theorem 5.18 shows that $x \in \widehat{\mathbb{R}}$ does not have a multiplicative inverse so

$$\frac{\widehat{\infty} - b}{\widehat{\infty} - a} \neq \left(\widehat{\infty} - b\right)\left(\widehat{\infty} - a\right)^{-1} \quad,$$

and we need a separate definition for the $\div$ operation. In this case, we resort to the limit

$$\frac{\widehat{\infty} - b}{\widehat{\infty} - a} = \lim_{x \to 0} \frac{\frac{1}{x} - b}{\frac{1}{x} - a} = \lim_{x \to 0} \frac{1 - bx}{1 - ax} = 1 \quad.$$

This is the result given in Axiom 5.10.

Example 6.5 This example treats the $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$ operation. If we axiomatize this operation with limit definition of infinity the we obtain

$$\left(\widehat{\infty} - b\right)\left(\widehat{\infty} - a\right) = \lim_{x \to 0} \left(\frac{1}{x} - b\right)\left(\frac{1}{x} - a\right)$$

$$= \lim_{x \to 0} \left(\frac{1}{x^2} - \frac{b + a}{x} + ba\right)$$

$$= \lim_{x \to 0} \left(\frac{1 - x\left(b + a\right) + x^2 ba}{x^2}\right) = \text{diverges}$$

$$= \infty \quad.$$

This is the value that appears in Axiom 5.10. If we gave any value other than that supported by the limit definition of infinity, then that would be contrived and arbitrary because we have used the limit definition for $\widehat{\mathbb{R}} \div \widehat{\mathbb{R}}$ in that same axiom. However, we could equally well choose some definition for $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$ and then require for consistency that $\widehat{\mathbb{R}}/\widehat{\mathbb{R}}$ be computed in the same way. Therefore, in this example we will demonstrate the invalidity of a few other possible definitions for $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$ to support the limit as the correct computation to determine the relevant quotient and product. Firstly, Axiom 3.8 states that ∞ does not have the distributive property of multiplication so $\widehat{\infty}$, likewise, does not have this property. Therefore, we cannot compute $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$ with the FOIL method. If we could do that, then we would obtain (assuming $\widehat{\infty}^2 = \widehat{\infty}$)

$$\left(\widehat{\infty} - b\right)\left(\widehat{\infty} - a\right) = \widehat{\infty} - \widehat{\infty} - \widehat{\infty} + ba \quad.$$

As written, this expression is undefined through $\widehat{\infty} - \widehat{\infty}$. If we rearranged the $\mathbb{R}_0$ term as

$$\widehat{\infty} - \widehat{\infty} - \widehat{\infty} + 2ba - ba = \left(\widehat{\infty} - ba\right) - \left(\widehat{\infty} - ba\right) - \left(\widehat{\infty} - ba\right) \quad,$$

then the non-associativity of addition in $\widehat{\mathbb{R}}$ gives a contradiction

$$\left[\left(\widehat{\infty} - ba\right) - \left(\widehat{\infty} - ba\right)\right] - \left(\widehat{\infty} - ba\right) = \left(\widehat{\infty} - ba\right) - \left[\left(\widehat{\infty} - ba\right) + \left(\widehat{\infty} - ba\right)\right]$$
$$0 - \left(\widehat{\infty} - ba\right) = \left(\widehat{\infty} - ba\right) - \left(\widehat{\infty} - 2ba\right)$$
$$-\left(\widehat{\infty} - ba\right) = ba \quad.$$

The expression on the left is not equal to the expression on the right. Indeed, if we subtract away the $\mathbb{R}_0$ part, then we obtain the familiar contradiction $-\widehat{\infty} = 0$. As another non-limit for computing $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$, we should examine the multiplicative absorptive property of $\widehat{\infty}$. Due to the lack of distributivity, it is impossible get either of $\left(\widehat{\infty} - b\right)$ or $\left(\widehat{\infty} - a\right)$ on its own multiplied by $\widehat{\infty}$. Therefore, the multiplicative absorptive property of $\widehat{\infty}$ is not relevant.

Remark 6.6 Consider $\mathbb{R}_0 \times \widehat{\mathbb{R}}$ as in

$$x\left(\widehat{\infty} - b\right) = \widehat{\infty} - xb \quad.$$

If x is a positive number, then, by Axiom 5.3, the magnitude of the product decreases as the magnitude of x increases. Therefore, there is some radical change in behavior as x increases from $\mathbb{R}_0$ to $\widehat{\mathbb{R}}$ because the product $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$ is greater than any $x \in \widehat{\mathbb{R}}$. This exotic behavior is studied more closely in Reference [3].

§7   Complex Numbers

Definition 7.1 The set of all complex numbers is

$$\mathbb{C} \quad \equiv \quad \{x + iy \mid x, y \in \mathbb{R}, \ i = \sqrt{-1}\}$$

Definition 7.2 The set of all complex numbers in the neighborhood of the origin is

$$\mathbb{C}_0 \quad \equiv \quad \{x + iy \mid x, y \in \mathbb{R}_0, \ i = \sqrt{-1}\}$$

Axiom 7.3 As ∞ does not absorb -1 in 1D, in 2D (meaning in $\mathbb{C}$) we have the condition that infinity absorbs neither ± 1 nor $\pm i$.

Definition 7.4 The affinely extended complex plane is

$$\overline{\mathbb{C}} \quad \equiv \quad \mathbb{C} \cup \{\pm\infty\} \cup \{\pm i\infty\} \quad.$$

Remark 7.5 As the extended real line $\overline{\mathbb{R}}$ has two distinct infinities, the extended complex plane $\overline{\mathbb{C}}$ has four: $\{+\infty, +i\infty, -\infty, -i\infty\}$.

Axiom 7.6 The multiplicative operations for $\pm\widehat{\infty}$ and $\pm i\,\widehat{\infty}$ with i are

$$\pm\widehat{\infty} \cdot i = i \cdot \pm\widehat{\infty} = \pm i\,\widehat{\infty}$$
$$\pm i\,\widehat{\infty} \cdot i = i \cdot \pm i\,\widehat{\infty} = \mp\widehat{\infty} \ \ .$$

Remark 7.7 The non-distributive property of $\pm\infty$ (Axiom 3.8) was practically redundant in 1D but for $z \in \mathbb{C}$ this feature gains significance. If $a, b \in \mathbb{R}_0$ then there exists a $c \in \mathbb{R}_0$ such that $a + b = c$. This allows us to mimic the distributive property through multiplicative absorption as

$$\widehat{\infty}(a + b) = c\widehat{\infty} = \widehat{\infty} \ \ .$$

To the contrary, if $z \in \mathbb{C}_0$, then

$$\widehat{\infty}(x + iy) = \text{undefined} \neq \text{sign}(x)\widehat{\infty} + i\,\text{sign}(y)\widehat{\infty}.$$

Axiom 7.8 The multiplicative operations for $\pm\widehat{\infty}$ with complex numbers $z \in \mathbb{C}_0$ are

$$\pm\widehat{\infty} \cdot z = z \cdot \pm\widehat{\infty} = \begin{cases} \pm\widehat{\infty} & \text{if} & \text{Re}(z) > 0 & \text{and} & \text{Im}(z) = 0 \\ \mp\widehat{\infty} & \text{if} & \text{Re}(z) < 0 & \text{and} & \text{Im}(z) = 0 \\ \pm i\,\widehat{\infty} & \text{if} & \text{Im}(z) > 0 & \text{and} & \text{Re}(z) = 0 \\ \mp i\,\widehat{\infty} & \text{if} & \text{Im}(z) < 0 & \text{and} & \text{Re}(z) = 0 \\ \text{undefined} & \text{if} & \text{Im}(z) \neq 0 & \text{and} & \text{Re}(z) \neq 0 \\ \text{undefined} & \text{if} & z = 0 \end{cases} \ \ .$$

Axiom 7.9 The multiplicative operations for $\pm i\,\widehat{\infty}$ with complex numbers $z \in \mathbb{C}_0$ are

$$\pm i\,\widehat{\infty} \cdot z = z \cdot \pm i\,\widehat{\infty} = \begin{cases} \pm i\,\widehat{\infty} & \text{if} & \text{Re}(z) > 0 & \text{and} & \text{Im}(z) = 0 \\ \mp i\,\widehat{\infty} & \text{if} & \text{Re}(z) < 0 & \text{and} & \text{Im}(z) = 0 \\ \mp\widehat{\infty} & \text{if} & \text{Im}(z) > 0 & \text{and} & \text{Re}(z) = 0 \\ \pm\widehat{\infty} & \text{if} & \text{Im}(z) < 0 & \text{and} & \text{Re}(z) = 0 \\ \text{undefined} & \text{if} & \text{Im}(z) \neq 0 & \text{and} & \text{Re}(z) \neq 0 \\ \text{undefined} & \text{if} & z = 0 \end{cases} \ \ .$$

Remark 7.10 The arithmetic operations for complex numbers $z \in \mathbb{C}$ whose real and/or imaginary parts are $\widehat{\mathbb{R}}$ numbers follow directly from the other axioms.

§8   The Riemann Hypothesis

Remark 8.1 The Riemann hypothesis dates to Riemann's 1858 paper [4]. Since the axioms of a complete ordered field date to Dedekind's 1872 paper

[5], it would be patently absurd to claim that the Riemann hypothesis is formulated in terms of the ordered field definition of $\mathbb{R}$. While we cannot directly show what definition of $\mathbb{R}$ Riemann had in mind when formulating his hypothesis, we can point out that his program of Riemannian geometry is a direct extension of Euclidean geometry. This qualitatively supports the notion that Riemann had in mind the cut-in-a-number-line definition of $\mathbb{R}$ given by Euclid in the Elements. When one examines the Elements [1], the very many diagrams, definitions, and postulates make it exceedingly obvious that Euclid's definition of $\mathbb{R}$ is exactly the one given here in Definition 2.2

$$\mathbb{R} \setminus x = (-\infty, x) \cup (x, \infty) \quad ,$$

formulated as the alternative identical statement

$$x \in \mathbb{R} \quad , \quad x > 0 \quad \implies \quad (0, \infty) = (0, x] \cup (x, \infty) \quad .$$

Riemann—the man himself being the premier mathematical analyst of the 19th century—went to no lengths whatsoever to formalize with rigor the definition of $\mathbb{R}$ used by him to formulate his hypothesis. What does this tell us that a man of the utmost standards of mathematical rigor did not even deem it worthwhile to mention his definition of $\mathbb{R}$? It tells us, in the opinion of this writer, that one should reasonably conclude that the specifics of the aspect of Riemann's hypothesis relating to the definition of the domain of $\zeta(z)$ were not highly relevant. Rather, the object of relevance would be the behavior $\zeta(z)$ at various z. It is reasonable to assume that when Riemann formulated his hypothesis, he had in mind that any definition of $\mathbb{R}$ consistent with the Euclid magnitude and also displaying the Archimedes property of real numbers would be sufficient. The domain of $\zeta(z)$, namely $\mathbb{C}$, would be constructed from two orthogonal copies of $\mathbb{R}$, one of them having the requisite phase factor i. As per Pugh [2] quoted in Section 1, if we prove that all $x \in \widehat{\mathbb{R}}$ satisfy the Archimedes property, then that should be sufficient reason to accept the present definition of $\mathbb{R}$ into applications regarding Riemann's hypothesis.

For some reason, the modern statement of the Archimedes property has evolved to include natural numbers in its predicate but this is not at all supported by the statement of the property as it is given in Euclid's Elements [1]. The modern statement depending on natural numbers is

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad nx > y \quad .$$

Numbers in the neighborhood of infinity do not conform the natural number statement of the Archimedes property but they do absolutely conform the statement that appears in Euclid's elements. Regarding the natural number statement of the property, consider Remark 6.6. Those remarks point out that for any $n \in \mathbb{N}$ and any positive $x \in \widehat{\mathbb{R}}$, we have $nx < x$. It follows that if $x < y$, then $nx \not> y$ and that, therefore, $x \in \widehat{\mathbb{R}}$ do not exhibit the Archimedes

property of real numbers. However, we will show in this section that the statement depending on natural numbers has no precedent in the ancient history of mathematics. Rather, it is only a modern (over-)simplification of the genuine Archimedes property of antiquity. Indeed, without reference to any one technical statement or another, the main gist of the Archimedes property is that there is no greatest real number. It is obvious that the present definition of $\mathbb{R}$ satisfies the main gist of the Archimedes property. The round bracket notation

$$\mathbb{R} \quad \equiv \quad (-\infty, \infty) \ ,$$

directly requires "no greatest element."

In this section, we will examine the Archimedes property to show that all $x \in \widehat{\mathbb{R}}$ do satisfy the property given in Euclid's Elements and that, therefore, they are fully qualified for applications to the Riemann hypothesis. Then we will show that the Riemann ζ function has infinitely many non-trivial zeros off the critical line in the neighborhood of infinity.

Definition 8.2 The statement of the Archimedes property which appears in Euclid's Elements, and which was attributed by Archimedes to his predecessor Eudoxus, *and* which is very often taken to be the definitive statement of the Archimedes property of real numbers, appears as Definition 4 in Book 5 of Euclid's Elements [1]. The original Greek is translated as follows.

"Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another."

Remark 8.3 As it appears in Euclid's Elements, the straightforward mathematical statement of the property should be

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z \in \mathbb{R} \quad \text{s.t.} \quad zx > y \ .$$

There is no mention of multiplication by a positive integer $n \in \mathbb{N}$. It is obvious that this property—the statement of the Archimedes property in which the multiplier is $z \in \mathbb{R}$ rather than $n \in \mathbb{N}$—holds for all $x, y \in \mathbb{R}$ as presently defined. If we have

$$x = 1 \ , \quad y = \widehat{\infty} - 1 \quad \implies \quad x < y \ ,$$

then choosing $z = \widehat{\infty} - 0.9$ gives $zx > y$. If we have

$$x = \widehat{\infty} - 2 \ , \quad y = \widehat{\infty} - 1 \quad \implies \quad x < y \ ,$$

then choosing $z = 1/3$ gives $zx > y$. It is obvious that the present definition of $\mathbb{R}$ defines a set which obeys the Archimedes property. For some reason, however, mathematicians often choose to express the property mathematically as

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad nx > y \ .$$

This is the statement of a property that $\mathbb{R}$ does not have in its current incarnation (Definition 2.3) because we needed to choose $z \notin \mathbb{N}$ for the two cases given above. However, the statement depending on $\mathbb{N}$ is **not** the Archimedes property of real numbers! Nowhere did Euclid mention integers but it is claimed, apparently, that Euclid's definition of multiplication should be taken only to mean multiplication by $n \in \mathbb{N}$ or, perhaps, that it should be obvious from the context that Euclid was writing about multiplication by natural numbers. Indeed, careful (or even cursory) examination of the context in Reference [1] shows no such thing.

The Archimedes property given here in Definition 8.2 appears as Definition 4 of Book 5 of Euclid's Elements [1]. We may directly extract from Definitions 1 and 2 of Book 5 that Euclid did not use a definition of multiplication restricted to $n \in \mathbb{N}$. Definition 1 of Book 5 is

> "A magnitude is a part of another magnitude, the lesser of the greater, when it measures the greater."

Fitzpatrick, the English translator of Euclid's Elements cited here as Reference [1], adds the following footnote to this definition.

> "In other words, α is said to be a part of β if $\beta = m\alpha$."

This makes it perfectly obvious that Euclid's multiplication was never restricted to $n \in \mathbb{N}$. Euclid was certainly aware that is possible to measure one length (magnitude) of, say, ten Archimedean length units, and another length having 25 such length units. This proves that the multiplier in Euclid's definitions was never intended to be restricted to $\mathbb{N}$.

However, so that we need not cite the translator's footnote in the determination that Euclid had no intention whatsoever to restrict implicitly his multiplier as $n \in \mathbb{N}$, we should also consider Book 5, Definition 2 [1].

> "And the greater is a multiple of the lesser whenever it is measured by the lesser."

Is a length of 25 units an integer multiple of a length of ten units? Obviously not. Are we to believe that Euclid meant to forbid the existence of the number 25 once one has discovered the number ten? Obviously not! A magnitude of 25 units is the greater of the lesser magnitude of ten units with multiplier 2.5. Surely this was known to Euclid!

We finish this remark with Fitzpatrick's footnote to Book 5, Definition 4 which is the Archimedes property proper. Fitzpatrick's footnote to,

> "Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another,"

is

> "In other words, α has a ratio with respect to β if $m\alpha > \beta$ and $n\beta > \alpha$, for some m and n."

Although it is common to use the variables m and n to refer to natural numbers, such is not the case in this context. If the multiplier was restricted as $m \in \mathbb{N}$, then it could not be said that four is a part of five, or two a part of three, and it would follow that, among four and five or two and three, neither is the greater and neither is the lesser. Clearly this would be an affront to reason! There is absolutely no historical precedent for any statements of the Archimedes property dependent on natural numbers. Such statements should be called "Archimedes properties of the second kind," or some such thing like, "the Archimedes property of natural numbers," to distinguish them from the Archimedes property of real numbers which famously appears in Euclid's Elements [1].

In closing, if we could add a second footnote to Definition 4 of Book 5, it would be the following.

> "The Archimedes property of real numbers states that there is no largest real number."

The $\widehat{\mathbb{R}}$ proven to be $\widehat{\mathbb{R}} \subset \mathbb{R}$ in Main Theorem 5.5 satisfies the requirement that real numbers have the Archimedes property. Now that we have properly motivated the application of $\widehat{\mathbb{R}}$ to the Riemann hypothesis, we will present the application in the remainder of this section.

Theorem 8.4 *If $b, y_0 \in \mathbb{R}_0$, if $z_0 = (\widehat{\infty} - b) + iy_0$, and if $\zeta(z)$ is the Riemann ζ function, then $\zeta(z_0) = 1$.*

Proof. Observe that the Dirichlet sum form of ζ [4] takes z_0 as

$$
\begin{aligned}
\zeta(z_0) &= \sum_{n=1} \frac{1}{n^{(\widehat{\infty}-b)+iy_0}} \\
&= \sum_{n=1} \frac{n^b}{n^{\widehat{\infty}}} \Big(\cos(y_0 \ln n) - i \sin(y_0 \ln n) \Big) \\
&= 1 + \sum_{n=2} 0 \Big(\cos(y_0 \ln n) - i \sin(y_0 \ln n) \Big) = 1 \quad . \qquad \text{🖉}
\end{aligned}
$$

Theorem 8.5 *The Riemann ζ function has non-trivial zeros at certain $z \in \mathbb{C}$ outside of the critical strip.*

Proof. Riemann's functional form of ζ [4] is

$$
\zeta(z) = \frac{(2\pi)^z}{\pi} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z) \quad .
$$

Theorem 8.4 gives $\zeta(\widehat{\infty}-b)=1$ when we set $y_0=0$ so we will use Riemann's equation to prove this theorem by solving for $z=-(\widehat{\infty}-b)+1$. (This value for z follows from $1-z=\widehat{\infty}-b$.) We have

$$\zeta\big[-(\widehat{\infty}-b)+1\big]=\lim_{z\to-(\widehat{\infty}-b)+1}\left(\frac{(2\pi)^z}{\pi}\sin\left(\frac{\pi z}{2}\right)\right)\lim_{z\to(\widehat{\infty}-b)}\Big(\Gamma(z)\zeta(z)\Big)$$

$$=\lim_{z\to-(\widehat{\infty}-b)+1}\Big(2\sin(\pi z/2)\Big)\lim_{z\to(\widehat{\infty}-b)}\Big((2\pi)^{-z}\Gamma(z)\zeta(z)\Big)\ .$$

For the limit involving Γ, we will compute the limit as a product of two limits. We separate terms as

$$\lim_{z\to(\widehat{\infty}-b)}\Big((2\pi)^{-z}\Gamma(z)\zeta(z)\Big)=\lim_{z\to(\widehat{\infty}-b)}\Big((2\pi)^{-z}\Gamma(z)\Big)\lim_{z\to(\widehat{\infty}-b)}\zeta(z)\ .$$

From Theorem 8.4, we know the limit involving ζ is equal to one. For the remaining limit, we will insert the identity and again compute it as the product of two limits. If z approaches $(\widehat{\infty}-b)$ along the real axis, then it follows from Axiom 5.10 that

$$1=\frac{z-(\widehat{\infty}-b)}{z-(\widehat{\infty}-b)}\ .$$

Inserting the identity yields

$$\lim_{z\to(\widehat{\infty}-b)}\Big((2\pi)^{-z}\Gamma(z)\Big)=\lim_{z\to(\widehat{\infty}-b)}\left((2\pi)^{-z}\Gamma(z)\frac{z-(\widehat{\infty}-b)}{z-(\widehat{\infty}-b)}\right)\ .$$

Let

$$A=\Gamma(z)\Big(z-(\widehat{\infty}-b)\Big)\ ,\qquad\text{and}\qquad B=\frac{(2\pi)^{-z}}{z-(\widehat{\infty}-b)}\ .$$

To get the limit of A into workable form, we will use the property $\Gamma(z)=z^{-1}\Gamma(z+1)$ to derive an expression for $\Gamma[z-(\widehat{\infty}-b)+1]$. If we can write $\Gamma(z)$ in terms of $\Gamma[z-(\widehat{\infty}-b)+1]$, then the limit as z approaches $(\widehat{\infty}-b)$ will be very easy to compute. Observe that

$$\Gamma\big[z-(\widehat{\infty}-b)+1\big]=\Gamma\big[z-(\widehat{\infty}-b)+2\big]\Big(z-(\widehat{\infty}-b)+1\Big)^{-1}\ .$$

By recursion we obtain

$$\Gamma\big[z-(\widehat{\infty}-b)+1\big]=\Gamma(z)\lim_{n\to(\widehat{\infty}-b)}\prod_{k=1}^n\Big(z-(\widehat{\infty}-b)+k\Big)^{-1}\ .$$

Rearrangement yields

$$\Gamma(z)=\Gamma\big[z-(\widehat{\infty}-b)+1\big]\lim_{n\to(\widehat{\infty}-b)}\prod_{k=1}^n\Big(z-(\widehat{\infty}-b)+k\Big)\ .$$

It follows that

$$A = \Gamma\big[z - (\widehat{\infty} - b) + 1\big] \lim_{n \to (\widehat{\infty}-b)} \prod_{k=0}^{n} \Big(z - (\widehat{\infty} - b) + k\Big) \ .$$

The limit of A is

$$\lim_{z \to (\widehat{\infty}-b)} A = \Gamma\big[(\widehat{\infty} - b) - (\widehat{\infty} - b) + 1\big] \lim_{n \to (\widehat{\infty}-b)} \prod_{k=0}^{n} \Big((\widehat{\infty} - b) - (\widehat{\infty} - b) + k\Big) \ .$$

Axiom 5.10 gives $(\widehat{\infty} - b) - (\widehat{\infty} - b) = 0$ so

$$\lim_{z \to (\widehat{\infty}-b)} A = \Gamma(1) \lim_{n \to (\widehat{\infty}-b)} \prod_{k=0}^{n} k = 0 \ .$$

Direct evaluation of the limit of B gives $0/0$ so we need to use L'Hôpital's rule which gives

$$\lim_{z \to (\widehat{\infty}-b)} B \overset{*}{=} \lim_{z \to (\widehat{\infty}-b)} \left(\frac{\dfrac{d}{dz}(2\pi)^{-z}}{\dfrac{d}{dz}\Big(z - (\widehat{\infty} - b)\Big)} \right)$$

$$= \lim_{z \to (\widehat{\infty}-b)} \frac{d}{dz} e^{-z\ln(2\pi)}$$

$$= -\ln(2\pi)\, e^{-(\widehat{\infty}-b)\ln(2\pi)}$$

$$= \frac{-1}{e^{\widehat{\infty}}}\ln(2\pi)\, e^{b\ln(2\pi)} = 0$$

Therefore, we find that the limit of AB is 0. It follows that

$$\zeta\big[-(\widehat{\infty} - b) + 1\big] = \lim_{z \to -(\widehat{\infty}-b)+1} 2\sin\left(\frac{\pi z}{2}\right) \times 0 = 0 \ . \qquad\qquad \varnothing$$

Definition 8.6 The Riemann hypothesis as defined by the Clay Mathematics Institute [6] is the following.

> "The non-trivial zeros of the Riemann ζ function have real parts equal to one half."

Definition 8.7 According to the Clay Mathematics Institute [6], the trivial zeros of ζ are the even negative integers.

Remark 8.8 The zeros demonstrated in Theorem 8.5 are neither on the critical line $\mathrm{Re}(z) = 1/2$ nor are they the negative even integers. Theorem 8.5, therefore, is the negation of the Riemann hypothesis.

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