Factorization Of Sum Of Squares Of Two Real Numbers

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Abstract

In Mathematics factorization of sum of squares of two real numbers is often needed. But there was no formula to do that. Hence I have proposed a new formula : Let a and b be two real numbers then the sum of their squares, that is $a^2 + b^2$ is equal to $a(a + b) - b(a - b)$. In this paper, we prove and examine the proposed formula.

Keywords: factorization of sum of two squares, sum of squares of two numbers, addition of two squares, factoring formula

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1. Preliminaries

1.1. Prime number:

A prime number is a natural number greater than 1 that has no positive divisors other than 1 and itself.
For example, 5 is prime number because 1 and 5 are its only positive integer factors.

1.2. Congruence modulo:

A is congruent to B modulo C, that is, $A \equiv B \pmod{C}$, if the difference between A and B, that is $A - B$ is divisible by C.
For example, 13 is congruent to 1 modulo 4, because $13 - 1 = 12$, and 12 is divisible by 4, hence $13 \equiv 1 \pmod{4}$.

2. Introduction

If we want to factorize the sum of squares of two real numbers, there was no formula to do that, not even in any Mathematical book/textbook. So, in this paper, we propose and prove the formula to factorize the sum of squares of two real numbers, also we will examine the formula with some examples. First we will study Fermat's theorem on sum of two squares and Pythagorean theorem.

2.1. Fermat's theorem on sum of two squares

Theorem. *Fermat's theorem on sum of two squares (see [1]) states that an odd prime p can be expressed as: $p = a^2 + b^2$ where a and b are integers, if and only if p is congruent to 1 modulo 4, that is,* $p \equiv 1 \pmod{4}$.

In other words, if $(p-1)$ is divisible by 4, then an odd prime p can be expressed as:

$$p = a^2 + b^2$$

where a and b are integers.

The prime numbers for which this is true are called Pythagorean primes.

For example, the primes 5, 13, 17, 29, 37 and 41 are all congruent to 1 modulo 4, and they can be expressed as sums of two squares in the following ways:

$$5 = 1^2 + 2^2, \quad 13 = 2^2 + 3^2, \quad 17 = 1^2 + 4^2, \quad 29 = 2^2 + 5^2, \quad 37 = 1^2 + 6^2, \quad 41 = 4^2 + 5^2$$

On the other hand, the primes 3, 7, 11, 19, 23 and 31 are all congruent to 3 modulo 4, and none of them can be expressed as the sum of two squares.

Fermat's theorem on sum of two squares is applicable to some prime numbers and can not be used to factorize $a^2 + b^2$.

2.2. Pythagorean theorem

The Pythagorean theorem (see [2]) was invented by the ancient Greek mathematician named Pythagoras during 500 B.C.

Theorem. *The Pythagorean theorem states that, in a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.*

The side opposite to the right angle (90°) is the longest side known as Hypotenuse.
The theorem can be written as an equation:

$$a^2 + b^2 = c^2$$

where c is hypotenuse and a and b are other sides of right triangle.

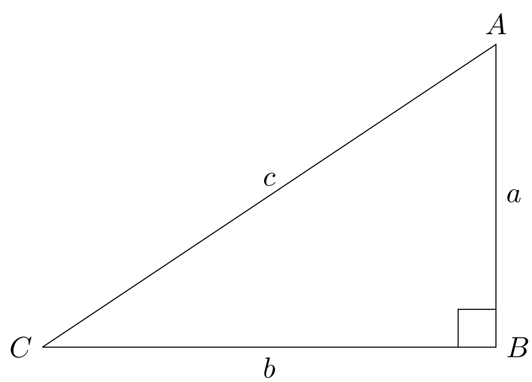

Figure 1: Right triangle $\triangle ABC$ right angled at B

In above Fig.1, $\triangle ABC$ is a right triangle, $\angle ABC$ is right angle and AC is hypotenuse.
Hence, by Pythagorean theorem,

$$AB^2 + BC^2 = AC^2$$
$$a^2 + b^2 = c^2$$

Pythagorean theorem can be used to find the unknown side lengths of a right triangle, but can not be used to factorize $a^2 + b^2$.

So, now we will discuss proposed formula.

3. Proposed Formula

Let a and b be two real numbers then the sum of their squares will be $a(a+b) - b(a-b)$. So, the formula can be expressed as:

$$a^2 + b^2 = a(a+b) - b(a-b)$$

Now we will prove proposed formula.

3.1. Proof of the proposed formula

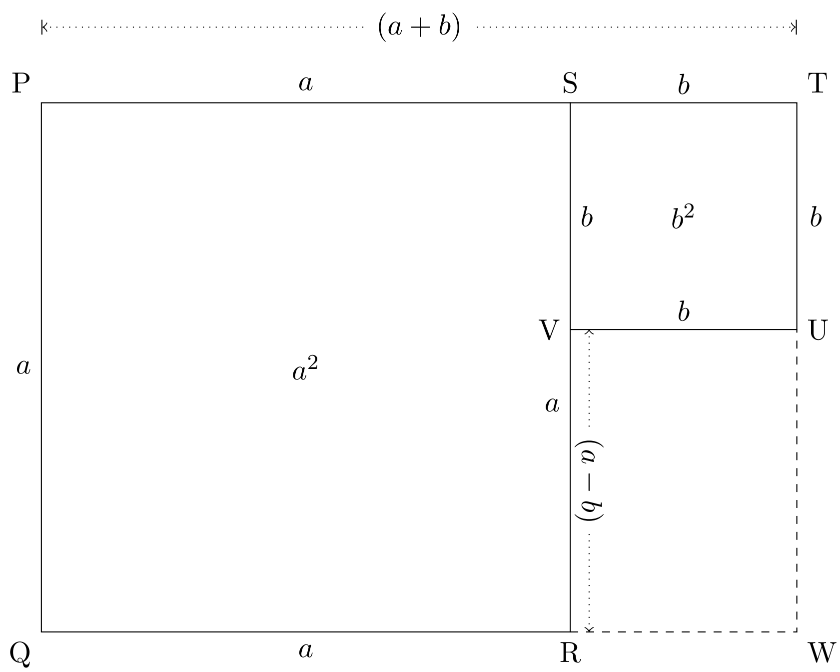

Figure 2: $\square PQRS$ and $\square STUV$ such that points P, S, T are collinear as well as points S, V, R are collinear.

In above Fig.2, $\square PQRS$ and $\square STUV$ are two squares, such that points P, S, T are collinear as well as points S, V, R are also collinear, also TU and QR are extended such that they intersect at point W.

Let

$$PQ = QR = RS = PS = a\,units \tag{3.1}$$

$$SV = VU = UT = ST = b\,units \tag{3.2}$$

Hence,

$$PT = PS + ST$$

putting values from Eqs. (3.1) and (3.2) in above equation,

$$PT = a + b \tag{3.3}$$

and

$$RV = RS - SV$$

putting values from Eqs. (3.1) and (3.2) in above equation,

$$RV = a - b \tag{3.4}$$

We know that area of square = $(side)^2$
So,

$$A(\square PQRS) = (PQ)^2$$

putting value from Eq. (3.1) in above equation,

$$A(\square PQRS) = a^2 \tag{3.5}$$

also,

$$A(\square STUV) = (SV)^2$$

putting value from Eq. (3.2) in above equation,

$$A(\square STUV) = b^2 \tag{3.6}$$

$\square PQWT$ and $\square RWUV$ are rectangles, and area of rectangle = $width \times length$.

So,

$$A(\square PQWT) = PQ \times PT$$

putting values from Eqs. (3.1) and (3.3) in above equation,

$$A(\square PQWT) = a(a + b) \tag{3.7}$$

and,

$$A(\square RWUV) = VU \times RV$$

putting values from Eqs. (3.2) and (3.4) in above equation,

$$A(\square RWUV) = b(a - b) \tag{3.8}$$

We want to find $a^2 + b^2$.
So, to find $a^2 + b^2$ that is $A(\square PQRS) + A(\square STUV)$, we have to subtract $A(\square RWUV)$ from $A(\square PQWT)$

$$A(\square PQRS) + A(\square STUV) = A(\square PQWT) - A(\square RWUV) \tag{3.9}$$

putting values from Eqs. (3.5), (3.6), (3.7) and (3.8) in above Eq. (3.9), we get,

$$a^2 + b^2 = a(a + b) - b(a - b)$$

Hence it is proved that $a^2 + b^2 = a(a + b) - b(a - b)$.

3.2. Examples

Now we will examine proposed formula with some examples.

3.2.1. Example 1:

We will find the sum of squares of -17 and 31
Here $a = -17$ and $b = 31$
So,

$$\begin{aligned} a^2 + b^2 &= a(a+b) - b(a-b) \\ (-17)^2 + 31^2 &= -17(-17+31) - 31(-17-31) \\ 289 + 961 &= -17(14) - 31(-48) \\ 1250 &= -238 + 1488 \\ 1250 &= 1250 \end{aligned}$$

3.2.2. Example 2:

In this example we will find the sum of squares of -3.127 and -17.35
Here $a = -3.127$ and $b = -17.35$
So,

$$\begin{aligned} a^2 + b^2 &= a(a+b) - b(a-b) \\ (-3.127)^2 + (-17.35)^2 &= -3.127(-3.127 - 17.35) - (-17.35)[-3.127 - (-17.35)] \\ 9.778129 + 301.0225 &= -3.127(-20.477) + 17.35(14.223) \\ 310.800629 &= 64.031579 + 246.76905 \\ 310.800629 &= 310.800629 \end{aligned}$$

3.2.3. Example 3:

In this example we will find the sum of squares of $\sqrt{2}$ and 4
Here $a = \sqrt{2}$ and $b = 4$
So,

$$\begin{aligned} a^2 + b^2 &= a(a+b) - b(a-b) \\ (\sqrt{2})^2 + 4^2 &= \sqrt{2}(\sqrt{2}+4) - 4(\sqrt{2}-4) \\ 2 + 16 &= (\sqrt{2})^2 + 4(\sqrt{2}) - 4(\sqrt{2}) + 16 \\ 18 &= 2 + 16 \\ 18 &= 18 \end{aligned}$$

4. Conclusion

In this paper, I have proposed and proved a formula to factorize the sum of squares of two real numbers, also the formula has been tested with different and complex values, which has produced expected and accurate results that makes sure that the proposed formula is flawless.

Hence, from above proof and examples, we can conclude that,

$$a^2 + b^2 = a(a+b) - b(a-b)$$

where a and b are two real numbers.

References

[1] DR Heath-Brown. Fermat's two squares theorem. *Invariant*, 11:3–5, 1984.

[2] Eli Maor. *The Pythagorean theorem: a 4,000-year history*. Princeton University Press, 2007.