A combined Poincaré and conformal Lie algebra

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Abstract

The Poincaré and conformal groups are contenders for the most fundamental spacetime symmetry group. An 8-dimensional rep, putting two 4-spinors together, makes a suitable platform to install matrix representations of these two fundamental groups. But some of their generators do not commute, so new generators are introduced to keep the algebra closed. The combined algebra then has 37 basis generators, a dozen more than needed for the Poincaré and conformal algebras. Interestingly, with two Lorentz subalgebras, one finds two distinct definitions of spin. For the adjoint representation, one set of Lorentz generators reduces to irreducible representations, all with integer spin. The other Lorentz group reduces to both integer and 'half-integer' spin irreducible representations. Also, one finds that the various representations confirm the spin rules for matrix translation generators with the spins of both Lorentz subgroups.

Keywords: Poincaré group; conformal group; spin 1/2; Lie algebra; adjoint rep

1 Introduction

The Poincaré and conformal groups have been well-studied, with a wide range of applications in a vast literature. A sample of applications include quantum field theory,[1] graphene,[2, 3] and theories of gravitation.[4, 5] Furthermore, the matrix representations (reps) of these groups here are based on the transformations of 4-spinors in the Dirac formalism, a bedrock underpinning many explanations of spin 1/2 particle behavior.[1, 6] Each of the ingredients has a high profile.

Individually, these topics have been well-studied. However, I am unaware of any attempts to combine the two algebras in an 8-dimensional matrix representation. The problem makes for an exercise that is complicated enough to display interesting twists.

While there are no applications in mind to motivate the work, the conformal group is largely regarded as being associated with massless fields since a nonzero mass would set a scale.[7] That makes the Poincaré group the go-to fundamental group for massive particle theories. One may speculate that the combined group can explain situations involving a massive and a massless fermion. However, this article remains focused on the algebra and there are no further attempts to develop applications.

To set the terminology, call the subgroup of spacetime rotations, *e.g.* rotations in 3-space and boosts, the "Lorentz" group. The Poincaré group contains the Lorentz group as a subgroup as well as the subgroup of translations of spacetime. The conformal group contains the Poincaré group as a subgroup and has, as

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well, dilations and inversions. The Poincaré group preserves spacetime scalar products while the conformal group preserves the ratios of spacetime scalar products which is less restrictive and allows re-scaling.

The result of combining an 8-dimensional matrix rep of the Poincaré group with a 4-dimensional matrix rep of the conformal group is a closed algebra generated by 8×8 matrices. The 4-dimensional conformal rep can be tucked into a 4×4 block of 8×8 matrices.

While there are 37 basis generators in the combined algebra, that includes the 10 basis generators of the Poincaré algebra and 15 generators of the conformal algebra. Extra generators are needed because the Poincaré and conformal generators require new generators to be defined so that all commutators between members of the combined algebra can be expressed as linear combinations of members, *i.e.* to make the combined algebra closed.

Sec. 2 discusses the overall structure of the combined group. Besides the issue of defining new generators to keep the algebra closed, it is interesting to look at the spin structure. Spin is a property of Lorentz groups. Since the Poincaré and conformal groups each have their own Lorentz subgroup, there are two types of spin defined in the combined group. For one type of spin, all members of the algebra transform as either tensors, vectors, or scalars under spacetime rotations. For the other type, some members transform as tensors, vectors or scalars, but others transform as if the Lorentz rep was 2-dimensional with spin $(0, 1/2)$ or $(1/2, 0)$. Just having one spin removes one of the halves in the spin of a vector, $(1/2, 1/2)$, leaving either $(0, 1/2)$ or $(1/2, 0)$.

Sec. 3 discusses the adjoint rep of the combined algebra, a collection of large, 37×37 matrices. By eigendecomposition of the adjoint rep, one can see the the spin structure of the algebra clearly. The adjoint rep has matrices for both versions of the Lorentz group, the one for the Poincaré algebra and the one for the conformal algebra. The adjoint rep of the Poincaré algebra's Lorentz subgroup reduces to integer spin irreducible reps (irreps), while the conformal group's Lorentz subgroup has spin 1/2 irreps as well as integer spin irreps.

Appendix A presents the choices of conventions and gives some background on Lie algebras. Much of Appendix A provides instructions to build the 8×8 matrix realization that we use to investigate the combined algebra. Appendix B collects the commutation relations of the combined algebra. Appendix C describes the process used to find the irreducible representations (irreps) of the various reducible Lorentz group reps. The results of reducing the two 8-spinor Lorentz reps are given at the end of Appendix C.

2 Combining the algebras

This section describes the two algebras and the algebra produced by combining them. To access a quick reference to basic, general facts about such algebras, consult Appendix A.

The 8-dimensional matrix rep of the Poincaré algebra has a ten generator basis, including six independent matrices from the angular momentum $J_8^{\mu\nu}$ that generate a Lorentz rep of spacetime rotations and the four linear momentum matrices P_8^{μ} that generate translations. They satisfy the Poincaré algebra,[6]

$$[J_8^{\mu\nu}, J_8^{\rho\sigma}] = -i\,(\eta^{\nu\rho} J_8^{\mu\sigma} + \eta^{\mu\sigma} J_8^{\nu\rho} - \eta^{\mu\rho} J_8^{\nu\sigma} - \eta^{\nu\sigma} J_8^{\mu\rho}) \tag{1}$$

$$[J_8^{\mu\nu}, P_8^{\rho}] = -i\,(\eta^{\nu\rho} P_8^{\mu} - \eta^{\mu\rho} P_8^{\nu}) \tag{2}$$

$$[P_8^{\mu}, P_8^{\nu}] = 0 \quad . \tag{3}$$

The commutation relations are homogeneous in P_8^{μ}. It follows that there is a free constant scale factor. We set the scale factor, and others like it, to unity to avoid clutter. Scale factors can be introduced by the interested reader.

In the realization in Appendix A, the 4-dimensional matrix rep of the conformal algebra acts nontrivially only on the second of the two 4-spinors in the 8-spinor. Thus the conformal generators occupy a 4×4 block in 8×8 matrices.

The conformal group has its own angular momenta $M^{\mu\nu}$ that generate a Lorentz rep, two sets of momenta K^{μ} and P^{μ} that generate two reps of translations, and a "dilation" matrix D that generates a rep of multiplication by a scale factor. The commutation relations of the algebra are [6, 8]

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}\right) \tag{4}$$

$$[M^{\mu\nu}, K^{\rho}] = -i\left(\eta^{\nu\rho}K^{\mu} - \eta^{\mu\rho}K^{\nu}\right) \quad ; \quad [M^{\mu\nu}, P^{\rho}] = -i\left(\eta^{\nu\rho}P^{\mu} - \eta^{\mu\rho}P^{\nu}\right) \tag{5}$$

$$[K^{\mu}, P^{\nu}] = +2i\left(\eta^{\mu\nu}D + M^{\mu\nu}\right) \tag{6}$$

$$[K^{\mu}, D] = +iK^{\mu} \quad ; \quad [P^{\mu}, D] = -iP^{\mu} \tag{7}$$

$$[M^{\mu\nu}, D] = [K^{\mu}, K^{\nu}] = [P^{\mu}, P^{\nu}] = [D, D] = 0 \quad . \tag{8}$$

In general, one could define the momenta, $K^{\mu} = k_K K^{\mu}$ and $P^{\mu} = k_P P^{\mu}$, with distinct, arbitrary scale factors, k_K and k_P. Unlike the Poincaré algebra, the conformal algebra is not homogeneous for these momenta. To preserve (6), the product needs to be unity, $k_K k_P = 1$, which requires the factors to be inverses, $k_K = k_P^{-1}$. The interested reader can fill in the consequences of more general choices for k_K and k_P. We choose unity for both factors, $k_K = 1$ and $k_P = 1$.

Since both sets of angular momentum matrices, $J_8^{\mu\nu}$ and $M^{\mu\nu}$, generate a rep of spacetime rotations, the combined group will have two reps of the Lorentz group. Since spin is a property of a Lorentz rep, there will be two distinct sets of spin, identified, for example, as spin_8 and spin_M. Likewise, any other quantities that differ for the $J_8^{\mu\nu}$ and $M^{\mu\nu}$ Lorentz reps will be labeled with a subscript 8 or M.

To begin with, we have $10 + 15 = 25$ linearly independent generators in the basis of the combined algebra. With the construction in Appendix A, the matrices $J_8^{\mu\nu}$ are nonzero in two 4×4 blocks along the diagonal. Each matrix P_8^{μ}, $M^{\mu\nu}$, and D, is nonzero in just one 4×4 block. And the two conformal momenta, *i.e.* K^{μ} and P^{μ}, are each nonzero in 2×2 blocks.

The nonzero commutators of the 2×2 block matrices with the others create problems. The effects include splitting some of the 4×4 block generators and requiring the creation of several new 2×2 block matrix generators to keep the combined algebra closed.

As noted, one consequence is the splitting of initial Poincaré momentum generator into two generators,

$$P_8^{\mu} = P_{8a}^{\mu} + P_{8b}^{\mu} \quad , \tag{9}$$

where, by Appendix A, P_{8a}^{μ} is nonzero in the 41 block and P_{8b}^{μ} is nonzero in the 32 block.

In Appendix A, the 8×8 matrices are sectioned off in 2×2 blocks. The 41 block notation indicates a 2×2 block in the first column of the fourth row.

And there are the newcomers, defined to keep the algebra closed. The new members to the algebra, $J_K^{\mu\nu}$, $J_P^{\mu\nu}$, D_k, and D_P are nonzero in 2×2 blocks in Appendix A. The subscript "K" labels $J_K^{\mu\nu}$ and D_K because they have nonzero commutators with K^{μ}, while $J_P^{\mu\nu}$ and D_P have nonzero commutators with P^{μ}.

One might expect the new angular momentum matrices $J_K^{\mu\nu}$ and $J_P^{\mu\nu}$ to have six linearly independent components like $J_8^{\mu\nu}$ and $M^{\mu\nu}$. However, a 2×2 matrix only has four components. Inspection shows that just three are linearly independent. For the basis, we take J_K^{ij} and J_P^{ij} with $1 \leq i < j \leq 3$, *i.e.* $ij \in \{12, 13, 23\}$.

With the new generators, there are many more generators than the original $10 + 15 = 25$ in the basis. Now the angular momenta, $J_8^{\mu\nu}$, $M^{\mu\nu}$, J_K^{ij}, and J_P^{ij}, number $6+6+3+3 = 18$ linearly independent matrices.

The linear momenta, *e.g.* P^{μ}_{8a}, P^{μ}_{8b}, P^{μ}, K^{μ}, each has four linearly independent matrices, so there are $4+4+4+4 = 16$, altogether. The three scalar matrix generators D, D_K, and D_P, brings the total to

$$N = 18 + 16 + 3 = 37 \quad , \tag{10}$$

so 12 more than the original 25.

The combined algebra is a Lie algebra with a basis of $N = 37$ matrix generators. Selecting a basis means choosing 37 linearly independent matrices from the collection $\{J^{\mu\nu}_8$, $M^{\mu\nu}$, J^{ij}_K, J^{ij}_P, P^{μ}_{8a}, P^{μ}_{8b}, P^{μ}, K^{μ}, D, D_K, $D_P\}$. The choice we use keeps the order indicated in the list. See Appendix A for a detailed list. Appendix A also defines a realization of 8×8 matrices that satisfy the combined algebra.

The commutation relations of the combined algebra are displayed in Appendix B.

It is interesting to compare the status of the generators as tensors, vectors, scalars, or, possibly, none of these with respect to the two reps of the Lorentz group determined by $J^{\mu\nu}_8$ and $M^{\mu\nu}$.

By inspecting the commutation relations with the $J^{\mu\nu}_8$ Lorentz generators, one can determine how the quantities will transform. As prototype commutation relations, look at $[J^{\mu\nu}_8, M^{\rho\sigma}]$ in (A.15) which signals that $M^{\mu\nu}$ transforms as a tensor and $[J^{\mu\nu}_8, P^{\rho}_8]$ in (2) which implies that P^{μ}_8 transforms as a vector. Similar commutation relations signal that quantities behave as tensors or vectors under spacetime rotations generated by $J^{\mu\nu}_8$. The commutators of scalars with $J^{\mu\nu}_8$ vanish.

By the commutation relations with $J^{\mu\nu}_8$ in (A.14) to (A.19), one concludes that all families in the collection $\{J^{\mu\nu}_8$, $M^{\mu\nu}$, J^{ij}_K, J^{ij}_P, P^{μ}_{8a}, P^{μ}_{8b}, P^{μ}, K^{μ}, D, D_K, $D_P\}$ transform as indicated by their spacetime indices μ,ν. This is no accident, the quantities were arranged and the notation was devised to make it so. The spacetime indices of the algebra's members indicate accurately their transformation behavior under the Lorentz rep generated by the angular momentum $J^{\mu\nu}_8$.

2.1 Prototypical spin 1/2 behavior

The situation changes with the $M^{\mu\nu}$-generated Lorentz rep. By comparing the prototype commutation relations $[J^{\mu\nu}_8, M^{\rho\sigma}]$ in (A.15) and $[J^{\mu\nu}_8, P^{\rho}_8]$ in (2) with the commutation relations (A.20) to (A.26) for $M^{\mu\nu}$ with other matrices, one sees that $M^{\mu\nu}$ acts as a tensor, K^{μ} and P^{μ} act as vectors, and D is a scalar under the $M^{\mu\nu}$-Lorentz rep. The others, $\{J^{\mu\nu}_8$, $J^{\mu\nu}_K$, $J^{\mu\nu}_P$, P^{μ}_{8a}, P^{μ}_{8b}, D_K, $D_P\}$ do not fit the prototypes.

The commutators of $J^{\mu\nu}_8$ with $M^{\mu\nu}$ are sums of $M^{\mu\nu}$ in a way that makes $M^{\mu\nu}$ a tensor under rotations generated by $J^{\mu\nu}_8$. Since the commutator is a sum of $M^{\mu\nu}$ matrices, and not $J^{\mu\nu}_8$, it follows that $J^{\mu\nu}_8$ is definitely not a tensor under the rotations generated by $M^{\mu\nu}$.

The rest, *i.e.* $\{J^{\mu\nu}_K$, $J^{\mu\nu}_P$, P^{μ}_{8a}, P^{μ}_{8b}, D_K, $D_P\}$ are neither tensors nor vectors nor scalars under the $M^{\mu\nu}$-Lorentz rep. Since only three of the $J^{\mu\nu}_K$ and three of the $J^{\mu\nu}_P$ are linearly independent, the list contains sixteen independent matrices.

To simplify the discussion in this section, let us make two vector-looking sets of matrices from the $\{J^{\mu\nu}_K$, $J^{\mu\nu}_P$, D_K, $D_P\}$. Define Q^{μ}_K and Q^{μ}_P by

$$Q^{\mu}_K \equiv \{J^{23}_K, J^{31}_K, J^{12}_K, iD_K\} \quad ; \quad Q^{\mu}_P \equiv \{J^{23}_P, J^{31}_P, J^{12}_P, iD_P\} \quad . \tag{11}$$

The two matrix quantities Q^{μ}_K and Q^{μ}_P look like vectors since they have one spacetime index μ, but one can quickly show they are not vectors under the $J^{\mu\nu}_8$ Lorentz rep. Note, for example, that $Q^{4}_K = iD_K$ is scalar and does not transform like the fourth component of a vector under under $J^{\mu\nu}_8$ Lorentz transformations.

The commutation relations of Q^{μ}_K and Q^{μ}_P with $M^{\mu\nu}$ are found to be

$$[M^{\mu\nu}, Q^{\rho}_K] = -\frac{i}{2}\left(\eta^{\nu\rho}Q^{\mu}_K - \eta^{\mu\rho}Q^{\nu}_K\right) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}Q_{K\,\sigma} \quad , \tag{12}$$

$$[M^{\mu\nu}, Q_P^{\rho}] = -\frac{i}{2}\left(\eta^{\nu\rho} Q_P^{\mu} + \eta^{\mu\rho} Q_P^{\nu}\right) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} Q_{P\,\sigma} \quad . \tag{13}$$

Comparing these with $[M^{\mu\nu}, P_{8a}^{\rho}]$ and $[M^{\mu\nu}, P_{8b}^{\rho}]$ in (A.23) and (A.24) shows that all the matrix quantities, $\{Q_K^{\mu}, Q_P^{\mu}, P_{8a}^{\mu}, P_{8b}^{\mu}\}$, obey similar commutation relations with $M^{\mu\nu}$. So we can discuss them all as one.

Consider the generators P_{8a}^{μ}. By (A.6), these matrices are nonzero in just one 2×2 block, the 41 block. One finds at the end of Appendix C that the Lorentz generator $M^{\mu\nu}$ reduces to spin $(0,1/2) \oplus (1/2,0)$, with spin $(0,1/2)$ in the 33-block and spin $(1/2,0)$ in the 44-block. There is no way the 41-block in P_{8a}^{μ} can see the 33 block of $M^{\mu\nu}$; *i.e.* the 33 block does not contribute to the matrix product of P_{8a}^{μ} and $M^{\mu\nu}$. Therefore, P_{8a}^{μ} must transform with spin $(1/2,0)$, the spin of the 44 $M^{\mu\nu}$ block. Similar comments apply to P_{8b}^{μ}, but resulting in spin $(0,1/2)$.

There is a general rule that applies here, see Theorem in Appendix A. Given a quantity Q and a generator X, if the products QX and XQ both vanish then the transformation $R = \exp(i\theta X)$ acts trivially on Q, and we have $RQ = QR = Q$.

Loosely speaking, since the matrix products of P_{8a}^{μ} and the 33 block of $M^{\mu\nu}$, vanish, P_{8a}^{μ} does not transform with the spin_M $(0,1/2)$ generated by the 33 block of $M^{\mu\nu}$. The 44 block contributes to the matrix product of $M^{\mu\nu}$ with P_{8a}^{μ}, and P_{8a}^{μ} transforms with spin_M $(1/2,0)$.

In contrast, the $J_8^{\mu\nu}$ Lorentz rep has two spin $(0,1/2)$ blocks, 11 and 33, and it has two spin $(1/2,0)$ blocks, 22 and 44. The nonzero 41 block of P_{8a}^{μ} has nonvanishing matrix products with the 11 and 44 blocks of $J_8^{\mu\nu}$, which results in P_{8a}^{μ} transforming with spin_8 $(1/2,1/2)$ for the $J_8^{\mu\nu}$ Lorentz rep.

The angular momenta $M^{\mu\nu}$ and quantities $\{Q_K^{\mu}, Q_P^{\mu}, P_{8a}^{\mu}, P_{8b}^{\mu}\}$, have commutation relations (12), (13), (A.23), and (A.24) that are prototypical of quantities that have nonvanishing matrix products with just one spin generator, either the spin $(0,1/2)$ generator or the spin $(1/2,0)$ generator.

We can now identify three types of transformation behavior under spacetime rotations of the Lorentz group. For a Lorentz rep with generators $j^{\mu\nu}$, one compares the commutator of the quantity Q, $[j^{\mu\nu}, Q]$, with prototypes. Those like $[J_8^{\mu\nu}, M^{\rho\sigma}]$ in (A.15) signal tensor, those like $[J_8^{\mu\nu}, P_8^{\rho}]$ in (2) signal vector, and those like $[M^{\mu\nu}, P_{8a}^{\rho}]$ in (A.23) or $[M^{\mu\nu}, P_{8b}^{\rho}]$ in (A.24) signal that Q is a vector-looking quantity that nevertheless transforms with spin 1/2.

In the following section, these properties are reinterpreted with the adjoint rep of the combined algebra.

3 Adjoint rep

The adjoint rep of a given algebra has no more information than the commutation relations. However, it offers a different perspective, since the properties of the commutation relations are reinterpreted as properties of the matrices of the adjoint rep. We suggest that a discussion of the adjoint rep may be an interesting way to view the properties of the combined algebra.

The adjoint rep of the combined algebra has 37 basis matrix generators T^a constructed from the structure constants $s^{ab}{}_{d}$ in (A.1),

$$(T^a)^b_d \equiv -i s^{ab}{}_{d} \quad , \tag{14}$$

where b, d are the row and column indices of the matrix T^a with $a, b, d \in \{1, 2, ..., N = 37\}$. The sequence of the 37 basis matrices T^a is set in detail in Appendix A, Table 5. Roughly, tensors are followed by vectors and then scalars, $\{J_8^{\mu\nu}, M^{\mu\nu}, J_K^{ij}, J_P^{ij}, P_{8a}^{\mu}, P_{8b}^{\mu}, P^{\mu}, K^{\mu}, D, D_K, D_P\}$. One can show that the matrices T^a obey the commutation relations in Appendix B.[6]

We repeat here that there are two Lorentz subalgebras, one generated by the $J_8^{\mu\nu}$ introduced with the Poincaré group and the conformal group version generated by the $M^{\mu\nu}$; see (A.14) and (A.20), respectively.

Spins referenced to the $J_8^{\mu\nu}$ based Lorentz rep are indicated by a subscript "8", as in spin $(A, B)_8$ or similar. The subscript M indicates $M^{\mu\nu}$-rep spins, *e.g.* spin $(A, B)_M$.

3.1 Spins via the $J_8^{\mu\nu}$ Lorentz rep

Let us start with the $J_8^{\mu\nu}$ rep of the Lorentz algebra, (A.14). The first six generators T^a, $a \in \{1, 2, ..., 6\}$, are the basis generators for $J_8^{\mu\nu}$. To decompose the spin of these 37×37 matrices, follow the reduction process in Appendix C. A similarity transformation, S_8, is found that yields new, but equivalent, matrices $\tilde{T}^a$,

$$\tilde{T}^a = S_8^{-1} T^a S_8 \quad . \tag{15}$$

The new $J_8^{\mu\nu}$ basis generators, $\tilde{T}^a$ for $a \in \{1, 2, ..., 6\}$, are block-wise diagonal, having been decomposed into irreps of the Lorentz algebra by the standard process. One finds that the irreps have spins $(A_n, B_n)_8$,

$$3(1,0) \oplus 3(0,1) \oplus 4(\frac{1}{2}, \frac{1}{2}) \oplus 3(0,0) \ . \tag{16}$$

There are 13 irreps.

As discussed in Appendix C, the similarity transformation S_8 is composed of eigenvectors v_m of two matrices, called A^2 and B^2 in the Appendix. By inspecting the nonzero components of the eigenvectors v_m, one can associate generators with each irrep in (16). The results are collected in Table 1.

Generator	$J_8^{\mu\nu}$ irrep(s)	Irrep spin$_8$	Generator	$J_8^{\mu\nu}$ Irreps	irrep spin$_8$
$J_8^{\mu\nu}$	I, IV	$(1,0)$, $(0,1)$	P^{μ}_{8a}, P^{μ}_{8b}	VII,$VIII$	$2(1/2,1/2)$
$M^{\mu\nu}$	II, V	$(1,0)$, $(0,1)$	K^{μ}, P^{μ}	IX,X	$2(1/2,1/2)$
J_K^{ij}	III	$(1,0)$	D, D_K, D_P	$X1$, XII, $XIII$	$3(0,0)$
J_P^{ij}	VI	$(0,1)$			

Table 1: Generators and spins associated with the irreps composing the adjoint rep of $J_8^{\mu\nu}$. The order of irreps is as in (16), *i.e.* irreps I, II, III have spin $(1,0)$, IV, V, VI have $(0,1)$, $VII, VIII, IX, X$ have $(1/2, 1/2)$, $XI, XII, XIII$ have spin $(0,0)$. All these irreps have integer spin, since, for every (A_n, B_n), one has $A_n + B_n$ equal to an integer. The irrep spins reflect the type of generator, *i.e.* $(0,1)$ and $(1,0)$ for tensors, $(1/2, 1/2)$ for vectors, and $(0,0)$ for scalars.

The results could have been anticipated based on the way the adjoint rep is constructed and the fact that all basis generators transform as either tensors, vectors or scalars under the $J_8^{\mu\nu}$ Lorentz transformations.

Consider, for example, the commutation relation (A.18). There the commutator $[J_8^{\mu\nu}, P^{\rho}_{8a}]$ is a sum of P^{μ}_{8a} generators. By Table 5, the six basis generators $J_8^{\mu\nu}$ are given the six indices $a = 1-6$, while the P^{μ}_{8a} have the four indices $a = 19-22$. Thus the adjoint rep of all six basis generators, $a = 1-6$, for $J_8^{\mu\nu}$ have nonzero components with P^{ρ}_{8a} row and column indices $b = 19-22$ and $d = 19-22$. This makes a 4×4 block in the same location along the diagonal of each of the six basis generators. From the table, we see that this is irrep VII, one of the 4-dimensional irreps with spin$_8$ $(1/2, 1/2)$.

Similar statements apply to the commutation relations of $J_8^{\mu\nu}$ with all the basis generators.

However one gets there, the spin$_8$ $(1,0)$ and $(0,1)$ irreps are 3-dimensional, the $(1/2, 1/2)$ irreps are 4-dimensional, and the scalars $(0,0)$ are 1-dimensional, so the total from (16) checks, $3(3) + 3(3) + 4(4) + 1(3) = 37$, as it must.

Next, check the new vector matrices $\tilde{T}^a$, (15), for $\{P^\mu_{8a},\ P^\mu_{8b},\ P^\mu,\ K^\mu\}$. It is known that vector matrices connect Lorentz irreps with spins $(A, B; C, D)_8$ where $C = A \pm 1/2$ and $D = B \pm 1/2$. [9, 10] With the $\tilde{T}^a$ one can determine the $(A, B; C, D)_8$ by inspection. The nonzero components of these matrices occur in blocks aligned with the irreps of the six basis generators. Simply locate the nonzero blocks and see which two irreps are connected. The results are collected in Table 2.

Vector$_8$	Irreps$_8$	$(A, B; C, D)_8$	Irreps$_8$	$(A, B; C, D)_8$
P^μ_{8a}	(I, VII)	$(1, 0; 1/2, 1/2)$	(II, VII)	$(1, 0; 1/2, 1/2)$
	(IV, VII)	$(0, 1; 1/2, 1/2)$	(XI, VII)	$(0, 0; 1/2, 1/2)$
P^μ_{8b}	$(I, VIII)$	$(1, 0; 1/2, 1/2)$	$(IV, VIII)$	$(0, 1; 1/2, 1/2)$
	$(V, VIII)$	$(0, 1; 1/2, 1/2)$	$(XI, VIII)$	$(0, 0; 1/2, 1/2)$
K^μ	(I, IX)	$(1, 0; 1/2, 1/2)$	(II, IX)	$(1, 0; 1/2, 1/2)$
	(IV, IX)	$(0, 1; 1/2, 1/2)$	(V, IX)	$(0, 0; 1/2, 1/2)$
	(XI, IX)	$(0, 0; 1/2, 1/2)$		
P^μ	(I, X)	$(1, 0; 1/2, 1/2)$	(II, X)	$(1, 0; 1/2, 1/2)$
	(IV, X)	$(0, 1; 1/2, 1/2)$	(V, X)	$(0, 0; 1/2, 1/2)$
	(XI, X)	$(0, 0; 1/2, 1/2)$		

Table 2: Vector matrices P^μ_{8a}, P^μ_{8b}, K^μ, P^μ connect the Lorentz irreps in (16). Note that the spins $(A, B; C, D)_8$ obey $C = A \pm 1/2$ and $D = B \pm 1/2$, as required for vector and momentum matrices.[9, 10] The subscript "8" mean that these spins and the term "vector" refer to the spacetime rotations generated by $J^{\mu\nu}_8$.

By Table 1, all the irreps $I - XIII$ have integer spins, meaning $A_n + B_n$ is an integer, which are suitable for their associated generators, either tensors, vectors, or scalars. This changes for the adjoint rep of the $M^{\mu\nu}$ Lorentz group, which is discussed next.

3.2 Spins via the $M^{\mu\nu}$ Lorentz rep

The evaluation of the spins$_M$ of the adjoint rep with respect to the set of $M^{\mu\nu}$ generators is more complicated. The spin 1/2 behavior discussed in Sec. 2.1 has consequences in this section.

The irreps of the adjoint rep for the $M^{\mu\nu}$ Lorentz algebra are found with eigendecomposition as described in Appendix C. Starting with the six $M^{\mu\nu}$ basis generators T^a for $a \in \{7, 8, ..., 12\}$, one finds a similarity transformation, S_M, that gives a new, primed set of basis generators,

$$T^{a'} = S^{-1}_M T^a S_M \quad . \tag{17}$$

The primed rep satisfies the algebra in Appendix B, just like the unprimed and tilde bases before it. But, now the six primed $M^{\mu\nu}$ basis generators are reduced to block diagonal form.

The spins $(A_n, B_n)_M$ of the irreps are found to be

$$I \oplus II \oplus (III, IV) \oplus (V - VIII) \oplus (IX - XII) \oplus (XIII - XIX) =$$
$$(1, 0) \oplus (0, 1) \oplus 2\left[(\frac{1}{2}, \frac{1}{2})\right] \oplus 4\left[(\frac{1}{2}, 0)\right] \oplus 4\left[(0, \frac{1}{2})\right] \oplus 7\left[(0, 0)\right] \quad . \tag{18}$$

Just as we saw before with $J_8^{\mu\nu}$, the nonzero components of the process's eigenvectors betray their association with the irreps. For example, the first eigenvector is found to have nonzero components at $a = 7$ and 12. By the sequence in Table 5, these indices identify M^{12} and M^{34}. Thus the structure constants for the commutation relations of $M^{\mu\nu}$ with M^{12} and M^{34} contribute to the spin$_M$ $(1,0)$ irrep, which is the first irrep in (18) with spin$_M$ $(1,0)$. See Table 3.

Generator	$M^{\mu\nu}$ irreps	Irrep spin$_M$	Generator	$M^{\mu\nu}$ irreps	Irrep spin$_M$
$J_8^{\mu\nu}$	I, II	$(1,0)$, $(0,1)$	P_{8a}^{μ}	VII,$VIII$	$2(1/2,0)$
$M^{\mu\nu}$	I, II	$(1,0)$, $(0,1)$	P_{8b}^{μ}	XI,XII	$2(0,1/2)$
J_K^{ij},D_K	IX, X	$2(0,1/2)$	K^{μ}, P^{μ}	IV, III	$2(1/2,1/2)$
J_P^{ij},D_P	V, VI	$2(1/2,0)$	$J_8^{\mu\nu}$, D	$XIII$-XIX	$7(0,0)$

Table 3: Generators and spins associated with the irreps in (18) composing the reducible adjoint rep of $M^{\mu\nu}$. To be identify a generator with an irrep, the irrep's eigenvectors must have nonzero components at the index corresponding to the generator. Note that the generators $\{P_{8a}^{\mu}$, P_{8b}^{μ}, J_K^{ij}, J_P^{ij}, D_K, $D_P\}$ discussed in Sec. 2.1 are related to spin 1/2 irreps.

The spin $(1,0)$ and $(0,1)$ irreps are 3-dimensional, the $(1/2,1/2)$ irreps are 4-dimensional, the $(1/2,0)$ and $(0,1/2)$ irreps are 2-dimensional, and the scalars $(0,0)$ are 1-dimensional, so one can check that the total adds up to 37. We have $2(3)+2(4)+4(2)+4(2)+1(7) = 37$, as it must.

The vector matrices connect spins $(A,B;C,D)_M$ where $C = A \pm 1/2$ and $D = B \pm 1/2$. With the transformed set of adjoint matrices, $T^{a'}$, one can determine the spins $(A,B;C,D)_M$ by inspection. By locating the nonzero components of the vector matrices, one can see which two irrep blocks are connected. The results are collected in Table 4.

Vector$_M$	Irreps$_8$	$(A,B;C,D)_8$	Irreps$_8$	$(A,B;C,D)_8$
K^{μ}	(I,IV)	$(1,0;1/2,1/2)$	(II,IV)	$(0,1;1/2,1/2)$
	(XIX,IV)	$(0,0;1/2,1/2)$		
P^{μ}	(I,III)	$(1,0;1/2,1/2)$	(II,III)	$(0,1;1/2,1/2)$
	(XIX,III)	$(0,0;1/2,1/2)$		

Table 4: Vector matrices K^{μ} and P^{μ} connect the irreps in (18). Note that the spins $(A,B;C,D)_M$ obey $C = A \pm 1/2$ and $D = B \pm 1/2$ which is required for vector and momentum matrices. The subscript "M"s mean that these spins and the term "vector" reference the spacetime rotations generated by $M^{\mu\nu}$.

By Table 3, integer and half-integer spin irreps are mixed in the decomposition of the adjoint rep of the $M^{\mu\nu}$ Lorentz subalgebra. These spin types are associated with particles obeying the identical particle statistics of bosons and fermions. By Table 1, the $J_8^{\mu\nu}$ Lorentz subalgebra consists entirely of boson-associated spins.

A Generator Matrices

This appendix gives the algebra members' definitions as 8-dimensional matrices. The section also contains convention settings and some background. A notebook detailing the construction, verifying the claims of the previous sections and the commutation relations of the following section is available on-line or by request.[13, 14]

Let the spacetime metric $\eta^{\mu\nu}$ be $\eta^{\mu\nu} = \eta_{\mu\nu} = \text{diag}\{+1,+1,+1,-1\}$. The indices $\mu, \nu \in \{1,2,3,4\}$ refer to Minkowski spacetime with $\mu = 4$ as the time index. The metric raises and lowers indices, e.g. $\sigma^{\mu} = \eta^{\mu\nu}\sigma_{\nu}$. The summation convention for repeated indices is in force, unless otherwise stated.

The Lie algebra discussed in the text has a basis of $N = 37$ generators. The generators are 8×8 matrices with complex components and the transformations R act on 8-component quantities ψ called "8-spinors", $\psi\,' = R\psi$. Many of the generators are not Hermitian, $X \neq X^{\dagger}$ its complex conjugate transpose, so the group they generate is not unitary. The distinction separating "members" of the algebra and "generators" of group transformations is often disregarded.

Recall some concepts.[6, 11, 12] Consider an $n \times n$ matrix Lie algebra that has a basis of N generators X^a, $\{X^1, X^2, ..., X^N\}$, and uses the commutator between any two member matrices M^a, M^b, *i.e.* $\left[M^a, M^b\right]$, as the "product" operation in the algebra. Linear combinations of members with complex coefficients are members. And the commutator of two member matrices is a member of the algebra. Thus the commutators of generators are expressible as linear combinations of the generators, called commutation relations,

$$\left[X^a, X^b\right] \equiv X^a X^b - X^b X^a = \sum_{c=1}^{N} i s^{ab}{}_{c} X^c \;, \tag{A.1}$$

where matrix multiplication is understood. The coefficients s^{ab}_{c} are called "structure constants." Generators are identified by superscripts such as a, b, c, with $a, b, c \in \{1, 2, ..., N\}$. In the associated transformation group, each generator, say X^a, generates a transformation $R^a(\theta) = \exp\left(\pm i\theta X^a\right)$, where θ is a real valued parameter. The sign is conventionally negative when a momentum generates a translation and positive otherwise.

Theorem: Given a quantity Q and a generator X, if the products QX and XQ both vanish then the transformation $R = \exp(i\theta X)$ acts trivially on Q, $RQ = QR = Q$. The proof is left for the interested reader.

Type$_8$	Generator #	X^a	Generator #	X^a	Notes
Tensors	$1 \leq a \leq 6$	$J_8^{\mu\nu}$	$7 \leq a \leq 12$	$M^{\mu\nu}$	$\mu\nu = (12, 13, 14, 23, 24, 34)$
	$13 \leq a \leq 15$	J_K^{ij}	$16 \leq a \leq 18$	J_P^{ij}	$ij = (12, 13, 23)$
Vectors	$19 \leq a \leq 22$	P_{8a}^{μ}	$23 \leq a \leq 26$	P_{8b}^{μ}	$\mu = (1, 2, 3, 4)$
	$27 \leq a \leq 30$	K^{μ}	$31 \leq a \leq 34$	P^{μ}	
Scalars	$a = 35$	D	$a = 36$	D_K	No spacetime index for scalars
	$a = 37$	D_P			

Table 5: The sequence of 37 basis generators. The "8" in "type$_8$" is a reminder that the terms "Tensor", "Vector", and "Scalar" refer to how the quantities transform under the spacetime rotations generated with $J_8^{\mu\nu}$, not $M^{\mu\nu}$. For examples, with $a = 11, 19, 36$, the generators are $X^{11} = M^{24}$, $X^{19} = P_{8a}^{1}$, $X^{36} = D_K$.

A.1 Define matrices

The matrices are all 8×8 matrices, with complex components. To define the matrices, it is convenient to address 2×2 matrix blocks in a 4-by-4 array. Thus the"41-block means the first 2×2 block in the fourth row and the "32-block" is the second block in the third row, *etc.* The nonzero 2×2 blocks of the matrices are defined in terms of the Pauli matrices, σ^μ, given by

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \sigma^y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \; ; \; \sigma^z = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \; ; \; \sigma^t = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} \quad . \tag{A.2}$$

In some expressions we use indices $\{1,2,3,4\}$ instead of $\{x,y,z,t\}$.

We now give nonzero 2×2 blocks of the matrices, starting with the tensors, $J_8^{\mu\nu}$, $M^{\mu\nu}$, $J_K^{\mu\nu}$, and $J_P^{\mu\nu}$. We define

$$\left(J_8^{ij}\right)_{11} = \left(J_8^{ij}\right)_{22} = \left(J_8^{ij}\right)_{33} = \left(J_8^{ij}\right)_{44} = \frac{1}{2}\epsilon^{ijk}\sigma^k \quad , \tag{A.3}$$

$$\left(J_8^{k4}\right)_{11} = -\left(J_8^{k4}\right)_{22} = \left(J_8^{k4}\right)_{33} = -\left(J_8^{k4}\right)_{44} = \frac{i}{2}\sigma^k \quad ,$$

$$\left(M^{ij}\right)_{33} = \left(M^{ij}\right)_{44} = \frac{1}{2}\epsilon^{ijk}\sigma^k \quad ; \quad \left(M^{k4}\right)_{33} = -\left(M^{k4}\right)_{44} = \frac{i}{2}\sigma^k \quad , \tag{A.4}$$

$$\left(J_K^{ij}\right)_K = -\frac{1}{2}\epsilon^{ijk}\sigma^k \; ; \; \left(J_K^{k4}\right)_K = -\frac{i}{2}\,\sigma^k \quad ; \quad \left(J_P^{ij}\right)_P = -\frac{1}{2}\epsilon^{ijk}\sigma^k \; ; \; \left(J_P^{k4}\right)_P = +\frac{i}{2}\,\sigma^k \quad . \tag{A.5}$$

While there are 16 matrices $J_8^{\mu\nu}$ for $1 \leq \mu, \nu \leq 4$, antisymmetry, *i.e.* $J_8^{\nu\mu} = -J_8^{\mu\nu}$, makes just six nontrivial and independent. For our basis we choose the 6 pairs $\mu\nu \in \{12, 13, 14, 23, 24, 34\}$ of nonrepeated integers from 1 to 4. Thus $J_8^{\mu\nu}$ and $M^{\mu\nu}$ each has 6 linearly independent generators.

But $J_K^{\mu\nu}$ and $J_P^{\mu\nu}$ have nonzero components only in one 2×2 matrix, the 31 and 42 block, respectively. At most four can be linearly independent. Inspection of (A.5) shows that the generator with space-space indices ij and the generator with indices $k4$, $k \neq i,j$, are proportional to the same matrix σ^k, $k \in \{1,2,3\}$. Thus just three of the $J_K^{\mu\nu}$ and three of the $J_P^{\mu\nu}$ are linearly independent. There are a total of $6+6+3+3$ $= 18$ linearly independent tensor generators J.

There are four sets of vector generators, P_{8a}^μ, P_{8b}^μ, K^μ, P^μ. The nonzero 2×2 blocks are

$$(P_{8a}^\mu)_{41} = i\,\sigma_\mu \; ; \; (P_{8b}^\mu)_{32} = -i\,\sigma^\mu \; ; \; (K^\mu)_{43} = -i\,\sigma_\mu \; ; \; (P^\mu)_{34} = +i\sigma^\mu \quad . \tag{A.6}$$

These are $4+4+4+4 = 16$ linearly independent matrix generators.

The nonzero components of the three scalar generators, *i.e.* D, D_K, D_P, occupy different 2×2 blocks,

$$(D)_{33} = -(D)_{44} = +\frac{i}{2}\,\sigma^4 \quad ; \quad (D_K)_K = +\frac{i}{2}\,\sigma^4 \quad ; \quad (D_P)_P = -\frac{i}{2}\,\sigma^4 \quad . \tag{A.7}$$

The three generators are linearly independent.

The tensors, vectors, and scalars defined above contribute to the basis generators. The total number of linearly independent generators is $18 + 16 + 3 = 37$. By construction, a linear combination of all 37 basis generators vanishes only for a set of null coefficients, confirming that the 37 generators are linearly independent.

A.2 Spacial inversion, parity

When acting on spacetime coordinates x^μ, spacial inversion, *i.e.* parity, produces $-x_\mu = -\eta_{\mu\nu}x^\nu$, since our spacetime metric $\eta_{\mu\nu}$ is the diagonal matrix diag$\{+1,+1,+1,-1\}$. Also, note that left- and right-handed quantities exchange under parity. The 8×8 parity matrix β we use has the following nonzero blocks,

$$\beta_{(12)} = \beta_{(21)} = \beta_{(34)} = \beta_{(43)} = \sigma^4 \quad . \tag{A.8}$$

The inverse of the parity matrix β is β, *i.e.* $\beta^2 = 1$.

The tensors $J_8^{\mu\nu}$ and $M^{\mu\nu}$ behave properly,

$$\beta J_8^{\mu\nu}\beta^{-1} = +J_{8\,\mu\nu} \quad ; \quad \beta M^{\mu\nu}\beta^{-1} = +M_{\mu\nu} \quad . \tag{A.9}$$

The tensors $J_K^{\mu\nu}$ and $J_P^{\mu\nu}$ behave understandably, but not properly. One has

$$\beta J_K^{\mu\nu}\beta^{-1} = +J_{P\,\mu\nu} \quad ; \quad \beta J_P^{\mu\nu}\beta^{-1} = +J_{K\,\mu\nu} \quad . \tag{A.10}$$

These two, $J_K^{\mu\nu}$ and $J_P^{\mu\nu}$, are nonzero in small 2×2 blocks and have opposite handedness. They switch places under parity.

Much the same happens with momenta. One finds that $\beta P_8^\mu\beta^{-1} = -P_{8\,\mu}$, as a vector should, but

$$\beta P_{8a}^\mu\beta^{-1} = -P_{8b\,\mu} \quad ; \quad \beta P_{8b}^\mu\beta^{-1} = -P_{8a\,\mu} \quad , \tag{A.11}$$

$$\beta K^\mu\beta^{-1} = -P_\mu \quad ; \quad \beta P^\mu\beta^{-1} = -K_\mu \quad . \tag{A.12}$$

The momenta P_{8a}^μ, P_{8b}^μ, K^μ, P^μ each has definite handedness. They react pairwise to spacial inversion.

Parity preserves "true" scalars, such as the scalar product of two vectors $x_\mu y^\mu$. "Pseudoscalars" change sign under spacial inversion. For the generators that are scalars under Lorentz transformations, one finds that

$$\beta D\beta^{-1} = -D \quad ; \quad \beta D_K\beta^{-1} = -D_P \quad ; \quad \beta D_P\beta^{-1} = -D_K \quad , \tag{A.13}$$

so there is the pair D_K and D_P, each with definite handedness. Only D acts as a pseudoscalar under parity, the two others have pseudoscalar like behavior, but pairwise.

B Commutation relations

The Lie algebra results from combining a Poincaré algebra and a conformal algebra. The 10 plus 15 basis generators of these two algebras are taken as given matrices.

Finding the commutators of the initial 25 Poincaré/conformal generators is straightforward. Some of the commutators require introducing new members of the algebra so that the commutators can be written as linear combinations of generators. The new members are introduced as required to keep the algebra closed.

The basis of the combined algebra contains 37 linearly independent generators. The algebra consists of linear combinations of the generators $\{J_8^{\mu\nu}, M^{\mu\nu}, J_K^{\mu\nu}, J_P^{\mu\nu}, P_{8a}^\mu, P_{8b}^\mu, K^\mu, P^\mu, D, D_K, D_P\}$. The commutator of any two members of the algebra is a member of the algebra.

Only the commutators that don't vanish are displayed in the following. Nonzero commutation relations with $J_8^{\mu\nu}$:

$$[J_8^{\mu\nu}, J_8^{\rho\sigma}] = -i\left(\eta^{\nu\rho}J_8^{\mu\sigma} + \eta^{\mu\sigma}J_8^{\nu\rho} - \eta^{\mu\rho}J_8^{\nu\sigma} - \eta^{\nu\sigma}J_8^{\mu\rho}\right) \tag{A.14}$$

$$[J_8^{\mu\nu}, M^{\rho\sigma}] = -i\left(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}\right) \tag{A.15}$$

$$[J_8^{\mu\nu}, J_K^{\rho\sigma}] = -i\,(\eta^{\nu\rho} J_K^{\mu\sigma} + \eta^{\mu\sigma} J_K^{\nu\rho} - \eta^{\mu\rho} J_K^{\nu\sigma} - \eta^{\nu\sigma} J_K^{\mu\rho}) \tag{A.16}$$

$$[J_8^{\mu\nu}, J_P^{\rho\sigma}] = -i\,(\eta^{\nu\rho} J_P^{\mu\sigma} + \eta^{\mu\sigma} J_P^{\nu\rho} - \eta^{\mu\rho} J_P^{\nu\sigma} - \eta^{\nu\sigma} J_P^{\mu\rho}) \tag{A.17}$$

$$[J_8^{\mu\nu}, P_{8a}^{\rho}] = -i\,(\eta^{\nu\rho} P_{8a}^{\mu} - \eta^{\mu\rho} P_{8a}^{\nu}) \quad ; \quad [J_8^{\mu\nu}, P_{8b}^{\rho}] = -i\,(\eta^{\nu\rho} P_{8b}^{\mu} - \eta^{\mu\rho} P_{8b}^{\nu}) \tag{A.18}$$

$$[J_8^{\mu\nu}, K^{\rho}] = -i\,(\eta^{\nu\rho} K^{\mu} - \eta^{\mu\rho} K^{\nu}) \quad ; \quad [J_8^{\mu\nu}, P^{\rho}] = -i\,(\eta^{\nu\rho} P^{\mu} - \eta^{\mu\rho} P^{\nu}) \tag{A.19}$$

The commutation relations of generators X with M,

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\,(\eta^{\nu\rho} M^{\mu\sigma} + \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho}) \tag{A.20}$$

$$\begin{aligned}[M^{\mu\nu}, J_K^{\rho\sigma}] &= -\frac{i}{2}\,(\eta^{\nu\rho} J_K^{\mu\sigma} + \eta^{\mu\sigma} J_K^{\nu\rho} - \eta^{\mu\rho} J_K^{\nu\sigma} - \eta^{\nu\sigma} J_K^{\mu\rho}) \\ &\quad + \frac{i}{2}\,(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma})\, D_K - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} D_K\end{aligned} \tag{A.21}$$

$$\begin{aligned}[M^{\mu\nu}, J_P^{\rho\sigma}] &= -\frac{i}{2}\,(\eta^{\nu\rho} J_P^{\mu\sigma} + \eta^{\mu\sigma} J_P^{\nu\rho} - \eta^{\mu\rho} J_P^{\nu\sigma} - \eta^{\nu\sigma} J_P^{\mu\rho}) \\ &\quad - \frac{i}{2}\,(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma})\, D_P - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} D_P\end{aligned} \tag{A.22}$$

$$[M^{\mu\nu}, P_{8a}^{\rho}] = -\frac{i}{2}\,(\eta^{\nu\rho} P_{8a}^{\mu} - \eta^{\mu\rho} P_{8a}^{\nu}) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_{8a\sigma} \tag{A.23}$$

$$[M^{\mu\nu}, P_{8b}^{\rho}] = -\frac{i}{2}\,(\eta^{\nu\rho} P_{8b}^{\mu} - \eta^{\mu\rho} P_{8b}^{\nu}) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_{8b\sigma} \tag{A.24}$$

$$[M^{\mu\nu}, K^{\rho}] = -i\,(\eta^{\nu\rho} K^{\mu} - \eta^{\mu\rho} K^{\nu}) \quad ; \quad [M^{\mu\nu}, P^{\rho}] = -i\,(\eta^{\nu\rho} P^{\mu} - \eta^{\mu\rho} P^{\nu}) \tag{A.25}$$

$$[M^{\mu\nu}, D_K] = -\frac{i}{2} J_K^{\mu\nu} \quad ; \quad [M^{\mu\nu}, D_P] = +\frac{i}{2} J_P^{\mu\nu} \tag{A.26}$$

The commutation relations of generators with J_K and J_P not listed previously:

$$[J_K^{\mu\nu}, K^{\rho}] = -\frac{i}{2}\,(\eta^{\nu\rho} P_{8a}^{\mu} - \eta^{\mu\rho} P_{8a}^{\nu}) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_{8a\sigma} \tag{A.27}$$

$$[J_P^{\mu\nu}, P^{\rho}] = -\frac{i}{2}\,(\eta^{\nu\rho} P_{8b}^{\mu} - \eta^{\mu\rho} P_{8b}^{\nu}) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_{8b\sigma} \tag{A.28}$$

$$[J_K^{\mu\nu}, D] = -\frac{i}{2} J_K^{\mu\nu} \quad ; \quad [J_P^{\mu\nu}, D] = -\frac{i}{2} J_P^{\mu\nu} \tag{A.29}$$

The commutation relations of generators with P_{8a}^{μ} and P_{8b}^{μ},

$$[P_{8a}^{\mu}, P^{\nu}] = -2i\,(\eta^{\mu\nu} D_K - J_K^{\mu\nu}) \tag{A.30}$$

$$[P_{8b}^{\mu}, K^{\nu}] = +2i\,(\eta^{\mu\nu} D_P + J_P^{\mu\nu}) \tag{A.31}$$

$$[P_{8a}^{\mu}, D] = +\frac{i}{2} P_{8a}^{\mu} \quad ; \quad [P_{8b}^{\mu}, D] = -\frac{i}{2} P_{8b}^{\mu} \tag{A.32}$$

The not-yet-listed commutation relations of generators with K and P,

$$[K^{\mu}, P^{\nu}] = +2i\,(\eta^{\mu\nu} D + M^{\mu\nu}) \tag{A.33}$$

$$[K^{\mu}, D] = +iK^{\mu} \quad ; \quad [K^{\mu}, D_K] = -\frac{i}{2} P_{8a}^{\mu} \quad ; \quad [P^{\mu}, D] = -iP^{\mu} \quad ; \quad [P^{\mu}, D_P] = +\frac{i}{2} P_{8b}^{\mu} \tag{A.34}$$

Finally, the remaining nonzero commutation relations that involve D,

$$[D, D_K] = +\frac{i}{2} D_K \quad ; \quad [D, D_P] = -\frac{i}{2} D_P \tag{A.35}$$

Commutators that vanish are not displayed.

C Reducing Lorentz reps

The standard process used to reduce Lorentz reps to their irreducible components, irreps, is outlined here for convenience and completeness.[6, 12] The irreps making up the adjoint reps of the $J_8^{\mu\nu}$- and $M^{\mu\nu}$-Lorentz algebra are displayed in (16) and (18). Results for the 8-dimensional matrix rep in Appendix A are provided at the end of this appendix.

Let the matrices $J^{\mu\nu}$ generate a rep of the Lorentz algebra. These matrices satisfy the usual Lorentz algebra commutation relations, similar to (A.14). Define generators J^i and K^i as

$$(J^1, J^2, J^3) \equiv (J^{23}, J^{31}, J^{12}) \quad ; \quad (K^1, K^2, K^3) \equiv (J^{14}, J^{24}, J^{34}) \quad . \tag{A.36}$$

The J^i and K^i generate rotations in space and boosts along coordinate axes. Next define A^i and B^i by

$$A^i \equiv \frac{1}{2}\left(J^i + iK^i\right) \quad ; \quad B^i \equiv \frac{1}{2}\left(J^i - iK^i\right) \quad . \tag{A.37}$$

One can show that A^i and B^i each represent the group of rotations in 3-space. One finds that A^i and B^i commute, so they are independent.

One calculates the square of the magnitude A^iA^i, which is the sum of the components squared because repeated indices are summed. If the A^i form an irrep of the Lorentz group, then the eigenvalues of the square A^iA^i are $A_n(A_n+1)$ for some integer or half integer A_n, *i.e.* $2A_n$ is an integer.

One also finds the square B^iB^i. Similar comments apply to B^i. Since A^i and B^i commute, they can share eigenvectors. A note of caution: linear combinations of eigenvectors with the same eigenvalues may be needed to get a collection of useful eigenvectors. This is especially true with eigenvectors whose eigenvalues vanish.

For an irrep with spin (A_n, B_n), there are a number of eigenvectors v_m of $A^2 = A^iA^i$ and B^2 with the same eigenvalues $A_n(A_n+1)$ and $B_n(B_n+1)$. The number of eigenvectors v_m for a particular irrep is $(2A_n+1)(2B_n+1)$. The nonzero components of the v_m tell us which generators contribute to the nth irrep.

The collection of eigenvectors can be organized as columns of a matrix S

$$S_n^i = v_n^i \quad . \tag{A.38}$$

The similarity transformation determined by S, can be shown to reduce the matrices $J^{\mu\nu}$ to block-diagonal form, with a Lorentz irrep in each block.

$$J^{\mu\nu\prime} \equiv S^{-1} J^{\mu\nu} S \quad . \tag{A.39}$$

The nonzero components of the eigenvector v_n occur in the range of indices for the corresponding block of its irrep. And, with that comment, the process is concluded.

Applying the process to the 37×37 matrices of the adjoint rep for $J_8^{\mu\nu}$ yields the irreps in (16). For the 37×37 matrices of the adjoint rep for $M^{\mu\nu}$, the result is presented in (18).

For the 8×8 matrix rep of the Lorentz group generated by the $J_8^{\mu\nu}$ the above process determines its spin$_8$ reduction into irreps,

$$I \oplus II \oplus III \oplus IV = (0, 1/2) \oplus (1/2, 0) \oplus (0, 1/2) \oplus (1/2, 0) \quad . \tag{A.40}$$

By the process of interrogating the eigenvectors of the irreps discussed in the text, one finds that P_{8a}^{μ} connects reps IV and I with spins $(A, B; C, D) = (1/2, 0; 0, 1/2)$, while P_{8b}^{μ} connects III and II with spins $(A, B; C, D)$

$= (0,1/2;1/2,0)$. One finds that the momenta P_P^μ and $P_{K\,\mu}$ connect irreducible Lorentz reps III and IV with spins $(1/2,0;0,1/2)$ for P_P^μ and $(0,1/2;1/2,0)$ for $P_{K\,\mu}$. These spins satisfy the rules $C = A \pm 1/2$ and $D = B \pm 1/2$ for irreps connected by vector and momentum matrices.

One finds that the spins $(A_n, B_n)_M$ of the $M^{\mu\nu}$ Lorentz rep are $4[(0,0)] \oplus (0,1/2) \oplus (1/2,0)$, *i.e.* four trivial reps plus a 4-dimensional rep based on the Dirac formalism. By the construction in Appendix A, it makes sense that four irreps are trivial because $M^{\mu\nu}$ has nonzero components only in a 4×4 corner of otherwise null 8×8 matrices. The momentum K^μ connects spins $(A,B;C,D)_M = (0,1/2;1/2,0)$ while P^μ connects spins $(A,B;C,D)_M = (1/2,0;0,1/2)$. The vector/momentum $(A,B;C,D)_M$ spin rule is again confirmed.

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