The Zeta Induction Theorem: The Simplest Equivalent to the Riemann Hypothesis?

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Abstract

This paper presents an uncommon variation of proof by induction. We call it deferred induction by recursion. To set up our proof, we state (but do not prove) the *Zeta Induction Theorem*. We then assume that theorem is true and provide an elementary proof of the Riemann Hypothesis (showing their equivalence).

1 Introduction

We define (but do not prove) the *Zeta Induction Theorem* below. Using that theorem, we provide a simple and (hopefully) interesting proof of the Riemann Hypothesis. Our proof uses "deferred induction by recursion". To be clear, we make no claim as to the usefulness of the *Zeta Induction Theorem* to the theory of the Riemann Zeta Function.

2 Definitions

In all that follows, the definitions below are assumed:

Definition. For $m, n \in \mathbb{N}$, define $A_m = \sum_{j=1}^{m} \left(\frac{1}{2}\right)^j$ and $B_{m,n} = \left(\frac{1}{2}\right)^{m+n}$

Definition. For $t \in \mathbb{R}_{>0}$, define $\epsilon(t) = \dfrac{1}{8.463 \cdot log(|t|+2)}$

Definition. Fix $t \in \mathbb{R}_{>0}$. For $s \in \mathbb{C}$, define the following open rectangles:

$$
\begin{aligned}
R_R(t) &= \tfrac{1}{2} < Re(s) < 1; & |Im(s)| < t \\
R_L(t) &= 0 < Re(s) < \tfrac{1}{2}; & |Im(s)| < t \\
R_\epsilon(t) &= 1 - \epsilon(t) < Re(s) < 1; & |Im(s)| < t \\
R_{m,n}(t) &= (A_m + B_{m,n}) < Re(s) < 1; & |Im(s)| < t
\end{aligned}
$$

Definition. $\zeta(s)$ is as defined in Riemann[1].

Definition. We define $ChooseIndex(S_k, \uparrow or \downarrow, \rightarrow limit, k, K, \delta)$ as follows. S_k is a real-valued sequence that is either monotone increasing ($\uparrow$) or monotone decreasing ($\downarrow$), with $\lim_{k\to\infty} S_k = limit$. For the given S_k, these facts are clear by inspection and are not separately proved. Therefore, for the given $\delta > 0$, there is a $K \in \mathbb{N}$ such that for all $k > K$ we have: (1) if monotone decreasing, then $0 \leq (S_k - limit) < \delta$, and (2) if monotone increasing, then $(limit - \delta) < S_k$. We assume that the given K is the K needed for the given δ, and that the given $k > K$.

3 The Zeta Induction Theorem

Theorem 1 (Zeta Induction Theorem)**.** *Let* $s \in \mathbb{C}$*;* $t \in \mathbb{R}_{>0}$*. If we assume* $\zeta(s) \neq 0$ *when* $s \in R_{m,n}(t)$*, then we have* $\zeta(s) \neq 0$ *when* $s \in R_{m,n+1}(t)$*.*

Proof. In this paper we assume (but do not prove) this theorem. □

4 Lemma

This lemma pulls together various statements, with proofs that are either well-known or straightforward. That way, those statements can be used subsequently without detracting from the flow of the discussion.

Lemma 1. *Let* $s \in \mathbb{C} \setminus \{1\}$*; fix* $t \in \mathbb{R}_{>0}$*. We have the following:*

- *i If* $\zeta(s) \neq 0$ *for all* $s \in R_R(t)$*, then* $\zeta(s) \neq 0$ *for all* $s \in R_L(t)$*.*
- *ii* $\zeta(s) \neq 0$ *for* $s \in R_\epsilon(t)$*.*
- *iii* $(A_m + B_{m,1}) = A_{m+1}$*.*
- *iv There exists an* $M \in \mathbb{N}$ *such that* $m > M \Rightarrow R_{m,1}(t) \subset R_\epsilon(t)$*.*
- *v There exists an* $N \in \mathbb{N}$ *such that* $n > N$ *and* $s \in R_{m,1}(t) \Rightarrow s \in R_{m+1,n}(t)$*.*

Proof.

- i From Riemann[1]: For $0 \leq Re(s) \leq 1$, if $\zeta(s) = 0$, then $\zeta(1-s) = 0$ (we call them twin zeros). Now assume $\zeta(s) = 0$ for some $s \in R_L(t) \cup R_R(t)$. We consider separately the real and imaginary parts of our twin zeros. *Real Parts*: $Re(s) + Re(1-s) = 1$. Set $\delta = \frac{1}{2} - Re(s)$, Then, $Re(s) = (\frac{1}{2} - \delta)$ and $Re(1-s) = (\frac{1}{2} + \delta)$. *Imaginary Parts*: $|Im(s)| = |Im(1-s)|$. In all cases, we have one of the twin zeros in $R_L(t)$ and the other in $R_R(t)$. Thus, with no zeros in $R_R(t)$ there can be no zeros in $R_L(t)$.

- ii From Ford[2]: $\zeta(\beta + it) \neq 0$ for $|t| \geq 3$ and $1 - \beta \leq \dfrac{1}{8.463 \cdot log(|t|)}$.

  Ford's statement still holds if we *increase* the size of the denominator, so $\epsilon(t)$ was defined by replacing $log(|t|)$ with $log(|t|+2)$. For all increasing $|t| \geq 0$, it is easily verified that $\epsilon(t) < 0.2$ and monotone decreasing. As revised by $\epsilon(t)$, Ford's statement extends to all $|t| \geq 0$ because, from Brent[3], there are no zeros in the $R_R(3)$ region. If $s \in R_\epsilon(t)$, we have $\epsilon(t) < \epsilon(Im(s))$, and therefore $\zeta(s) \neq 0$.

- iii As defined: $(A_m + B_{m,1}) = \sum_{j=1}^{m} \left(\frac{1}{2}\right)^j + \left(\frac{1}{2}\right)^{m+1} = \sum_{j=1}^{m+1} \left(\frac{1}{2}\right)^j = A_{m+1}$.

- iv We $ChooseIndex(A_{m+1}, \uparrow, \rightarrow 1, m+1, M, \delta = \epsilon(t))$. Thus $1 - \epsilon(t) < A_{m+1}$. Using (iii), we have $1 - \epsilon(t) < (A_m + B_{m,1})$. Hence, $R_{m,1}(t) \subset R_\epsilon(t)$.

- v Fix $s \in R_{m,1}(t)$ and fix $\epsilon = Re(s) - (A_m + B_{m,1})$. To set $B_{m+1,n} < \epsilon$, we now $ChooseIndex(B_{m+1,n}, \downarrow, \rightarrow 0, n, N, \delta = \epsilon)$. Using (iii), we have: $(A_{m+1} + B_{m+1,n}) < (A_{m+1} + \epsilon) = ((A_m + B_{m,1}) + \epsilon) = Re(s)$. But $(A_{m+1} + B_{m+1,n}) < Re(s)$ means $s \in R_{m+1,n}(t)$. □

5 The Riemann Hypothesis

Theorem 2 (Riemann Hypothesis)**.** *Let* $s \in \mathbb{C} \setminus \{1\}$*, with* $Re(s) \in [0,1] \setminus \{\frac{1}{2}\}$*. Then,* $\zeta(s) \neq 0$.

Proof. From Hadamard[4]: $\zeta(s) \neq 0$ for $Re(s) \in \{0,1\}$. So, we limit our proof to $Re(s) \in (0,1) \setminus \{\frac{1}{2}\}$. Fix $t \in \mathbb{R}_{>0}$. We first show $\zeta(s) \neq 0$ for $s \in R_R(t) \cup R_L(t)$.

Step 1-A (The First Interval). We start by ***assuming*** that $\zeta(s) \neq 0$ when $s \in R_{1,1}(t)$. Now set $m = 1$ and apply the *Zeta Induction Theorem*. It follows by induction that, for all $n \in \mathbb{N}, \zeta(s) \neq 0$ when $s \in R_{1,n}(t)$.

Step 1-B (The Right Strip). Fix $s \in R_R(t)$ and fix $\epsilon = (Re(s) - A_1) > 0$. Now $ChooseIndex(B_{1,n}, \downarrow, \to 0, n, N, \delta = \epsilon)$. Then $A_1 = \frac{1}{2} < (A_1 + B_{1,n}) < (A_1 + \epsilon) = Re(s)$. But $(A_1 + B_{1,n}) < Re(s)$ implies $s \in R_{1,n}(t)$, so by Step 1-A we have $\zeta(s) \neq 0$. Hence, $\zeta(s) \neq 0$ for all $s \in R_R(t)$.

Step 1-C (The Left Strip). From Lemma 1(i): $\zeta(s) \neq 0$ for $s \in R_L(t)$.

Step 2 (The Second Interval). One problem remains. We *assumed* that $\zeta(s) \neq 0$ for $s \in R_{1,1}(t)$. To prove that, we will now ***assume*** that $\zeta(s) \neq 0$ for $s \in R_{2,1}(t)$. Now set $m = 2$ and apply the *Zeta Induction Theorem*. It follows by induction that, for all $n \in \mathbb{N}, \zeta(s) \neq 0$ when $s \in R_{2,n}(t)$. We have therefore shown that $\zeta(s) \neq 0$ when $s \in R_{1,1}(t)$ because by Lemma 1(v) there is an n such that $s \in R_{1,1}(t)$ implies $s \in R_{2,n}(t)$.

Step 3 (Recursion). We can continue our recursive augment as many times as we like. To prove that $\zeta(s) \neq 0$ when $s \in R_{m,1}(t)$ we need only assume $\zeta(s) \neq 0$ when $s \in R_{m+1,1}(t)$ and then apply the *Zeta Induction Theorem* and Lemma 1(v). But our desired result is eventually established by Lemma 1(ii) and (iv), with no further need for recursion, because there exists an M such that for $m > M, R_{m,1}(t) \subset R_\epsilon(t)$, and we have $\zeta(s) \neq 0$ for $s \in R_\epsilon(t)$.

Step 4 (Wrapping Up). We have established the theorem for $s \in R_R(t) \cup R_L(t)$. But t was arbitrarily chosen, so the result holds for all $t \in \mathbb{R}_{>0}$. □

6 Discussion

Our "proof" of the Riemann Hypothesis (RH) uses deferred induction by recursion, with each inductive step depending, recursively, on a subsequent inductive step. An alternate (but less interesting) approach is also possible. We can recurse in the opposite direction (without deferred induction). We select m using Lemma 1(iv) and have $\zeta(s) \neq 0$ for $s \in R_{m,1}(t) \subset R_\epsilon(t)$. By the *Zeta Induction Theorem* (ZI), $\zeta(s) \neq 0$ for all $R_{m,n}(t)$. Then, using Lemma 1(v) we have $\zeta(s) \neq 0$ for $s \in R_{m-1,1}(t)$. Again applying ZI, we recurse until we reach $R_{1,1}(t)$ and $R_{1,n}(t)$, thereby covering all of $R_R(t)$.

ZI is just one short step away from simply assuming RH. So it should come as no surprise that ZI and RH are equivalent. Proof/disproof of one proves/disproves the other. We showed ZI implies RH. Clearly, RH implies ZI because $\zeta(s) \neq 0$ for $s \in$ all $R_{m,n+1}(t)$. Both are disproved only if $\zeta(s) = 0$ for some $Re(s) \in (0,1) \setminus \{\frac{1}{2}\}$. That said, proof of ZI almost certainly requires direct proof of RH.

References

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