FLUID STATE OF THE ELECTROMAGNETIC FIELD

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Abstract: In this work we discuss the nature of the electromagnetic field and show, by using Maxwell field equations, that its steady state is a fluid state. Similar to the fluid state of Dirac quantum particles that we discussed in our previous work, the quantum particles of an electromagnetic field, i.e. photons, can be formulated in terms of velocity potentials and stream functions in three dimensions in which the electric field and magnetic field are identified with the velocity fields of the fluid flows.

It is observed in physics that the electromagnetic field also exhibits the wave-particle duality. Like matter wave, this apparently contradictory dual manifestation of the quantum particles of the electromagnetic field, i.e., photons, may be due to the fact that the photons may also have internal substructures which could not be probed within the present experimental facilities. In the current formulation of quantum physics, photons are assumed to be massless and chargeless quantum particles despite the fact that they are able to interact electromagnetically with other charged particles. In fact, the assumption of masslessness of photons can be shown to have no fundamental role in the formulation of the theory of special relativity which can be formulated without the need of introduction of the postulation of the universal speed of light in vacuum [1] [2] [3] [4]. On the other hand, the assumption of being chargeless can be confirmed from experiments that show that a photon γ can be converted into a pair of an electron e^- and a positron e^+ as $\gamma \to e^- + e^+$ and, inversely, when an electron and a positron collide they create a pair of neutral photons as $e^- + e^+ \to \gamma + \gamma$. In general, the state of being chargeless of a physical object can be ascertained simply by assuming that the total charge of the object is equal to zero. However, this does not rule out the possibility that there may be an equal amount of positive and negative charges exist inside the particle if it has an internal structure. With this view then it can be suggested that photons are not structureless point particles but rather physical objects with internal geometric and topological structures that can exhibit geometric interactions as physical interactions [5] [6] [7]. In one of his classic papers, Einstein proposed that the electromagnetic field is not a wave propagating through space but rather is composed of discrete wave packets [8]. With this particular picture of photons as wave packets we have shown that they can be viewed as differentiable manifolds whose geometric and topological structures can be described by solutions of a wave equation. However, as in the case of Dirac quantum particles, the physical aspect of the problem of wave-particle duality can only be satisfactorily addressed in terms of the physical dynamics rather than the mathematical formulation that can be used to describe the physical phenomenon in which the characters of the physical objects involved cannot be specified. In this work we will address this problem and show that the electromagnetic quantum particles in steady motion can also be described using Maxwell field equation as in a state of fluid in the sense that the electric field and the magnetic field can be deduced simply as the velocity fields of a fluid flow which is conveniently described by velocity potentials and stream functions in three dimensions. The electromagnetic substance of the fluid that is associated with the charge of a quantum particle could be different from the material substance that is associated with the inertial mass of the particle. Since transverse waves need a medium rigid enough to propagate which fluids cannot provide therefore we may speculate either the characteristics of the electromagnetic fluid is different from those of normal fluids or the three-dimensional space that we are living in may in fact be only the surface of a higher-dimensional space. This situation is similar to our discussions in our previous works that the spatiotemporal manifold may be a fibre bundle in which the dynamics that we observe only occurs in the fibre but not in the base space.

For a comparison to the formulation of fluid state of Dirac quantum particles, we give a brief review of our formulation of Maxwell field equations from a system of linear first order partial differential equations. In our work on the fluid state of Dirac quantum particles we show that the internal physical dynamics of these particles can be described by the dynamical equations of the theory of classical fluids and the equations can be derived from Dirac relativistic equation in quantum mechanics. From our approach to a mathematical formulation of Dirac and Maxwell field equations, we have shown both of these equations with an external field can be formulated covariantly from a general system of linear first order partial differential equations [9] [10]

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{r}\frac{\partial \psi_i}{\partial x_j} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} b_{ij}^{r} V_j + c_i^r\right)\psi_i + d^r, \qquad r = 1, 2, \dots, n \tag{1}$$

The system of equations given in Equation (1) can be rewritten in a matrix form as

$$\left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)\psi = -i\left(\sum_{i=1}^{n} qB_iV_i + m\sigma\right)\psi + J \tag{2}$$

where $\psi = (\psi_1, \psi_2, \dots, \psi_n)^T$, $\partial\psi/\partial x_i = (\partial\psi_1/\partial x_i\,, \partial\psi_2/\partial x_i\,, \dots, \partial\psi_n/\partial x_i)^T$ with A_i, B_i, σ and J are matrices representing the quantities a_{ij}^r, b_{ij}^r, c_j^r and d^r, which are assumed to be constant in this work. While the quantities q, m and J represent physical entities related directly to the physical properties of the particle, the quantities V_i represent an external field, such as the potentials of an electromagnetic field. If we apply the operator $\sum_{i=1}^{n} A_i\, \partial/\partial x_i$ on the left on both sides of Equation (2) and with the assumption that the coefficients a_{ij}^k, b_i^r and c^r are constants then we obtain

$$\left(\sum_{i=1}^{n} {A_i}^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n}\sum_{j>i}^{n}\left(A_iA_j + A_jA_i\right)\frac{\partial^2}{\partial x_i \partial x_j}\right)\psi = -m^2\sigma^2\psi - im\sigma J + \sum_{i=1}^{n} A_i \frac{\partial J}{\partial x_i} \tag{3}$$

In physics, physical equations, such as Maxwell and Dirac equations, are formed by selecting the matrices A_i so that each component ψ_i satisfies a wave equation similar to the Klein-Gordon equation. The simplest case is to form Dirac equation simply by setting $A_i^2 = \pm 1$ and $A_i A_j + A_j A_i = 0$. Dirac equation for an arbitrary field can be formulated from the system of linear first order partial differential equations given in Equation (2) by setting $B_i = A_i = \gamma_i$, $\sigma = 1$, $J = 0$. For Dirac field, the wavefunction ψ has four components $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ and the matrices γ_i are the Dirac matrices. In this case, in terms of the operators γ^μ, Equation (2) becomes

$$\left(\sum_{i=1}^{4} \gamma_i \frac{\partial}{\partial x_i}\right)\psi = -i\left(\sum_{i=1}^{4} q\gamma_i V_i + m\right)\psi \tag{4}$$

Equation (4) can be written in a covariant form as Dirac equation for an arbitrary field as [11]

$$\left(\gamma^\mu\left(i\partial_\mu - qV_\mu\right) - m\right)\psi = 0 \tag{5}$$

On the other hand, for the case of Maxwell field equations of electromagnetism, we have to rely on the established equations in classical electrodynamics to determine the matrices A_i [12]. There could also be a general method to determine the matrices A_i so that physical fields could be determined and classified. For the case of the electromagnetic field, we consider from the general equation given in Equation (2) a reduced form given as

$$\left(A_1 \frac{\partial}{\partial t} + A_2 \frac{\partial}{\partial x} + A_3 \frac{\partial}{\partial y} + A_4 \frac{\partial}{\partial z}\right)\psi = A_5 J \tag{6}$$

where $\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6)^T$ in which the electric field $\mathbf{E} = (\psi_1, \psi_2, \psi_3)$, the magnetic field $\mathbf{B} = (\psi_4, \psi_5, \psi_6)$, and $J = (j_1, j_2, j_3, j_4, j_5, j_6)^T$. As shown in the appendix, the matrices A_i for the electromagnetic field can also be specified by requiring that the equation given in Equation (3) be reduced to a wave equation. With the form of the matrices A_i given in appendix, the system of field equations given in Equation (6) can be written explicitly as a system of linear first order partial differential equations as follows

$$-\frac{\partial \psi_1}{\partial t} + \frac{\partial \psi_6}{\partial y} - \frac{\partial \psi_5}{\partial z} = \mu j_1 \tag{7}$$

$$-\frac{\partial \psi_2}{\partial t} + \frac{\partial \psi_4}{\partial z} - \frac{\partial \psi_6}{\partial x} = \mu j_2 \tag{8}$$

$$-\frac{\partial \psi_3}{\partial t} + \frac{\partial \psi_5}{\partial x} - \frac{\partial \psi_4}{\partial y} = \mu j_3 \tag{9}$$

$$\frac{\partial \psi_4}{\partial t} + \frac{\partial \psi_3}{\partial y} - \frac{\partial \psi_2}{\partial z} = \mu j_4 \tag{10}$$

$$\frac{\partial \psi_5}{\partial t} + \frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_3}{\partial x} = \mu j_5 \tag{11}$$

$$\frac{\partial\psi_6}{\partial t}+\frac{\partial\psi_2}{\partial x}-\frac{\partial\psi_1}{\partial y}=\mu j_6 \tag{12}$$

For simplicity we also set $\epsilon\mu = 1$. From the system of equations given in Equations (7-12), using Gauss's law $\nabla.\mathbf{E} = \rho_e/\epsilon$, we obtain the following wave equations for the components of the electric and magnetic field

$$\frac{\partial^2\psi_i}{\partial t^2}-\frac{\partial^2\psi_i}{\partial x^2}-\frac{\partial^2\psi_i}{\partial y^2}-\frac{\partial^2\psi_i}{\partial z^2}=-\mu\frac{\partial j_i}{\partial t} \quad \text{for} \quad i=1,2,3 \tag{13}$$

$$\frac{\partial^2\psi_i}{\partial t^2}-\frac{\partial^2\psi_i}{\partial x^2}-\frac{\partial^2\psi_i}{\partial y^2}-\frac{\partial^2\psi_i}{\partial z^2}=-\frac{\partial j_i}{\partial t} \quad \text{for} \quad i=4,5,6 \tag{14}$$

By inserting physical constants, the system of equations given in Equations (7-12) can be rewritten in vector form as Maxwell field equations as

$$\nabla.\mathbf{E}=\frac{\rho_e}{\epsilon} \tag{15}$$

$$\nabla.\mathbf{B}=\mu\rho_m \tag{16}$$

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=-\mu\mathbf{j}_m \tag{17}$$

$$\nabla\times\mathbf{B}-\epsilon\mu\frac{\partial\mathbf{E}}{\partial t}=\mu\mathbf{j}_e \tag{18}$$

where we have also assumed the existence of the magnetic monopole written in terms of the magnetic current $\mathbf{j}_m$.

As in the case of Dirac quantum particles, we now discuss the steady internal fluid motion of a quantum particle of the electromagnetic field therefore we rewrite the field equations given in Equations (7-12) for such state of motion as follows

$$\nabla.\mathbf{E}=0 \tag{19}$$

$$\nabla\times\mathbf{E}=0 \tag{20}$$

$$\nabla.\mathbf{B}=0 \tag{21}$$

$$\nabla\times\mathbf{B}=0 \tag{22}$$

From the system of field equations given for the steady electric and magnetic field it is seen that the two fields are completely independent when they are regarded as physical entities in three-dimensional Euclidean space. However, as we showed for the case of Dirac quantum particles that despite the Dirac field components are independent when considered in two-dimensional space, in three-dimensional space they are actually connected. Also as discussed in our previous works that quantum particles associated with the electromagnetic field may exist in four-dimensional space therefore we may speculate that the field components of the electric field and the magnetic field may also be connected in the fourth dimension of the spatial space. We will discuss in more details on this topic later on. Now, it is also noted that there are two mathematical conditions, namely $\boldsymbol{\nabla}.\mathbf{V} = 0$ and $\boldsymbol{\nabla} \times \mathbf{V} = 0$, that we would need to consider as a starting point in our formulation, where the quantity $\mathbf{V}$ represents either the electric field $\mathbf{E}$ or the magnetic field $\mathbf{B}$. In physics, the mathematical condition $\boldsymbol{\nabla}.\mathbf{V} = 0$ seems to be more fundamental because it is related to a source or a dynamical condition of a physical field. In particular in fluid dynamics it represents the condition of continuity of a flow. On the other hand, the mathematical condition $\boldsymbol{\nabla} \times \mathbf{V} = 0$ describes the infinitesimal rotation of a vector field. Nonetheless, in the following we will discuss both cases when we consider each of the mathematical condition as a starting point.

First, let us start our discussion with the mathematical condition $\boldsymbol{\nabla} \times \mathbf{V} = 0$, which simply assumes the irrotational property of a fluid. In this case if we let $\mathbf{V} = \nabla V$, where V is a scalar function, then the fundamental condition $\boldsymbol{\nabla} \times \mathbf{V} = 0$ is satisfied automatically since $\boldsymbol{\nabla} \times (\nabla V) \equiv 0$. From the mathematical condition $\boldsymbol{\nabla}.\mathbf{V} = 0$, we obtain $\nabla^2 V = 0$. In this case the scalar function V can be identified with a velocity potential. To be specific, let us consider the case when the vector field $\mathbf{V}$ is identified as an electric field, i.e. $\mathbf{V} = \mathbf{E}$. This particular identification specifies the electric field $\mathbf{E}$ as a velocity field. This is a static vector field. It is worth mentioning here that this velocity field is only related to the substance in a fluid state that is associated with the charge. If we consider a charged particle of mass m as a small sphere and the electric field as the velocity of a viscous fluid then we may speculate that the force of viscosity $\mathbf{F}$ on the charged particle also obeys Stokes law of drag force in the classical fluid dynamics as $\mathbf{F} = 6\pi\mu a\mathbf{E}$, where μ is the dynamic viscosity of the fluid of charge, a is the radius of the charged particle. If the constant $6\pi\mu a$ is identified as the charge q of the sphere then we regain the Newton's second law for the dynamics of a charged particle of mass m moving in an electric field with an acceleration $\mathbf{a} = d^2\mathbf{r}/dt^2$ as $m\,d^2\mathbf{r}/dt^2 = q\mathbf{E}$. Even though we can describe the interaction between an electric field and a charged particle as a dragging process that obeys the Stokes law of drag force, there is an important feature that distinguishes it from the classical fluid dynamics is the dragging process can occur not only in the direction of the fluid flow but also in the opposite direction of the flow. Namely, if the charge $q > 0$ then we have the normal drag with positive viscosity, but if $q < 0$ then we have a process of drag with negative viscosity. Intrinsically, the two way dragging may be due the fact that the charge of a quantum particle is related to its geometric and topological structures that result in a motion which effectively corresponds to a negative viscosity. We may get an insight into the dragging process of a charged particle in an electric field by analysing the dimension of the physical entities in the Newton's second law $m\,d^2\mathbf{r}/dt^2 = q\mathbf{E}$. If we write $[q]$ as the dimension of the charge q then we obtain an equation for the dimensions as $ML/T^2 = [q]L/T$, from which we obtain $[q] = M/T$. With this dimension of the charge q with respect to the electric field we may interpret charge as a change of mass with respect to time. If the charge is constant then the particle which possesses the charge changes an equal amount of mass per unit time. Even though the mass that we have mentioned is related to the fluid flow of the electric field, its nature is unknown therefore we may call it an electromagnetic mass. Furthermore, if q is negative then the particle is losing its electromagnetic mass with a constant rate, but if q is positive then the particle is gaining its electromagnetic mass also with a constant rate. With this picture, we may view an atom as a dynamical system of source and sink in fluid dynamics and this could be happening in the fourth spatial dimension.

Besides the vector potential V, the motion of an electric fluid can also be expressed by means of a stream function if the flow is axisymmetric. Consider an electric field **E** as the velocity field of a steady fluid flow around a charged particle which is seen as an obstacle which may have a sophisticated geometric and topological structure. However, in the following we will consider a charged particle as a spherical object of radius a. If we take the origin at the centre of the sphere and the flow is parallel to the z-axis, then in the spherical coordinates (r, θ, ϕ), where $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$, the components E_z in the direction of z and E_{xy} in the (x, y)-plane of the electric field can be expressed in terms of a stream function ψ as

$$E_z = -\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}, \qquad E_{xy} = \frac{1}{\rho}\frac{\partial\psi}{\partial z} \tag{23}$$

where $\rho = \sqrt{x^2 + y^2}$, and with specified boundary conditions the stream function ψ of the flow can be found as [13]

$$\psi = -\frac{1}{2}E\left(1 - \frac{3}{2}\frac{a}{r} + \frac{1}{2}\frac{a^3}{r^3}\right)r^2\sin^2\phi \tag{24}$$

However, as in the case of Dirac quantum particles, the problem that we are interested in is how to construct a physical structure so that the quantum particles of an electric field can exhibit a standing wave, which requires the existence of two fluid flows in opposition direction. We may assume that the fluid flow in opposite direction may be related to a static magnetic field that accompanies the electric field **E**. If we let $\mathbf{B} = \nabla(V_B)$ then we also have $\nabla \times \mathbf{B} = \nabla \times \left(\nabla(V_B)\right) \equiv 0$. According to the classical electrodynamics the force **F** acting on a charge q by the magnetic field **B** is given by $m\, d^2\mathbf{r}/dt^2 = q\mathbf{v} \times \mathbf{B}$, therefore if the direction of motion of the charge is parallel to the magnetic field **B** then we have $\mathbf{F} = 0$. As in the case of the electric field, we may get an insight into the dragging process of a charged particle in a magnetic field by analysing the dimension of the physical entities in the Newton's second law $m\, d^2\mathbf{r}/dt^2 = q\mathbf{v} \times \mathbf{B}$. The equation for the dimensions of this force law is $ML/T^2 = [q](L/T)(L/T)$, from which we obtain $[q] = M/L$. As expected, the charge q that involves the magnetic field has different dimension from the charge that involves the electric field. In this case we may interpret charge as a change of its electromagnetic mass with respect to distance. If the charge is constant then the particle which possesses the charge changes an equal amount of its electromagnetic mass per unit distance.

Even though we have been able to show that the electric and magnetic field can be formulated as the velocity fields of some kind of fluid and when combined together they are able to establish a standing wave, however, the two fields are completely unrelated. In the following we will show that this difficulty can be resolved by showing that an electric field and a magnetic field which are opposite to each other can be formed from the same two steam functions in three dimensions. In order to show this we start our formulation by using the condition $\boldsymbol{\nabla}.\mathbf{V}=0$ as a fundamental condition from which three-dimensional stream functions for the quantum particles of the electromagnetic field can be constructed. From vector calculus, it can be shown that if we let $\mathbf{V}=\nabla\times\mathbf{A}$, where $\mathbf{A}$ is a vector function, then the condition $\boldsymbol{\nabla}.\mathbf{V}=0$ is satisfied automatically due to the identity $\boldsymbol{\nabla}.(\nabla\times\mathbf{A})\equiv 0$. Now, let ψ and ϕ be two scalar functions so that we can write the vector function $\mathbf{A}$ in terms of these two functions as $\mathbf{A}=\psi\nabla\phi$. Then, since $\nabla\times(\nabla\phi)\equiv 0$ we obtain the following identity [14]

$$\mathbf{V}=\nabla\times(\psi\nabla\phi)=\psi\nabla\times(\nabla\phi)+\nabla\psi\times\nabla\phi=\nabla\psi\times\nabla\phi \tag{25}$$

The result given in Equation (25) shows that the field $\mathbf{V}$ is tangent to the surfaces defined by the equations $\psi=\text{constant}$, and $\phi=\text{constant}$. Therefore, according to the theory of fluids, if the functions ψ and ϕ are identified as three-dimensional stream functions then the field $\mathbf{V}$ is identified with a velocity field. As discussed above, if we identify the vector field $\mathbf{V}$ either with the electric field $\mathbf{E}$ or the magnetic field $\mathbf{B}$ then the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are also regarded as velocity fields of an electromagnetic fluid. Using the vector identity $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})=\boldsymbol{\nabla}(\boldsymbol{\nabla}.\mathbf{A})-\nabla^2\mathbf{A}$ then from the condition $\boldsymbol{\nabla}\times\mathbf{V}=0$ of irrotational field we obtain $\nabla(\boldsymbol{\nabla}.\mathbf{A})-\nabla^2\mathbf{A}=0$. Using the Stokes theorem $\oint_\Gamma \mathbf{F}.d\boldsymbol{\Gamma}=\iint_S \nabla\times\mathbf{F}.d\mathbf{S}$, a volume flow V associated with these stream functions can be obtained as

$$V=\iint_S \mathbf{F}.d\mathbf{S}=\iint_S \nabla\times(\psi\nabla\phi).d\mathbf{S}=\oint_{\partial S}\psi\nabla\phi.d\mathbf{l}=(\psi_1-\psi_2)(\phi_1-\phi_2) \tag{26}$$

Consider the equation of continuity with the velocity field $\mathbf{V}$ given as

$$\frac{\partial\rho}{\partial t}+\nabla.(\rho\mathbf{V})=0 \tag{27}$$

Therefore the condition $\boldsymbol{\nabla}.\mathbf{V}=0$ is equivalent to assuming that the fluid is steady and incompressible. If we consider a physical quantity Q with material density ρ, such as mass or charge, which is incompressible, then instead of the volume flow V we can define a flow of the physical quantity Q as

$$Q=\iint_S \rho\mathbf{V}.d\mathbf{S}=\iint_S \rho\nabla\times(\psi\nabla\phi).d\mathbf{S}=\oint_{\partial S}\rho\psi\nabla\phi.d\mathbf{l}=\rho(\psi_1-\psi_2)(\phi_1-\phi_2) \tag{28}$$

If we identify the vector field $\mathbf{V}$ with an electric field, i.e, $\mathbf{V}=\mathbf{E}$ then the electric field in this case is defined in terms of the stream functions ψ and ϕ as $\mathbf{E}=\nabla\times(\psi\nabla\phi)=\nabla\psi\times\nabla\phi$. There is an important feature in this definition that we now want to exploit further, namely the order in which the vector field $\mathbf{V}$ is defined in terms of the stream functions ψ and ϕ. For given two stream functions ψ and ϕ in fact we can obtain two different vector fields depending the order of the two scalar functions ψ and ϕ from which the vector function $\mathbf{A}$ is defined, namely, we can have either $\mathbf{A}=\psi\nabla\phi$ or $\mathbf{A}=\phi\nabla\psi$. If we define an electric field as $\mathbf{E}=\nabla\times(\psi\nabla\phi)=\nabla\psi\times\nabla\phi$ then we still have the option to define another vector field as $\mathbf{V}=\nabla\times(\phi\nabla\psi)=\nabla\phi\times\nabla\psi$. A very important property of this new vector field is that it has the same magnitude as the electric field **E** but pointing in opposite direction. Therefore, as discussed before, we may identify the new vector field with a magnetic field **B**, namely $\mathbf{B} = \nabla \times (\phi \nabla \psi) = \nabla \phi \times \nabla \psi$, and the two fluids flowing in opposite direction can form a standing wave, which can be identified with the quantum particle, or photon, of an electromagnetic field.

We have shown that an electric field **E** can be constructed from two stream functions ψ and ϕ as $\mathbf{E} = (\psi_1, \psi_2, \psi_3) = \nabla \psi \times \nabla \phi$. And, similarly, a magnetic field **B** can also be constructed from the same two stream functions as $\mathbf{B} = (\psi_4, \psi_5, \psi_6) = \nabla \phi \times \nabla \psi$. These two velocity fields, which are equal in magnitude and opposite direction, are supposed to be observable in the three-dimensional Euclidean space. However, as discussed in our previous works that even though we exist as three-dimensional physical objects that can only be able to observe other physical objects in three spatial dimensions, from the fact that the observable universe seems to exist as an expanding 3-sphere S^3 it is reasonable to suggest that there may exist four-dimensional physical objects, such as the electromagnetic field, whose fourth dimensional components we are unable to observe. In this case we may speculate that the electric field and magnetic field are also related in a fourth spatial w-dimension, as in the case that we showed that two-dimensional components of a Dirac field are related in a third spatial dimension. Using the form of the relationship between the components of Dirac field we assume that the electric field and the magnetic field are related in a fourth spatial dimension w as follows

$$\frac{\partial \Psi_4}{\partial w} = k\Psi_1, \qquad \frac{\partial \Psi_1}{\partial w} = -k\Psi_4 \tag{29}$$

$$\frac{\partial \Psi_5}{\partial w} = k\Psi_2, \qquad \frac{\partial \Psi_2}{\partial w} = -k\Psi_5 \tag{30}$$

$$\frac{\partial \Psi_6}{\partial w} = k\Psi_3, \qquad \frac{\partial \Psi_3}{\partial w} = -k\Psi_6 \tag{31}$$

where k is an undetermined constant. From Equations (29-31) we obtain

$$\frac{\partial^2 \Psi_i}{\partial^2 w} + k^2 \Psi_i = 0 \qquad \text{for} \qquad i = 1,2,3,4,5,6 \tag{32}$$

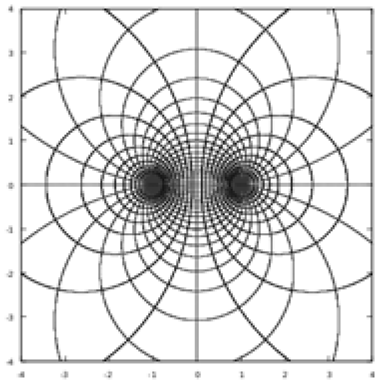
The undetermined physical quantity k may be identified with the electromagnetic mass of a quantum particle. Solutions to Equation (32) can be written in the form

$$\Psi_i = \psi_i(x, y, z)\sin(kw) \qquad \text{for} \qquad i = 1, 2, 3 \tag{33}$$

$$\Psi_i = \psi_i(x, y, z)\cos(kw) \qquad \text{for} \qquad i = 4, 5, 6 \tag{34}$$

where the functions ψ_i are the components of the electric field and the magnetic field defined as $\mathbf{E} = (\psi_1, \psi_2, \psi_3) = \nabla \psi \times \nabla \phi$ and $\mathbf{B} = (\psi_4, \psi_5, \psi_6) = \nabla \phi \times \nabla \psi$. The fact that the electric field and magnetic field are related as $\mathbf{B} = -\mathbf{E}$ can also be verified from the above equations by showing that $\psi_4(x, y, z) = -\psi_1(x, y, z)$, $\psi_5(x, y, z) = -\psi_2(x, y, z)$ and $\psi_6(x, y, z) =$

$-\psi_3(x, y, z)$. As an example, also as a possible representation of the geometric structure of the quantum particle of an electromagnetic field, we now show how two stream functions can be constructed using the toroidal coordinates. It is shown that toroidal coordinates can be obtained from a two-dimensional bipolar coordinate system by rotating about the axis separating its two foci as shown in the following figures



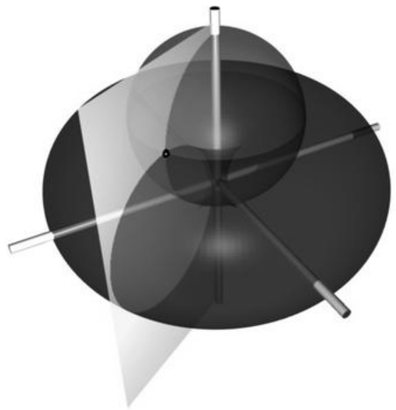


Figure { SEQ Figure * ARABIC }: Bipolar coordinate system Figure { SEQ Figure * ARABIC }: Toroidal coordinates

In geometry, the relationship between the Cartesian coordinates (x, y, z) and the toroidal coordinates (ψ, ϕ, φ) is given as follows [15]

$$x = \frac{a\sinh\phi\cos\varphi}{\cosh\phi - \cos\psi}, \qquad y = \frac{a\sinh\phi\sin\varphi}{\cosh\phi - \cos\psi}, \qquad z = \frac{a\sin\psi}{\cosh\phi - \cos\psi} \tag{35}$$

where the domains of the toroidal coordinates are given as $0 \le \psi < 2\pi$, $0 \le \phi < \infty$, and $0 \le \varphi < 2\pi$. From the relations given in Equation (35), it can be shown that

$$\psi = \cot^{-1}\left(\frac{x^2 + y^2 + z^2 - a^2}{2az}\right) \tag{36}$$

$$\phi = \coth^{-1}\left(\frac{x^2 + y^2 + z^2 + a^2}{2a\sqrt{x^2 + y^2}}\right) \tag{37}$$

The surfaces of $\psi = \psi_0$ correspond to 2-spheres given by the equation $x^2 + y^2 + (z - a\cot\psi_0)^2 = a^2/\sin^2\psi_0$, and the surfaces of $\phi = \phi_0$ correspond to 2-tori given by the equation $z^2 + \left(\sqrt{x^2 + y^2} - a\coth\phi_0\right)^2 = a^2/\sinh^2\phi_0$, where ψ_0 and ϕ_0 are constants. Therefore, according to what that have been discussed above for the theory of fluids, the functions ψ and ϕ can be identified as three-dimensional stream functions.

Appendix

In this appendix we give the form of the matrices A_i that are used to form Maxwell field equations from a general system of linear first order partial differential equations. The matrices A_i are given as follows

$$A_1=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\quad A_2=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&-1&0&0&0\\0&1&0&0&0&0\end{pmatrix}\quad A_3=\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\-1&0&0&0&0&0\end{pmatrix}$$

$$A_4=\begin{pmatrix}0&0&0&0&-1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&-1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\quad A_5=\begin{pmatrix}\mu&0&0&0&0&0\\0&\mu&0&0&0&0\\0&0&\mu&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\tag{1}$$

We then obtain the following results

$$A_1^2=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\quad A_2^2=\begin{pmatrix}0&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\end{pmatrix}\quad A_3^2=\begin{pmatrix}-1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-1\end{pmatrix}$$

$$A_4^2=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\end{pmatrix}\quad A_5^2=\begin{pmatrix}\mu^2&0&0&0&0&0\\0&\mu^2&0&0&0&0\\0&0&\mu^2&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}$$

$$A_1A_2+A_2A_1=\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{pmatrix}\quad A_1A_3+A_3A_1=\begin{pmatrix}0&0&1&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&1&0&0\end{pmatrix}\quad A_2A_3+A_3A_2=\begin{pmatrix}0&0&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{pmatrix}$$

$$A_1A_i+A_iA_1=0\quad \text{for}\quad i=2,3,4\tag{2}$$

If we apply the differential operator $(A_1\,\partial/\partial t+A_2\,\partial/\partial x+A_3\,\partial/\partial y+A_4\,\partial/\partial z)$ to Equation (6) then we arrive at

$$\left(\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\frac{\partial^2}{\partial t^2}+\begin{pmatrix}0&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\end{pmatrix}\frac{\partial^2}{\partial x^2}+\begin{pmatrix}-1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-1\end{pmatrix}\frac{\partial^2}{\partial y^2}\right.$$
$$+\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\end{pmatrix}\frac{\partial^2}{\partial z^2}+\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{pmatrix}\frac{\partial^2}{\partial x\partial y}+\begin{pmatrix}0&0&1&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&1&0&0\end{pmatrix}\frac{\partial^2}{\partial x\partial z}$$
$$\left.+\begin{pmatrix}0&0&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{pmatrix}\frac{\partial^2}{\partial y\partial z}\right)\psi=-\left(\begin{pmatrix}\mu&0&0&0&0&0\\0&\mu&0&0&0&0\\0&0&\mu&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\frac{\partial}{\partial t}\right)J\tag{3}$$

From the above equation, using Gauss's law $\mathbf{\nabla}.\mathbf{E} = \rho_e/\epsilon$ we obtain the following wave equations for the components of the electric field $\mathbf{E} = \left(E_x, E_y, E_z\right) = (\psi_1, \psi_2, \psi_3)$ and the magnetic field $\mathbf{B} = \left(B_x, B_y, B_z\right) = (\psi_4, \psi_5, \psi_6)$

$$\frac{\partial^2\psi_i}{\partial t^2} - \frac{\partial^2\psi_i}{\partial x^2} - \frac{\partial^2\psi_i}{\partial y^2} - \frac{\partial^2\psi_i}{\partial z^2} = -\mu\frac{\partial j_i}{\partial t} \quad \text{for} \quad i = 1, 2, 3 \tag{4}$$

$$\frac{\partial^2\psi_i}{\partial t^2} - \frac{\partial^2\psi_i}{\partial x^2} - \frac{\partial^2\psi_i}{\partial y^2} - \frac{\partial^2\psi_i}{\partial z^2} = -\frac{\partial j_i}{\partial t} \quad \text{for} \quad i = 4, 5, 6 \tag{5}$$

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