ANOTHER PROOF FOR CATALAN'S CONJECTURE

ABSTRACT. In 2002 Preda Mihailescu used the theory of cyclotomic fields and Galois modules to prove Catalan's Conjecture. In this short paper, we give a very simple proof. We first prove that no solutions exist for $a^x - b^y = 1$ for $a, b > 1$ and $x, y > 2$. Then we prove that when $x = 2$ the only solution for y is $y = 3$.

Introduction Catalan's Conjecture was first made by Belgian mathematician Eugne Charles Catalan in 1844, and states that 8 and 9 (2^3 and 3^2) are the only consecutive powers, excluding 0 and 1. That is to say, that the only solution in the natural numbers of $a^x - b^y = 1$ for $a, b > 1$, $x, y > 0$ is $x = 3$, $a = 2$, $y = 2$, $b = 3$. In other words, Catalan conjectured that $3^2 - 2^3 = 1$ is the only nontrivial solution. It was finally proved in 2002 by number theorist Preda Mihailescu making extensive use of the theory of cyclotomic fields and Galois modules.

Theorem 0.1. *To demonstrate that the only solution in the natural numbers of $a^x - b^y = 1$ for $a, b > 1$, $x, y > 0$ is $x = 3$, $a = 2$, $y = 2$, $b = 3$.*

Lemma 0.2. *To demonstrate that no solutions in the natural numbers exist for $a^x - b^y = 1$ for $a, b > 1$, $x, y > 2$.*

Proof. We first assume that solutions do exist to this equation for $a, b > 1$, $x, y > 2$.

Then we observe that:

$$a^x - b^y = (a + b)(a^{x-1} - b^{y-1}) - ab(a^{x-2} - b^{y-2}). \tag{0.1}$$

We can therefore state our equation as follows:

$$(a + b)(a^{x-1} - b^{y-1}) - 1 = ab(a^{x-2} - b^{y-2}). \tag{0.2}$$

But note that $(a^{x-1} - b^{y-1})$ and $(a^{x-2} - b^{y-2})$ are both divisible by $(a - b)$. Therefore, as long as all four exponents are greater than zero, i.e. $x, y > 2$, it follows that neither side is divisible by $(a - b)$ unless $a - b = 1$, for $(a + b)$ is then divisible by 1.

So let $b = a - 1$, such that from (0.2):

$$(2a - 1)(a^{x-1} - (a - 1)^{y-1}) - 1 = a(a - 1)(a^{x-2} - (a - 1)^{y-2}). \tag{0.3}$$

(0.4)

$$\rightarrow 2a^x - a^{x-1} - 2a(a-1)^{y-1} + (a-1)^{y-1} - 1 = a^x - a^{x-1} - a^2(a-1)^{y-2} + a(a-1)^{y-2},$$

$$\rightarrow a^x - 1 = (a - 1)^{y-2}(a - a^2) + (a - 1)^{y-1}(2a - 1), \tag{0.5}$$

Date: Aug 2018.

2010 *Mathematics Subject Classification.* Primary 11D41.

Key words and phrases. Number Theory, Catalan's Conjecture, Mihailescu's Theorem.

$$\rightarrow a^x - 1 = (a-1)^{y-2}[(a-a^2) + (a-1)(2a-1)], \tag{0.6}$$

$$\rightarrow a^x - 1 = (a-1)^{y-2}(3a - a^2 - 1). \tag{0.7}$$

But we know that $a^x - 1 = b^y$, from which it follows that:

$$b^y = (a-1)^{y-2}(3a - a^2 - 1), \tag{0.8}$$

And since $b = (a-1)$, it follows that:

$$(a-1)^y = (a-1)^{y-2}(3a - a^2 - 1), \tag{0.9}$$

$$\rightarrow (3a - a^2 - 1) = (a-1)^2, \tag{0.10}$$

$$\rightarrow (3a - a^2 - 1) = (a^2 - 2a + 1) \tag{0.11}$$

$$\rightarrow 2a^2 - 5a + 2 = 0, \tag{0.12}$$

$$\rightarrow (2a-1)(a-2) = 0, \tag{0.13}$$

From this it follows either that:

$$a = \frac{1}{2}, \tag{0.14}$$

which is not an integer solution, or that:

$$a = 2. \tag{0.15}$$

But since $b = a - 1$ it follows that:

$$b = 1. \tag{0.16}$$

But this is disallowed by the parameters of our proof requiring that $a, b > 1$.

Therefore there can be no solutions for exponents when both exponents are greater than 2 for the equation $a^x - b^y = 1$, leading to a contradiction in our initial assumption. Therefore the lemma is true.

It follows from this that at least x or y must equal 2 (and its respective base be square-free).

Lemma 0.3. *To demonstrate that when $x = 2$ no solutions in the natural numbers exist for $a^x - b^y = 1$ for $a, b > 1$, $x, y > 0$ other than $x = 3$, $a = 2$, $y = 2$, $b = 3$.*

It follows from Lemma 0.2 that at least x or y must be a square (and its base be square-free). Therefore let $x = 2$. It is still true that $b = a - 1$. So, using our equation in (0.2), we can state:

$$(a+b)(a - b^{y-1}) - 1 = ab(1 - b^{y-2}). \tag{0.17}$$

$$\rightarrow a^2 + a(a-1) - a(a-1)^{y-1} - (a-1)^y - 1 = a(a-1) - a(a-1)^{y-1}. \tag{0.18}$$

$$\rightarrow a^2 - 1 = (a-1)^y. \tag{0.19}$$

$$\rightarrow (a+1)(a-1) = (a-1)^y. \tag{0.20}$$

$$\rightarrow (a+1) = (a-1)^{y-1}. \tag{0.21}$$

However, since the bases on both sides differ in value by just 2, there are only two non-trivial values for a that avoid different prime factors on both sides, namely $a = 2$ and $a = 3$. But if $a = 2$, then $(2 + 1) \neq 1^{y-1}$. On the other hand, if $a = 3$, then it follows that:

$$4 = 2^{y-1}. \tag{0.22}$$

$$\rightarrow y = 3. \tag{0.23}$$

And from the original equation, we can now say that:

$$3^2 - 1 = b^3. \tag{0.24}$$

We know anyway that if $a = 3$ and $b = a - 1$ it follows that $b = 2$, all of which satisfies the original equation:

$$3^2 - 2^3 = 1. \tag{0.25}$$

Thus, it follows that the only solution in the natural numbers of $a^x - b^y = 1$ for $a, b > 1$, $x, y > 0$ is $x = 3$, $a = 2$, $y = 2$, $b = 3$. □

References

The Rectory, Village Road, Waverton, Chester Ch3 7QN, UK
E-mail address: julianbeauchamp47@gmail.com