MODULAR LOGARITHMS UNEQUAL

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Abstract. The main idea of this article is simply calculating integer functions in module. The algebraic in the integer modules is studied in completely new style. By a careful construction the result that two finite numbers is with unequal logarithms in a corresponding module is proven, which result is applied to solving a kind of high degree diophantine equation.

Contents

In this paper p, p_i are primes. m, m' are great enough. All numbers that are indicated by Latin letters are integers unless with further indication. $C(z)$ mean constant independent of z. $F(z)$ means variable F is the function dependent of z. The formula $a << b$ means that b is far greater than a.

1. Function in module

Theorem 1.1. *Define the congruence class in the form:*

$$[a]_q := [a+kq]_q, \forall k$$

$$[a=b]_q : [a]_q = [b]_q$$

$$[x]_{qq'} = [a]_q[b]_{q'} : [x=b]_q, [x=b]_{q'}, (q,q')=1$$

then

$$[a+b]_q = [a]_q + [b]_q$$

$$[ab]_q = [a]_q \cdot [b]_q$$

$$[a+c]_q[b+d]_{q'} = [a]_q[b]_{q'} + [c]_q[d]_{q'}, (q,q')=1$$

$$[ka]_q[kb]_{q'} = k[a]_q[b]_{q'}, (q,q')=1$$

$$[a^k]_q[b^k]_{q'} = ([a]_q[b]_{q'})^k, (q,q')=1$$

Definition 1.2. Function of $x \in \mathbf{Z}$: $c + \sum_{i=1}^{m} c_i x^i$ is called power-analytic (i.e power series), it's denoted by $P(x)$.

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Theorem 1.3. *Power-analytic functions modulo p are all the functions from mod p to mod p*

$$[x^0 = 1]_p$$

$$[f(x) = \sum_{n=0}^{p-1} f(n)(1 - (x-n)^{p-1})]_p$$

Theorem 1.4. *(Modular Logarithm) Define*

$$[lm_a(x) := y]_{p^{m-1}(p-1)} : [a^y = x]_{p^m}$$

$$[E := \sum_{i=0}^{n} \frac{p^i}{i!}]_{p^m}$$

n is sufficiently great. then

$$[E^x = \sum_{i=0}^{n} \frac{p^i x^i}{i!}]_{p^m}$$

$$[lm_E(px+1) = \sum_{i=1}^{n} \frac{(-1)^{i+1} p^{i-1}}{i} x^i]_{p^{m-1}}$$

$$[Q(q) lm(1+xq) = \sum_{i=1} (xq)^i (-1)^{i+1}/i]_{q^m}$$

$$Q(q) := \prod_i [p_i]_{p_i^m}, \forall p_i : p_i | q$$

Define

$$[lm(x) := lm_e(x)]_{p^{m-1}}$$

e is the generating element in mod p and meets

$$[e^{1-p^m} = E]_{p^m}$$

To prove the theorem, one can contrast the coefficients of E^x and $E^{lm(1+px)}$ to those of real exponents of $exp(px)$ and $exp(log(px+1))$.

Definition 1.5. $P(q)$ is the product of all the distinct prime factors of q.

Definition 1.6.

$$[lm(px) := plm(x)]_{p^m}$$

Definition 1.7.

$$y := \overline{[x]}_q : [y = x]_q, -q/2 < y \le q/2$$

2. Unequal Logarithms of Two Numbers

Theorem 2.1. *If*

$$a + P(q)b \le q$$

$$a > b > 0$$

$$P^2(q) | q$$

$$(a,b) = (a,q) = (b,q) = (a-b,q) = 1$$

then

$$[lm(a) \ne lm(b)]_{q/P(q)}$$

Proof. Define

$$r := P(q)$$
$$[v+1 := 1 - p_i^m]_{p_i^m(p_i-1)}, v > 0, p_i|q$$

Presume

$$q' = \prod_i (a^{v+1} - b^{v+1}, p_i^m), q|q'$$

Set

$$0 \le x, x' < q'$$
$$0 \le y, y' < q'r + r$$
$$d := (x - x', q^m)$$
$$l := \prod_i [\frac{a^{v+1}}{b^{v+1}}]_{p_i^m}$$

Consider

$$[lax - by = lax' - by' = q'rU]_{q'^2} \tag{2.1}$$
$$(x, y, x', y') = (b, a, b, a)$$

After checking the freedom and determination of variables and the symmetry between $(x, y), (x', y')$, and with the Drawer Principle, we can find two *distinct* points $(x, y), (x', y')$ satisfying these conditions.

Make for some z

$$[lax - kby = lax' - kby']_{p_i^m}$$
$$[k = \frac{u}{b(by - by')} := 1 + q^2 z/d]_{p_i^m}$$
$$K := \frac{\overline{[u^{p_i-1}]}_{p_i^m}}{b^{p_i-1}(by - by')^{p_i-1}}$$

Therefore

$$[l^{p_i-1}(ax - ax')^{p_i-1} = K(by - by')^{p_i-1}]_{p_i^m}$$
$$[a^{p_i-1}(ax - ax')^{p_i-1} = Kb^{p_i-1}(by - by')^{p_i-1}]_{p_i^m}$$
$$[a^{p_i-1}(ax - ax')^{p_i-1} = \overline{[u^{p_i-1}]}_{p_i^m}]_{p_i^m}$$

Because

$$|a^{p_i-1}(ax - ax')^{p_i-1} - \overline{[u^{p_i-1}]}_{p_i^m}| < p_i^m$$

then

$$Z^{p_i-1} := a^{p_i-1}(ax - ax')^{p_i-1} = \overline{[u^{p_i-1}]}_{p_i^m}$$

Vary m on this formula

$$Z^{p_i-1} = \overline{[u^{p_i-1}]}_{p_i^{m'}}, m' << m$$

Hence

$$\overline{[\overline{[u]}_{p_i^{m'}}^{p_i-1}]}_{p_i^{m'}} = \overline{[\overline{[u]}_{p_i^m}^{p_i-1}]}_{p_i^m}$$
$$\overline{[\overline{[u]}_{p_i^m}^{p_i-1}]}_{p_i^{m'}} = \overline{[\overline{[u]}_{p_i^m}^{p_i-1}]}_{p_i^m}$$

Then

$$\overline{[u]}_{p_i^m}^{p_i-1} << p_i^m$$
$$Z^{p_i-1} = \overline{[u]}_{p_i^m}^{p_i-1}$$

$$Z = \overline{[u]}_{p_i^m}$$

This means

$$[a^2(x - x') = kb^2(y - y')]_{p_i^m}$$

It's invalid unless

$$q'|d$$

So that

$$[ax - by = ax' - by']_{q'^2}$$
$$|(ax - by) - (ax' - by')| < q'^2$$
$$ax - by = ax' - by'$$
$$x - x' = y - y' = 0$$

It's invalid.

If (q', p_i^m) is great enough then

$$a^{p_i - 1} = b^{p_i - 1}$$

It's invalid. □

Remark 2.2. We can find $(d, p_i^m) = (q', p_i^m)$ is impossible under only the construction 2.1 and without the condition $(l - 1, q^m) = q'$, because if not make

$$(x, y, x', y') \to (x, y, x', y') + q'z'(b, la, b, la)$$

to set

$$[lax - kby = 0]_{p_i^m}$$

then

$$[xy' = x'y]_{p_i^m}$$

hence

$$[by(l\frac{ax}{by} - 1) = by'(l\frac{ax'}{by'} - 1)]_{(q'^2 r, p_i^m)}$$

It's invalid.

Theorem 2.3. *For prime p and positive integer q the equation*

$$a^p + b^p = c^q$$

has no integer solution (a, b, c) such that $(a, b) = (b, c) = (a, c) = 1, a, b > 0$ if $p > 8, q > 2$.

Proof. Make logarithm on a, b in mod c^q. The conditions are sufficient for a controversy. Prove on the module $(a - b, c)^m$ or the other part of module. □

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