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Abstract:

This paper shows the importance of two properties, which are at the base of the Riemann hypothesis.
The key point of all the reasoning about the validity of the Riemann hypothesis is in the fact that only if the Riemann hypothesis is true, these two properties, which are satisfied by the non-trivial zeros, are both true.
In fact, only if these two properties are both true , all non-trivial zeros lie on the critical line.

The lines (------------------------) divide the main parts of the paper.

Two properties at the base of the Riemann hypothesis: an argument for its truth

The Riemann zeta function is defined by the Dirichlet series:

0. $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, for every complex number s with real part Re(s) greater than 1.

However, this function can be analytically continued to a holomorphic function on the whole complex s-plane, except for a simple pole at s=1, and it satisfies the following functional equation:

1. $\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\ \zeta(1-s)$ for $s \in$ C \ {0, 1}, where s is a complex number and Γ is the gamma function. [1]

First of all, we study the factors of the functional equation:

1.A) 2^s**:**

The factor 2^s is an exponential function of the form $y = a^s$, with $a = 2$ and $s \in$ C. Since the base of the exponent is different from 0, there is no value of s such that cancels the factor 2^s and therefore the functional equation **(1.)**.

2.A) π^{s-1}**:**

Doing a similar reasoning to that used for the factor **(1.A),** we can assert that there is no value of s such that cancels the factor π^{s-1} and therefore the functional equation **(1.).**

3.A) $\sin\left(\frac{\pi s}{2}\right)$**:**

Since the factor $\sin\left(\frac{\pi s}{2}\right)$ is a goniometric function of the form y=sin(x), it cancels the functional equation **(1.)** when the argument of the sin equals to $n\pi$, with $n \in$ **Z**.

In this case we have that:

$$\frac{\pi s}{2} = n\pi$$

$$\pi s = 2n\pi$$

$s = 2n$, with $n \in$ **Z**

Because the Riemann zeta function has no zeros in the half-plane Re(s)>1 and on the line Re(s) = 0 [1], we get:

$s = -2n$, with $n \in$ N \ {0}.

Thus, when $s = -2n$, with $n \in$ N \ {0}, the functional equation **(1.)** equals 0. Therefore:

$\zeta(-2n) = 0$ **,** with $n \in$ **N** \ {0}

$s = -2n$, with $n \in$ N \ {0} are the trivial zeros.

4.A) $\Gamma(1-s)$**:**

Since the gamma function has the property to be non- zero everywhere, there is no value of s such that cancels the factor $\Gamma(1-s)$ and therefore the functional equation **(1.)**.[2]

5.A) $\zeta(1-s)$ **:**

For the zero-product property, the functional equation **(1.)** equals 0 if $\zeta(1-s)$ =*0*. Therefore

$\zeta(1-s) = \zeta(s) = 0$

In addition, when $\zeta(1-s)$ =0, s is not a trivial zero. In fact:

Let s be a trivial zero. We substitute it in the functional equation **(1.):**

$\zeta(-2n) = 2^{-2n}\pi^{-2n-1}\sin(-n\pi)\Gamma(1+2n)\zeta(1+2n)$

$\zeta(1+2n) = \zeta(-2n) = 0$

$\zeta(1+2n) = \zeta(-2n) = 0$ is an absurd, since $\zeta(1+2n) \neq 0$ for the fact that the Riemann zeta function has no zeros in the half-plane Re(s)>1.

At this point, from the study of the functional equation **(1.)** we follow this thought:

- Since the equation **(1.)** is valid for $s \in$ C \ {0, 1} [1], it will be satisfied also by the non-trivial zeros, which have 0<Re(s) <1.
- Since it is satisfied by non-trivial zeros, when *s* is a non-trivial zero one between 2^s , π^s , $\sin\left(\frac{\pi s}{2}\right)$, $\Gamma(1-s)$and $\zeta(1-s)$ will cancel the equation **(1.)**.
- Since 2^s , π^s and $\Gamma(1-s)$ don't cancel the equation **(1.)** and $\sin\left(\frac{\pi s}{2}\right)$only for the trivial zeros, which are of the form $s = -2n$ for $n \in$ N \ {0} [for what we have said previously],when *s* is a non-trivial zero only $\zeta(1-s)$ will cancel the equation **(1.)**.
- Since only $\zeta(1-s)$ cancels the equation **(1.)** for non-trivial zeros, when *s* is a non-trivial zeros $\zeta(1-s) = 0$ and for the zero-product property $\zeta(s) = 0$.
- Thus, we deduce that $\zeta(s) = \zeta(1-s)$, when *s* is a non-trivial zero.

From the previous reasoning we get the first property of non-trivial zeros, which is:

$$\textbf{2.}\ \zeta(s)=\zeta(1-s)$$

From the equation **(2.)** we observe that:

2.A) Non-trivial zeros are symmetric with respect to the point $P\left(\frac{1}{2};0\right)$

Now, we introduce the mirror symmetry formula of the Riemann zeta function:

$$\textbf{3.}\ \zeta(\overline{s})=\overline{\zeta(s)}$$ [1] [3]

Following a similar thought to that done before…

- Since the equation **(3.)** is valid for $s \in C \setminus \{1\}$ [1], it will be satisfied also by the non-trivial zeros.
- Hence, we deduce that $\zeta(\overline{s})=\overline{\zeta(s)}$, when s is a non-trivial zero.

… we obtain the second property of non-trivial zeros, which is:

$$\textbf{4.}\ \zeta(\overline{s})=\overline{\zeta(s)}$$

From the equation **(4.)** we observe that:

4.A) The equation **(4.)** implies that non-trivial zeros are symmetric with respect to the real line. In fact:

- When s is a non-trivial zero, $\zeta(s)=0$.
- Since $\zeta(s)=0$, $\overline{\zeta(s)}=0$ and so $\zeta(s)=\overline{\zeta(s)}$
- Since $\overline{\zeta(s)}=\zeta(\overline{s})$, $\zeta(s)=\zeta(\overline{s})$
- Thus, since $\zeta(s)=\zeta(\overline{s})$, non-trivial zeros are symmetric with respect to the real line.

So for the observation **4.A,** we can rewrite the equation **(4.)** in this way for the non-trivial zeros:

$$\textbf{5.}\ \zeta(s)=\zeta(\overline{s})$$

At this point, we have shown that non-trivial zeros satisfy both equations **(2.)** and **(5.).** Hence the following system is valid for non-trivial zeros:

$$\textbf{6.}\begin{cases}\zeta(s) & =\zeta(1-s)\\ \zeta(s) & =\zeta(\overline{s})\end{cases}$$

From now on, we will consider non-trivial zeros in couples so that:

$$\textbf{7.}\ Z_n=\left(s_{1(n)};s_{2(n)}\right),$$

where Z_n is nth couple of non-trivial zeros, $s_{1(n)}$ is the first non-trivial zero of the nth couple and $s_{2(n)}$ is the second non-trivial zero of the nth couple.

Now, we rewrite the system **(6.)**, using the formula **(7.)**:

$$\textbf{8.}\begin{cases} Z_n = (s_n; 1 - s_n) \\ Z_n = (s_n; \bar{s}_n) \end{cases}$$

From the system **(8.)** we observe that:

8.A) A couple Z_n of non-trivial zeros is made by two non-trivial zeros so that let $i\,t_1$ and $i\,t_2$ their imaginary parts, $|i\,t_1| = |i\,t_2|$.

8.B) For the first equation of the system **(8.)** the two non-trivial zeros of a couple Z_n are symmetric with respect to the point P$\left(\frac{1}{2};0\right)$, according to the observation **2.A**.

8.C) For the second equation of the system **(8.)** the two non-trivial zeros of a couple Z_n are symmetric with respect to the real line, according to the observation **4.A**.

8.D) The combination between the observations **8.B** and **8.C** implies the symmetry about the critical line $x = \frac{1}{2}$ of the non-trivial zeros.

From the observations **8.B)** and **8.C)** we obtain the key statement, which is at the base of the Riemann hypothesis:

9. *" The non-trivial zeros of the Riemann zeta function are situated in the complex plane, arranged in couples, symmetric with respect to the point P*$\left(\frac{1}{2};0\right)$ *and to the real line ,so that let* $i\,t_1$ *and* $i\,t_2$ *the imaginary parts of the two non-trivial zeros of the couple,* $|i\,t_1| = |i\,t_2|$*".*

At this point, we show that only if the real part of the non-trivial zeros is equal to $\frac{1}{2}$, that is when the Riemann hypothesis is true, the statement **(9.)** is satisfied by all couples of non-trivial zeros Z_n.

In order to prove the statement **(9.)** we consider the following system, in which also the symmetry about the critical line $x = \frac{1}{2}$ is expressed:

$$\textbf{10.}\begin{cases} \begin{cases} \zeta(z_j) = \zeta(s) \\ \zeta(z_k) = \zeta(1-s) \end{cases} & \textit{simmetry about } P(\tfrac{1}{2};0) \\ \begin{cases} \zeta(z_j) = \zeta(s) \\ \zeta(z_k) = \zeta(\bar{s}) \end{cases} & \textit{simmetry about the real line} \\ \begin{cases} \zeta(z_j) = \zeta(s) \\ \zeta(z_k) = \zeta(1-\bar{s}) \end{cases} & \textit{simmetry about the critical line} \end{cases}$$

for every couples of non-trivial zeros (z_j, z_k) with imaginary part so that, $|i\,t_j| = |i\,t_k|$.

Proof of the statement (9.):

We study all the cases, also the limit ones (a limit case is when $z_j \equiv z_k$):

First of all, we exclude the impossible ones (those cases in which at least one of the three conditions is not satisfied):

A)
$$\begin{cases} \begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(1-s) \end{cases} \\ \begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(\overline{s}) \end{cases} \\ \begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(1-\overline{s}) \end{cases} \end{cases}$$

B)
$$\begin{cases} \begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(1-s) \end{cases} & \wedge\, \zeta(z_j)\equiv\zeta(s)\equiv\zeta(z_k)\equiv\zeta(1-s) \\ \begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(\overline{s}) \end{cases} \\ \begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(1-\overline{s}) \end{cases} \end{cases}$$

C)
$$\begin{cases} \begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(1-s) \end{cases} & \wedge\, \zeta(z_j)\equiv\zeta(s)\equiv\zeta(z_k)\equiv\zeta(1-s) \\ \begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(\overline{s}) \end{cases} & \wedge\, \zeta(z_j)\equiv\zeta(s)\equiv\zeta(z_k)\equiv\zeta(\overline{s}) \\ \begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(1-\overline{s}) \end{cases} \end{cases}$$

D)
$$\begin{cases} \begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(1-s) \end{cases} & \wedge\, \zeta(z_j)\equiv\zeta(s)\equiv\zeta(z_k)\equiv\zeta(1-s) \\ \begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(\overline{s}) \end{cases} \\ \begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(1-\overline{s}) \end{cases} & \wedge\, \zeta(z_j)\equiv\zeta(s)\equiv\zeta(z_k)\equiv\zeta(1-\overline{s}) \end{cases}$$

$$
\textbf{E)}\ \left\{\begin{array}{ll}
\begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(1-s) \end{cases} & \\
\begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(\overline{s}) \end{cases} & \wedge\, \zeta(z_j)\equiv\zeta(s)\equiv\zeta(z_k)\equiv\zeta(\overline{s}) \\
\begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(1-\overline{s}) \end{cases} & \wedge\, \zeta(z_j)\equiv\zeta(s)\equiv\zeta(z_k)\equiv\zeta(1-\overline{s})
\end{array}\right.
$$

Now we study the possible cases (those cases, in which all conditions are satisfied):

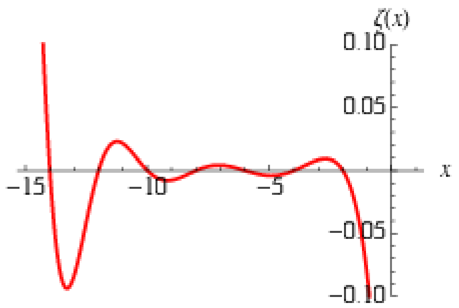
$$
\textbf{F)}\ \left\{\begin{array}{ll}
\begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(1-s) \end{cases} & \wedge\, \zeta(z_j)\equiv\zeta(s)\equiv\zeta(z_k)\equiv\zeta(1-s) \\
\begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(\overline{s}) \end{cases} & \wedge\, \zeta(z_j)\equiv\zeta(s)\equiv\zeta(z_k)\equiv\zeta(\overline{s}) \\
\begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(1-\overline{s}) \end{cases} & \wedge\, \zeta(z_j)\equiv\zeta(s)\equiv\zeta(z_k)\equiv\zeta(1-\overline{s})
\end{array}\right.
$$

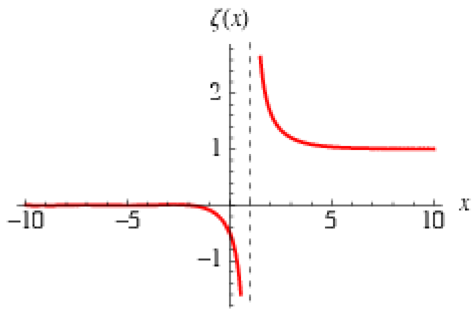
The only solution of the case F) is $z_j \equiv z_k \equiv \frac{1}{2}$. However, since $\zeta(\frac{1}{2}) = -1{,}46...$ and so $\neq 0$ [1], we can exclude it, because we will have no non-trivial zero.

$$
\textbf{G)}\ \left\{\begin{array}{ll}
\begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(1-s) \end{cases} & \\
\begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(\overline{s}) \end{cases} & \wedge\, \zeta(z_j)\equiv\zeta(s)\equiv\zeta(z_k)\equiv\zeta(\overline{s}) \\
\begin{cases} \zeta(z_j)=\zeta(s) \\ \zeta(z_k)=\zeta(1-\overline{s}) \end{cases} &
\end{array}\right.
$$

Also this case can be excluded, since the Riemann zeta function has no zeros on the real line in the critical strip.

This graph of $\zeta(s)$, where s is a real number, shows it **[4]**:





H) $\begin{cases} \begin{cases} \zeta(z_j) = \zeta(s) \\ \zeta(z_k) = \zeta(1-s) \end{cases} \\ \begin{cases} \zeta(z_j) = \zeta(s) \\ \zeta(z_k) = \zeta(\overline{s}) \end{cases} \\ \begin{cases} \zeta(z_j) = \zeta(s) \\ \zeta(z_k) = \zeta(1-\overline{s}) \end{cases} \qquad \wedge \zeta(z_j) \equiv \zeta(s) \equiv \zeta(z_k) \equiv \zeta(1-\overline{s}) \end{cases}$

Since the previous possible cases F and G are excluded because of the fact that we have obtained no non-trivial zero, only the case H, which represents the Riemann hypothesis, is valid; in fact the case H is the only one that satisfies all the conditions and has non-trivial zeros.

Therefore the statement **(9.)** is confirmed and this is the argument for the truth of the Riemann hypothesis.

The fact that we have a limit case in H, regarding the symmetry with respect to the critical line $x = \frac{1}{2}$, implies that this symmetry is actually an unnecessary condition and this is why we have two properties at the base of the Riemann hypothesis.

References:

[1]: *https://de.wikipedia.org/wiki/Riemannsche_%CE%B6-Funktion*
[2]: *https://en.wikipedia.org/wiki/Gamma_function*
[3]: *http://functions.wolfram.com/ZetaFunctionsandPolylogarithms/Zeta/04/02/01/*
[4]: *http://mathworld.wolfram.com/RiemannZetaFunction.html*