A rational cover

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Abstract

This letter demonstrates in an elegant way that Cantor's postulate that there's a bijection between any two countable infinite sets is flawed.

There's an interesting theorem in contemporary mathematics stating that one can cover all the rational numbers in the interval $[0, 1]$ by non-vanishing intervals whose union is of length less than 1. The proof goes as follows.

One can list all the rationals in interval $[0, 1]$ in an infinitely long list like this:

$$\begin{array}{llll} \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ \vdots \end{array}$$

Now cover each of these rationals by an interval, each of length 3^{-n}, with n being naturals, and let numbers n count rationals in the order as written above. So, cover point $1/2$ on the number line by an interval of length $1/3$, cover $1/3$ by an interval of length $1/9$, cover $2/3$ by an interval of length $1/27$ and so on.

There are obviously countably many rationals in interval $[0, 1]$, since we've listed them all just now in the list above. There are also countably many naturals n. Hence, one can indeed cover each rational from the above list by an interval of length 3^{-n}. We notice that each of the covering intervals is of a non-vanishing length 3^{-n} for any n.

This way, the union of all the intervals of lengths 3^{-n} is at most of length

$$\sum_{n \in \mathbb{N}} 3^{-n} = \frac{1}{2} \tag{1}$$

This is a well known result.

Now, consider this.

There are infinitely many rationals in interval $[0, 1]$. Actually, there is a rational between any two distinct reals. Just consider any two distinct reals written in a decimal form. There's always a number with finite number of decimals that fits in-between any two reals. For instance, between 0.1 and 0.11 one can fit 0.105. This is obviously a trivial conclusion. So one can fit a rational between any two distinct reals.

Now, if the cover of all rationals in interval $[0, 1]$ is of length $1/2$, whilst the interval $[0, 1]$ is obviously of length 1, then there are plenty of pairs of reals not being covered. Since one can fit a rational between any two reals, it's obvious that this covering does not cover all the rationals in interval $[0, 1]$ then.

To illustrate this point further, please consider Figure 1.

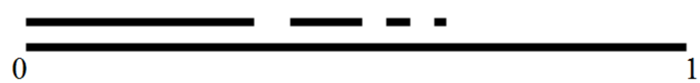

Figure 1: *The unit interval* $[0, 1]$ *and first four covering intervals above it.*

So we're covering the unit interval with some smaller intervals. If there's a gap between covering intervals, as depicted in Figure 1, then one can find rational numbers on the number line beneath the gap. These rational numbers beneath the gap between two adjacent covering intervals are obviously not covered.

Thus, in order to cover the rationals underneath the gaps, one must avoid making any gaps during the covering process. If there was just one gap between any two covering intervals, one could certainly find rationals underneath the gap on the number line that are not covered. This is so because rational numbers are dense in $\mathbb{R}$. One can always find a rational number between any two distinct real numbers.

Thus, one must not allow for any gaps between covering intervals. Since we're being economic, we decide to start the covering process by placing the first covering interval on top of the very beginning of the unit interval, as depicted in Figure 1. This way the very first covering interval covers as much length of the unit interval as possible. We then continue to cover the unit interval with all the other covering intervals, minding not to leave any gaps between any two covering intervals. Continuing this way, we can cover the unit interval in one way and one way only, as depicted in Figure 2.

Obviously, we've covered only one half of the unit interval this way, since the length of all the covering intervals altogether is $\sum_{n \in \mathbb{N}} 3^{-n} = 1/2$, and since there can be no gaps. And this is the most economic way of covering the unit interval. And yet there's half of the unit interval not covered. Do not notice that the requirement that there can be no gaps leaves no other possibility, once the first covering interval is laid on top of the beginning of the unit interval, as

Figure 2: *The unit interval* $[0, 1]$ *and all the covering intervals of lengths* 3^{-n} *laid one next to another leaving no gaps between them.*

seen from left to right.

So what went wrong? This conclusion that one can cover rationals in interval $[0, 1]$ by a cover of arbitrary length is considered rigorous and true by contemporary mathematics. And it's obviously wrong. So what went wrong?

What went wrong is that there aren't as many rationals in interval $[0, 1]$ as there are naturals altogether. There are more rationals than naturals. Just consider the list above, listing some rationals in interval $[0, 1]$. The first column reads

$$\frac{1}{2}$$
$$\frac{1}{3}$$
$$\frac{1}{4}$$
$$\frac{1}{5}$$
$$\vdots$$

So there's obviously a bijection between the first column above and all the naturals n.

But there are more rationals. By looking at the list, it seems that there are about $O(N^2)$ rationals for all sufficiently large naturals N. There are definitely more rationals in the list than there are naturals: just look at it.

Now, if we discard Cantor's postulate that there's a bijection between any two countable infinite sets, then we see where the problem is: the covering with covers of lengths 3^{-n} cannot cover all the rationals in interval $[0, 1]$. It only covers first N, as N grows without bounds. So some rationals are simply left uncovered, since there are infinitely more rationals in interval $[0, 1]$ than there are naturals overall.

This demonstrates in an elegant way that Cantor's postulate that there's a bijection between any two countable infinite sets is flawed.