# The curvature and dimension of a fractal surface

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**Abstract**

The curvature of a surface can lead to fractional dimension. In this paper, the properties of the 2-sphere surface of a three-dimensional ball and the $2.x$-surface of a three-dimensional fractal set are considered. Tessellation is used to approximate each surface, primarily because the $2.x$-surface of a three-dimensional fractal set is otherwise non-differentiable (having no well-defined surface normal).

## 1 Tessellation of surfaces

Approximating the surface of a three-dimensional shape via triangular tessellation (a mesh) allows us to calculate the surface's dimension $D \in [2.0, 3.0)$.

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First we calculate, for each triangle, the average dot product of the triangle's normal $\hat{n}_i$ and its three neighbouring triangles' normals $\hat{o}_1$, $\hat{o}_2$, $\hat{o}_3$:

$$d_i = \frac{\hat{n}_i \cdot \hat{o}_1 + \hat{n}_i \cdot \hat{o}_2 + \hat{n}_i \cdot \hat{o}_3}{3}. \tag{1}$$

Because we assume that there are three neighbours per triangle, the mesh must be closed (no cracks or holes).

Then we calculate the normalized measure:

$$m_i = \frac{1.0 - d_i}{2.0}. \tag{2}$$

Once $m_i$ has been calculated for all triangles, we can then calculate the average normalized measure $\lambda$, where $t$ is the number of triangles:

$$\lambda = \frac{\sum_{i=1}^{t} m_i}{t}. \tag{3}$$

The dimension of the surface is

$$D = 2.0 + \lambda. \tag{4}$$

Analogous is the dimension of a line $D = 1.0 + \lambda$, where $D \in [1.0, 2.0)$. See Figures 1 and 2.

In this paper, Marching Cubes [1] is used to generate the triangular tessellations. The full C++ code can be found at [2].

For a 2-sphere, the *local* curvature vanishes as the size of the triangles decreases. This results in a dimension of 2.0, which is to be expected from a non-fractal surface. See Figures 3, 4, and 5.

On the other hand, for the $2.x$-surface of a three-dimensional fractal set, the local curvature does not vanish. This results in a dimension greater than 2.0, but not equal to or greater than 3.0, which is to be expected from a fractal surface. See Figures 6, 7, 8, and 9.

As far as we know, this method of calculating the fractal dimension of a surface is novel.

# References


[1] `http://paulbourke.net/geometry/polygonise/`

[2] `https://github.com/sjhalayka/meshdim`

Figure 1: The average dot product of neighbouring line segments is $d_i = (0.0 + 0.0)/2 = 0.0$. This leads to a normalized measure of $m_i = (1.0 - d_i)/2.0 = 0.5$, which in turn leads to an average normalized measure of $\lambda = 0.5$. The dimension is $D = 1.0 + \lambda = 1.5$.

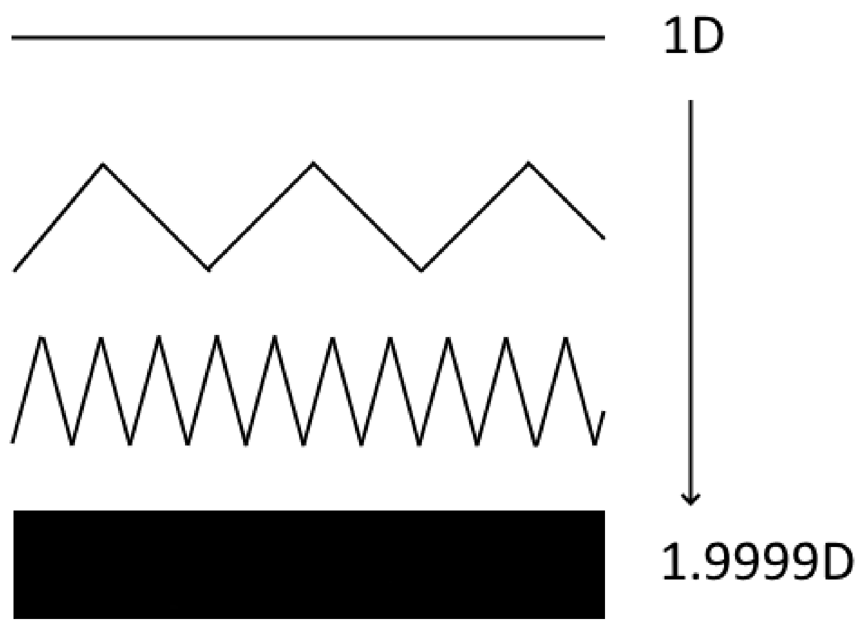


Figure 2: A line as it goes from dimension 1.0 (at top) to 1.9999 (at bottom). In the end, where the dimension is 1.9999, the result is practically a rectangle.

Figure 3: Low resolution surface for the iterative equation is $Z = Z^2$. The surface's dimension is 2.02.

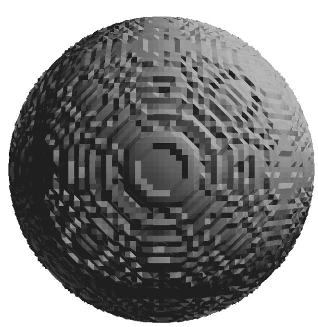

Figure 4: Medium resolution surface for the iterative equation is $Z = Z^2$. The surface's dimension is 2.06.

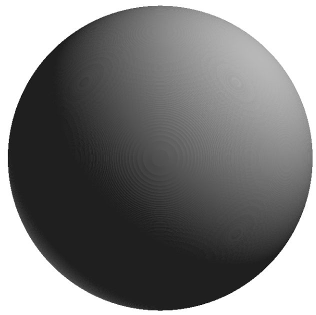

Figure 5: High resolution surface for the iterative equation is $Z = Z^2$. The surface's dimension is 2.0.

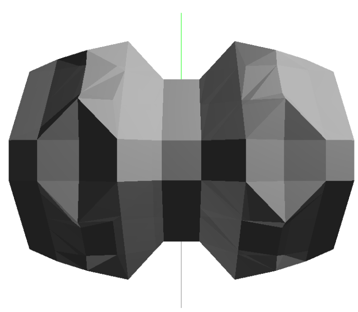

Figure 6: Low resolution surface for the iterative equation is $Z = Z\cos(Z)$. The surface's dimension is 2.05.

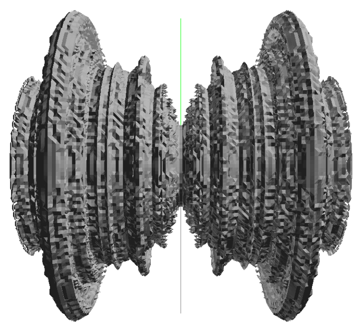

Figure 7: Medium resolution surface for the iterative equation is $Z = Z\cos(Z)$. The surface's dimension is 2.11.

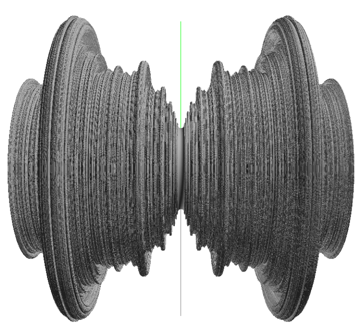

Figure 8: High resolution surface for the iterative equation is $Z = Z\cos(Z)$. The surface's dimension is 2.08.

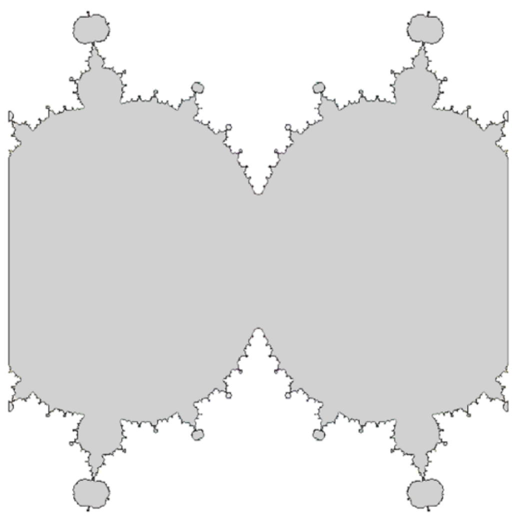

Figure 9: A two-dimensional slice of $Z = Z\cos(Z)$, showing the fractal nature of the set.