The curvature and dimension of a closed surface

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Abstract

The curvature of a closed surface can lead to fractional dimension. In this paper, the properties of the 2-sphere surface of a three-dimensional ball and the $2.x$-dimensional surface ($2.x$-surface) of a three-dimensional fractal set are considered. Tessellation is used to approximate each surface, primarily because the $2.x$-surface of a three-dimensional fractal set is otherwise non-differentiable (having no well-defined surface normals).

1 Tessellation of closed surfaces

Approximating the $2.x$-surface of a three-dimensional shape via triangular tessellation (a mesh) allows us to calculate the $2.x$-surface's dimension $D \in (2.0, 3.0)$.

First we calculate, for each triangle, the average dot product of the triangle's normal $\hat{n}_i$ and its three neighbouring triangles' normals $\hat{o_1}$, $\hat{o_2}$, $\hat{o_3}$:

$$d_i = \frac{\hat{n}_i \cdot \hat{o_1} + \hat{n}_i \cdot \hat{o_2} + \hat{n}_i \cdot \hat{o_3}}{3}. \tag{1}$$

Because we assume that there are three neighbours per triangle, the mesh must be *closed* (no cracks or holes, precisely two triangles per edge).

Then we calculate the normalized measure:

$$m_i = \frac{1 - d_i}{2}. \tag{2}$$

Once m_i has been calculated for all triangles, we can then calculate the average normalized measure λ, where t is the number of triangles:

$$\lambda = \frac{\sum_{i=1}^{t} m_i}{t}. \tag{3}$$

The dimension of the closed surface is:

$$D = 2 + \lambda. \tag{4}$$

In this paper, Marching Cubes [1] is used to generate the $2.x$-dimensional triangle meshes. The full C++ code for this paper can be found at [2].

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2 Conclusions

For a 2-sphere, the *local* curvature all but vanishes as the maximum triangle edge length L decreases:

$$\lim_{L\to 0} \lambda(L) = 0. \tag{5}$$

To decrease L, one must increase the sampling resolution r (an integer, greater than or equal to 2), where g_{max} is the sampling grid maximum and g_{min} is the sampling grid minimum:

$$L = \sqrt{3}\underbrace{\left(\frac{g_{max} - g_{min}}{r - 1}\right)}_{\text{step size}}. \tag{6}$$

This results in a dimension of practically (but never quite) 2.0, which is to be expected from a non-fractal surface. See Figures 1 - 3.

On the other hand, for the $2.x$-surface of a three-dimensional fractal set, the local curvature does not vanish:

$$\lim_{L\to 0} \lambda(L) \neq 0. \tag{7}$$

This results in a dimension considerably greater than 2.0, but not equal to or greater than 3.0, which is to be expected from a fractal surface. See Figures 4 - 7.

As far as we know, this method of calculating the dimension of a closed surface is novel.

References

[1] http://paulbourke.net/geometry/polygonise/

[2] https://github.com/sjhalayka/meshdim

Figure 1: Low resolution ($r = 10$) surface for the iterative equation is $Z = Z^2$. The surface's dimension is 2.02.

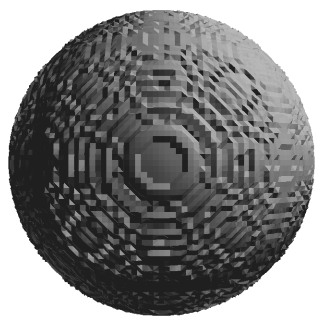

Figure 2: Medium resolution ($r = 100$) surface for the iterative equation is $Z = Z^2$. The surface's dimension is 2.06.

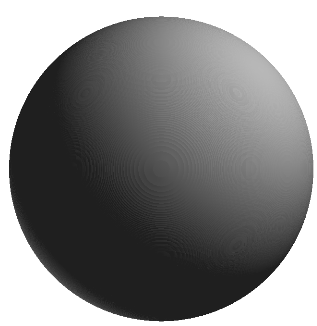

Figure 3: High resolution ($r = 1000$) surface for the iterative equation is $Z = Z^2$. The surface's dimension is practically 2.0.

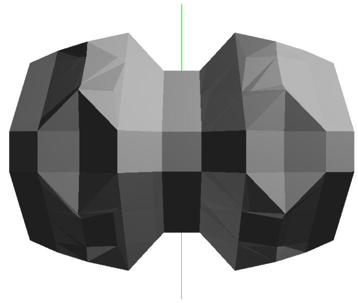

Figure 4: Low resolution ($r = 10$) surface for the iterative equation is $Z = Z\cos(Z)$. The surface's dimension is 2.05.

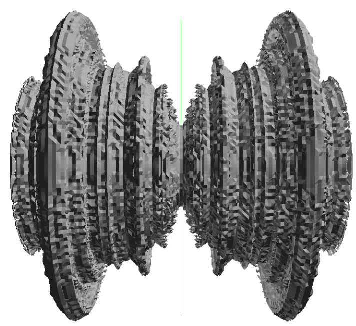

Figure 5: Medium resolution ($r = 100$) surface for the iterative equation is $Z = Z\cos(Z)$. The surface's dimension is 2.11.

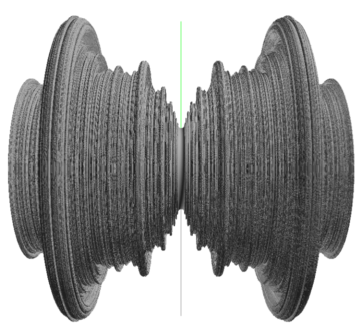

Figure 6: High resolution ($r = 1000$) surface for the iterative equation is $Z = Z\cos(Z)$. The surface's dimension is 2.08.

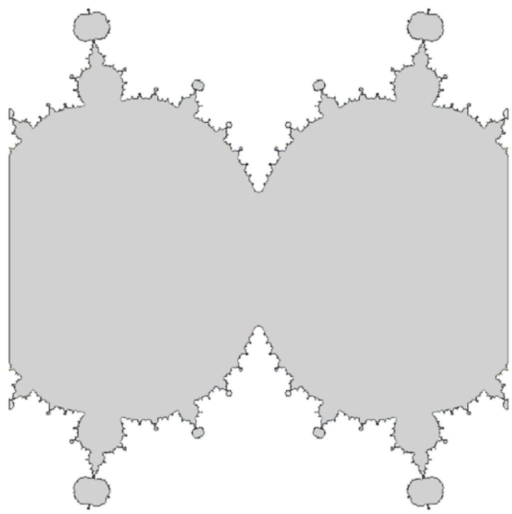

Figure 7: A two-dimensional slice of $Z = Z\cos(Z)$, showing the fractal nature of the set.