A RESOLUTION OF THE BROCARD-RAMANUJAN PROBLEM

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Abstract. We identify equivalent restatements of the Brocard-Ramanujan diophantine equation, $(n!+1) = m^2$; and employing the properties and implications of these equivalencies, prove that for all $n > 7$, there are no values of n for which $(n!+1)$ can be a perfect square.

1. Introduction

In a question first posed in 1876, in the journal Nouvelle Correspondance Mathematique, Henri Brocard asked, "*For which values of the integer x is the expression, $[(1 \cdot 2 \cdot 3 \cdot 4 \cdot \ \ldots \ \cdot x) + 1]$, a perfect square?*"[1]. This product of the sequence of integers from 1 to x is known as "the factorial of x" or "x factorial", denoted "$x!$" (hereafter we shall dispense with x and substitute n its place). Brocard had previously observed that for certain values of "n", $n!$ plus 1 was a perfect square.

$$n = 4: \quad (n!+1) = [(1{\cdot}2{\cdot}3{\cdot}4)+1] = (24+1) = 25 = 5^2$$

$$n = 5: \quad (n!+1) = [(1{\cdot}2{\cdot}3{\cdot}4{\cdot}5)+1] = (120+1) = 121 = 11^2$$

$$n = 7: \quad (n!+1) = [(1{\cdot}2{\cdot}3{\cdot}4{\cdot}5{\cdot}6{\cdot}7)+1] = (5040+1) = 5041 = 71^2.$$

Srinivasa Ramanujan, unaware of Brocard's earlier journal question, independently made this same observation in 1913: "*The number $(1+n!)$ is a perfect square for the values, 4, 5, 7, of n. Find other solutions"* [2, 3]. Both mathematicians sought the answer to the additional question: Are 4, 5, and 7 the only values of n for which $(n!+1)$ is a perfect square, and if so, why these and no others?

Theorem 1.1. *For all positive integers m and n, except $n = 4$, $n = 5$, and $n = 7$, there are no other values of n for which $(n!+1) = m^2$.*

Proof. Consider, that for every positive integer n, $n! = [(1{\cdot}2{\cdot}3{\cdot} \ldots {\cdot}(n-1){\cdot}n)$ is also the product of $[n \cdot (n-1)!]$; and for all $n \geq 2$, the product of $[(n-2)! \cdot ((n-1) \cdot n)]$.

Table 1. **Alternative Calculations of $n!$ for Selected Values of n**

n	$n! = [n \cdot (n-1)!]$	$n! = [(n-2)! \cdot ((n-1) \cdot n)]$
1	$1! = (1 \cdot 0!) = (1 \cdot 1) = 1$	- - -
2	$2! = (2 \cdot 1!) = (2 \cdot 1) = 2$	$[0! \cdot (1 \cdot 2)] = (1 \cdot 2) = 2$
3	$3! = (3 \cdot 2!) = [3 \cdot (1 \cdot 2)] = (3 \cdot 2)$	$[1! \cdot (2 \cdot 3)] = (1 \cdot 6) = 6$
4	$4! = (4 \cdot 3!) = [4 \cdot (1 \cdot 2 \cdot 3)] = (4 \cdot 6)$	$[2! \cdot (3 \cdot 4)] = (2 \cdot 12) = 24$
5	$5! = (5 \cdot 4!) = [5 \cdot (1 \cdot 2 \cdot 3 \cdot 4)] = (5 \cdot 24)$	$[3! \cdot (4 \cdot 5)] = (6 \cdot 20) = 120$
6	$6! = (6 \cdot 5!) = [6 \cdot (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5)] = (6 \cdot 120)$	$[4! \cdot (5 \cdot 6)] = (24 \cdot 30) = 720$
7	$7! = (7 \cdot 6!) = [7 \cdot (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6)] = (7 \cdot 720)$	$[5! \cdot (6 \cdot 7)] = (120 \cdot 42) = 5040$

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And for all $n > 1$, $n!$ is a product of 2, and where $(n! + 1) = m^2$, m can only be an odd integer.

Then where $(n! + 1) = m^2$, $n! = (m^2 - 1) = [(m - 1)(m + 1)]$, and $(m - 1)$ and $(m + 1)$ are both even. Let the notation, "$2^{\geq 2}$", be read as "a power of 2 greater than or equal to 2^2".

With m an odd integer then m can be expressed in the form $m = (2x + 1)$ where x is an *odd* or *even* positive integer. If x is even, $(m - 1) = ((2x + 1) - 1) = 2x$ is a product of $2^{\geq 2}$; and $(m + 1) = ((2x + 1) + 1) = (2x + 2) = 2(x + 1)$ is a product of only 2^1. If x is odd, $(m - 1) = ((2x + 1) - 1) = 2x$ is the product of a power of 2 of only 2^1; and $(m + 1) = ((2x + 1) + 1) = (2x + 2) = 2(x + 1)$, a product of $2^{\geq 2}$.

And one of $(m - 1)$ and $(m + 1)$ is always the product of a power of 2 of only 2^1 and the other a product of $2^{\geq 2}$.

Then where $n! = (m^2 - 1)$, with $[(m + 1) - (m - 1)] = 2$, $(m^2 - 1)$ is the product of consecutive even integers; with each of the consecutive even integers further expressible as a product of 2^1— and in order for $n! = (m^2 - 1)$ and $(m - 1)$ and $(m + 1)$ to have a difference of 2, the co-factors of our 2^1 multipliers can only be consecutive integers of opposite parity having a difference of 1:

$$4! = (1{\cdot}2{\cdot}3{\cdot}4) = 24 = (4 \cdot 6) = [(2{\cdot}2) \cdot (2{\cdot}3)]$$
$$5! = (1{\cdot}2{\cdot}3{\cdot}4{\cdot}5) = 120 = (10 \cdot 12) = [(2{\cdot}5) \cdot (2{\cdot}6)] = [(2{\cdot}2) \cdot (5{\cdot}6)]$$
$$7! = (1{\cdot}2{\cdot}3{\cdot}4{\cdot}5{\cdot}6{\cdot}7) = 5040 = (70 \cdot 72) = [(2{\cdot}35) \cdot (2{\cdot}36)] = [(2{\cdot}2) \cdot (35{\cdot}36)].$$

And as

$$1! = 1,$$
$$2! = (1 \cdot 2) = 2,$$
$$3! = (1{\cdot}2{\cdot}3) = (2 \cdot 3) = 6,$$

cannot be expressed in the form, $[(2{\cdot}2) \cdot (odd{\cdot}even)]$ or $[(2{\cdot}2) \cdot (even{\cdot}odd)]$, where *odd* and *even* are sequential integers, then for all $n < 4$, $n!$ cannot equal $(m^2 - 1)$.

With $n!$ of all $n \geq 4$ expressible as a product of $(n - 2)! \cdot ((n - 1) \cdot n)$, where the factors $(n-1)$ and n are consecutive positive integers; and for all $(n-2) > 3$, $(n-2)!$ equal to the factorial of a lesser n value, also expressible as $[(2{\cdot}2) \cdot (odd{\cdot}even)]$ or $[(2{\cdot}2) \cdot (even{\cdot}odd)]$— where $(odd{\cdot}even)$ or $(even{\cdot}odd)$ may or may not be consecutive integers; then if we further designate the $(n - 2)!$ co-factors of $(2 \cdot 2)$ as a and b, with $a < b$, such that $(n - 2)! = [(2{\cdot}2) \cdot (a{\cdot}b)]$; and let $c = (n - 1)$ and $d = n$, then $n! = [(2{\cdot}2) \cdot (a{\cdot}b){\cdot}(c{\cdot}d)]$.

Reassociating the factors, $(a \cdot b){\cdot}(c \cdot d)$, into the product-pairs, $(a \cdot c)$ and $(b \cdot d)$, or $(a \cdot d)$ and $(b \cdot c)$, then $n! = (m^2 - 1)$ if and only if one of our product-pairs gives us consecutive integers of opposite parity with a difference of 1.

With $a < b$ and $c < d$, then the product-pair of greatest difference in magnitude is that of $(a{\cdot}b){\cdot}(c{\cdot}d))$— the difference between the product of the two lesser factors and the product of the two greatest factors. Followed by that of the greater integer in $(a \cdot b)$ *times* the greater integer in $(c \cdot d)$, and the lesser integer in $(a \cdot b)$ *times* the lesser integer in $(c \cdot d)$— i.e., $(b \cdot d)(a \cdot c)$. With the product-pair of least difference being that of the lesser integer in $(a \cdot b)$ *times* the greater integer in $(c \cdot d)$; and the greater integer in $(a \cdot b)$ *times* the lesser integer in $(c \cdot d)$— i.e., $(a \cdot d){\cdot}(b \cdot c)$.

And given that $n! = (m^2 - 1)$ only if the products of the reassociated $(a \cdot b){\cdot}(c \cdot d)$ co-factors of $(2{\cdot}2)$ differ by *one*, it is the product-pair of least difference, $(a{\cdot}d) \cdot (b{\cdot}c)$, that will reveal if we have consecutive integers and $n! = [(2{\cdot}2){\cdot}(a{\cdot}d){\cdot}(b{\cdot}c)] = (m^2{-}1)$.

Then where $n! = (m^2-1) = [(m-1)\cdot(m+1)] = (2ad\cdot 2bc)$; and $[(m+1)-(m-1)] = 2$, with $(2bc-2ad) = [2\cdot(bc-ad)]$; then $[2\cdot(bc-ad)]$ can equal 2 only if $(bc-ad) = 1$. That is, where a and b, and c and d, are consecutive integers; with $b = (a+1)$ and $d = (c+1)$, then $ad = [a\cdot(c+1)] = (ac+a)$, and $bc = [(a+1)\cdot c] = (ac+c)$, and

$$(bc-ad) = [(ac+c)-(ac+a)] = (c-a);$$

and for all $(n-2) \geq 4$, $(c-a) = 1$ only if $c = (a+1) = b$, $bc = c^2$, and $a = (d-2) = (c-1)$, giving us

$$(bc-ad) = (c^2-ad) = [c^2-(c-1)(c+1)] = [c^2-(c^2-1)].$$

For $n = 4$ and $n = 5$ where $(n-2) < 4$, with, respectively, $((n-1)\cdot n) = (3\cdot 4)$ and $((n-1)\cdot n) = (4\cdot 5)$, we have that the integer composition of each $(n-2)!$ is exactly that required to complete the $n!$ factor sequence:

$$\begin{aligned} 4! &= [(n-2)!\cdot(3\cdot4)] = [(1\cdot2)\cdot(3\cdot4)] = (1\cdot4)\cdot(2\cdot3) \\ &= (2\cdot2)\cdot(2\cdot3); \\ 5! &= [(n-2)!\cdot(4\cdot5)] = [(2\cdot3)\cdot(4\cdot5)] = [(2\cdot5)\cdot(3\cdot4)] = [(2\cdot5)\cdot(2\cdot6)] \\ &= (2\cdot2)\cdot(5\cdot6). \end{aligned}$$

And as noted by Brocard and Ramanujan, $(4!+1)$ and $(5!+1)$ are both perfect squares.

Our next n value is that of $n = 6$, with $(n-2)! = [(2\cdot2)\cdot(a\cdot b)] = [(2\cdot2)\cdot(2\cdot3)]$; and

$$6! = (1\cdot2\cdot3\cdot4\cdot5\cdot6) = [4!\cdot(5\cdot6)] = [(2\cdot2)\cdot(a\cdot b)\cdot(c\cdot d)] = [(2\cdot2)\cdot(2\cdot3)\cdot(5\cdot6)].$$

And with $a \neq (d-2)$ and $b \neq c$; then $ad = (2\cdot6)$ and $bc = (3\cdot5)$ are not consecutive integers— $(bc-ad) = (15-12) = 3$; and 6! cannot equal (m^2-1).

Which brings us to $n = 7$, where $(n-2)! = [(2\cdot2)\cdot(a\cdot b)] = (2\cdot2)\cdot(5\cdot6)$:

$$\begin{aligned} 7! &= (1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7) = [5!\cdot(6\cdot7)] = [(2\cdot2)\cdot(5\cdot6)\cdot(6\cdot7)] \\ &= [(2\cdot2)\cdot(5\cdot7)\cdot(6\cdot6)] = [(2\cdot2)\cdot(35\cdot36)]. \end{aligned}$$

And $(7!+1)$ is a perfect square.

But with $6! = [(2\cdot2)\cdot(12\cdot15)]$, and $7! = [(2\cdot2)\cdot(35\cdot36)]$; and with a and b of $(n-2)!$ continuously increasing with each increase in n (Note: For a and b of the least difference in the calculations of $n!$ below, a and b of $(n-2)!$ are ordered, $a < b$):

$$\begin{aligned} 8! &= [6!\cdot(7\cdot8)] = [(2\cdot2)\cdot(\mathbf{12\cdot15})\cdot(7\cdot8)] = [(2\cdot2)\cdot(12\cdot8)\cdot(15\cdot7)] = [(2\cdot2)\cdot(96\cdot105)]; \\ 9! &= [7!\cdot(8\cdot9)] = [(2\cdot2)\cdot(\mathbf{35\cdot36})\cdot(8\cdot9)] = [(2\cdot2)\cdot(35\cdot9)\cdot(36\cdot8)] = [(2\cdot2)\cdot(315\cdot288)]; \\ 10! &= [8!\cdot(9\cdot10)] = [(2\cdot2)\cdot(\mathbf{96\cdot105})\cdot(9\cdot10)] = \ \ldots\ = [(2\cdot2)\cdot(960\cdot945)]; \\ 11! &= [9!\cdot(10\cdot11)] = [(2\cdot2)\cdot(\mathbf{288\cdot315})\cdot(10\cdot11)] = \ \ldots\ = [(2\cdot2)\cdot(3168\cdot3150)]; \\ 12! &= [10!\cdot(11\cdot12)] = [(2\cdot2)\cdot(\mathbf{945\cdot960})\cdot(11\cdot12)] = \ \ldots\ = [(2\cdot2)\cdot(11340\cdot10560)]; \\ &\ldots\ , \end{aligned}$$

then the disparity between a and b of $(n-2)!$, and $c = (n-1)$ and $d = n$, continuously increases; and for all $n > 7$, a can never equal $(n-2)$ and b can never equal c; and $n!$ can never again equal (m^2-1).

What is intriguing is that $n! = (m^2-1) = [(m-1)\cdot(m+1)]$ also implies that every product of four sequential positive integers, plus 1, is a perfect square. That is, if we allow a, b, c, d to be four consecutive integers, then ad and bc are consecutive even integers, and every $[(a\cdot b\cdot c\cdot d)+1] = [(m^2-1)+1]$ is a perfect square.

Without question, $(4!+1) = [(1\cdot 2\cdot 3\cdot 4)+1]$, $(5!+1) = [(2\cdot 3\cdot 4\cdot 5)+1]$, and $(7!+1) = [(70\cdot 72)+1] = [((7{\cdot}10)\cdot(8{\cdot}9))+1] = [(7{\cdot}8{\cdot}9{\cdot}10)+1]$, all satisfy this perfect-square criterion. Which further implies that $(n!+1) = m^2$ only if $n!$ can be expressed as the product of four sequential positive integers.

Then the observations of Brocard and Ramanujan (with all such four consecutive integer $(a{\cdot}b{\cdot}c{\cdot}d)$ products, plus 1, a perfect square) can be alternatively stated as those products of four consecutive integers that can also be expressed as factorials of n. And with $n!$ a product of n, we have that n must be a factor of $(a{\cdot}b{\cdot}c{\cdot}d)$.

The question then becomes, can the properties of such four consecutive integer products incontrovertibly establish why only $4, 5, 7$ and no others?

Decomposing $(7{\cdot}8{\cdot}9{\cdot}10)$ into its prime componsnts, we have

$$\begin{aligned}[7{\cdot}(2{\cdot}2{\cdot}2){\cdot}(3{\cdot}3){\cdot}(2{\cdot}5)] &= [(6{\cdot}7)\cdot(2{\cdot}2){\cdot}(3){\cdot}(2{\cdot}5)] = [(5{\cdot}6{\cdot}7)\cdot(2{\cdot}2){\cdot}(3){\cdot}(2)]\\ &= [(4{\cdot}5{\cdot}6{\cdot}7)\cdot(3{\cdot}2)] = (2{\cdot}3{\cdot}4{\cdot}5{\cdot}6{\cdot}7) = 5040\\ &= 7!,\end{aligned}$$

and we see that the factors, "$8, 9, 10$", are simply the recombining of the prime elements of those factors of 7! less than 7. Then (setting 6! aside for just a moment) what is it that prevents the factorials of $n > 7$ from being reconstructed from the prime composition of their n-based products of $(a{\cdot}b{\cdot}c{\cdot}d)$?

Where $n = 6$ and $6! = (1{\cdot}2{\cdot}3{\cdot}4{\cdot}5{\cdot}6) = 720$, the range of values for $(a{\cdot}b{\cdot}c{\cdot}d)$, incorporating n, are those of $(3{\cdot}4{\cdot}5{\cdot}6) = 360$; $(4{\cdot}5{\cdot}6{\cdot}7) = 840$; $(5{\cdot}6{\cdot}7{\cdot}8) = 1680$, and $(6{\cdot}7{\cdot}8{\cdot}9) = 3024$. Clearly none of the $(a{\cdot}b{\cdot}c{\cdot}d)$ products for $n = 6$ are equal to 6!.

But note that for all $n > 3$, of the four possible $(a{\cdot}b{\cdot}c{\cdot}d)$ products incorporating n, the first three $(a{\cdot}b{\cdot}c{\cdot}d)$ products of n are a repetition of the last three $(a{\cdot}b{\cdot}c{\cdot}d)$ products of $(n-1)$ (i.e., d of the *second* $(a{\cdot}b{\cdot}c{\cdot}d)$ product of $(n-1)$ increments to n and remains a factor through the fourth and final $(a{\cdot}b{\cdot}c{\cdot}d)$ product). Let the symbol, "$\in$" be read as "*is a member of the set*".

$$\begin{aligned}
3!&:\quad (a{\cdot}b{\cdot}c{\cdot}d) \in \{(0{\cdot}1{\cdot}2{\cdot}3), (1{\cdot}2{\cdot}3{\cdot}4), (2{\cdot}3{\cdot}4{\cdot}5), (3{\cdot}4{\cdot}5{\cdot}6)\}.\\
4!&:\quad (a{\cdot}b{\cdot}c{\cdot}d) \in \{(1{\cdot}2{\cdot}3{\cdot}4), (2{\cdot}3{\cdot}4{\cdot}5), (3{\cdot}4{\cdot}5{\cdot}6), (4{\cdot}5{\cdot}6{\cdot}7)\}.\\
5!&:\quad (a{\cdot}b{\cdot}c{\cdot}d) \in \{(2{\cdot}3{\cdot}4{\cdot}5), (3{\cdot}4{\cdot}5{\cdot}6), (4{\cdot}5{\cdot}6{\cdot}7), (5{\cdot}6{\cdot}7{\cdot}8)\}.\\
6!&:\quad (a{\cdot}b{\cdot}c{\cdot}d) \in \{(3{\cdot}4{\cdot}5{\cdot}6), (4{\cdot}5{\cdot}6{\cdot}7), (5{\cdot}6{\cdot}7{\cdot}8), (6{\cdot}7{\cdot}8{\cdot}9)\}.\\
7!&:\quad (a{\cdot}b{\cdot}c{\cdot}d) \in \{(4{\cdot}5{\cdot}6{\cdot}7), (5{\cdot}6{\cdot}7{\cdot}8), (6{\cdot}7{\cdot}8{\cdot}9), (7{\cdot}8{\cdot}9{\cdot}10)\}.
\end{aligned}$$

With $(7{\cdot}8{\cdot}9{\cdot}10)$ equal to 7!, and $8! = (8\cdot 7!) = [8\cdot(7{\cdot}8{\cdot}9{\cdot}10)]$; and for $n = 8$, the products of $(a{\cdot}b{\cdot}c{\cdot}d)$ within $\{(5{\cdot}6{\cdot}7{\cdot}8), (6{\cdot}7{\cdot}8{\cdot}9), (7{\cdot}8{\cdot}9{\cdot}10), (8{\cdot}9{\cdot}10{\cdot}11)\}$, then the greatest product of $(a{\cdot}b{\cdot}c{\cdot}d)$ for $n = 8$ is that of $(8{\cdot}9{\cdot}10{\cdot}11) = [11\cdot(8{\cdot}9{\cdot}10)]$; while $n! = [(8{\cdot}7)\cdot(8{\cdot}9{\cdot}10)] = [56\cdot(8{\cdot}9{\cdot}10)]$; and for $n = 8$, every product of $(a{\cdot}b{\cdot}c{\cdot}d)$ is less than $n!$.

Given that the last three products of $(a{\cdot}b{\cdot}c{\cdot}d)$ for $(n-1) = 8$ are the first three $(a{\cdot}b{\cdot}c{\cdot}d)$ products of $n = 9$, then where the final $(a{\cdot}b{\cdot}c{\cdot}d)$ product for $(n-1)$ is less than $(n-1)!$, the first three $(a{\cdot}b{\cdot}c{\cdot}d)$ products of n are less than $n!$; and it is then only the fourth and final $(a{\cdot}b{\cdot}c{\cdot}d)$ product of $n = 9$ that we need to evaluate against 9!— and if that $(a{\cdot}b{\cdot}c{\cdot}d)$ product is less than 9!, then we need only consider the fourth and final $(a{\cdot}b{\cdot}c{\cdot}d)$ product of $n = 10$; and if that $(a{\cdot}b{\cdot}c{\cdot}d)$ product is less than 10!, only the fourth and final $(a{\cdot}b{\cdot}c{\cdot}d)$ product of $n = 11\ldots$ ad infinitum.

If we denote the greatest $(a{\cdot}b{\cdot}c{\cdot}d)$ product of n as $(a{\cdot}b{\cdot}c{\cdot}d)_1$ (with $a = n$), and the greatest $(a{\cdot}b{\cdot}c{\cdot}d)$ product of $(n-1)$ as $(a{\cdot}b{\cdot}c{\cdot}d)_2$ (where $a = (n-1)$) —with each element or sub-group of $(a{\cdot}b{\cdot}c{\cdot}d)_1$ or $(a{\cdot}b{\cdot}c{\cdot}d)_2$ assigned the same subscript as the $(a{\cdot}b{\cdot}c{\cdot}d)_1$ or $(a{\cdot}b{\cdot}c{\cdot}d)_2$ factor sequence from which it is extracted (recall that the factors $(a \cdot b \cdot c)_1$ and $(b \cdot c \cdot d)_2$ are the same)— then the increase from $(a{\cdot}b{\cdot}c{\cdot}d)_2$ to $(a{\cdot}b{\cdot}c{\cdot}d)_1$ is equal to $[(d_1 - a_2) \cdot (a{\cdot}b{\cdot}c)_1] = [(d_1 - a_2) \cdot (b{\cdot}c{\cdot}d)_2]$, where $(d_1 - a_2) = 4$.

Then in order for any $(a \cdot b \cdot c \cdot d)_1$ to equal $n!$, the growth from $(a \cdot b \cdot c \cdot d)_2$ to $(a \cdot b \cdot c \cdot d)_1$ must equal $[n! - (a \cdot b \cdot c \cdot d)_2)]$, such that

$$(a \cdot b \cdot c \cdot d)_1 = [(a \cdot b \cdot c \cdot d)_2 + ((d_1 - a_2) \cdot (b \cdot c \cdot d)_2))] = n!.$$

But with $(n-1)$ increasing with each increase in n, and

$$[(d_1 - a_2) \cdot (b{\cdot}c{\cdot}d)_2] = [(d_1 - a_2)/(n-1) \cdot (a \cdot b \cdot c \cdot d)_2],$$

and $(d_1 - a_2) = 4$, a constant; then $(4/(n-1)$ is an ever decreasing quantity, and instead of $[(4/(n-1)) \cdot (a \cdot b \cdot c \cdot d)_2)]$ increasing in relation to $n!$ (as per the need for $(d_1 - a_2)$ to equal $[n! - (a \cdot b \cdot c \cdot d)_2])\ldots$

$n = 7; (n-1) = 6$: $[7! - (6{\cdot}7{\cdot}8{\cdot}9)_2]/7! = [(5040 - 3024)/5040] = (2016/5040) = 0.4$

$n = 8; (n-1) = 7$: $[8! - (7{\cdot}8{\cdot}9{\cdot}10)_2]/8! = [(40320 - 5040)/40320] = (35280/40320) = 0.875$

$n = 9; (n-1) = 8$: $[9! - (8{\cdot}9{\cdot}10{\cdot}11)_2]/9! = [(362880 - 7920)/362880] = (354960/362880) = 0.978$

$\ldots,$

just the opposite occurs, with $[(4/(n-1)) \cdot (a \cdot b \cdot c \cdot d)_2)]$ continuously diminishing in respect to $n!$. Beginning with $(n-1) = 6$ and $n = 7$:

$[(4/6) \cdot (6{\cdot}7{\cdot}8{\cdot}9)]/7! = [(0.666 \cdot 3024)/5040] = (2016/5040) = 0.4$

$[(4/7) \cdot (7{\cdot}8{\cdot}9{\cdot}10)]/8! = [(0.5714 \cdot 5040)/40320] = (2880/40320) = 0.07143$

$[(4/8) \cdot (8{\cdot}9{\cdot}10{\cdot}11)]/9! = [(0.5 \cdot 7920)/362880] = (3960/362880) = 0.01091$

$[(4/9) \cdot (9{\cdot}10{\cdot}11{\cdot}12)]/10! = [(0.444 \cdot 11880)/3628800] = (5280/3628800) = 0.00146$

$[(4/10) \cdot (10{\cdot}11{\cdot}12{\cdot}13)]/11! = [(0.4 \cdot 17160)/39916800] = (6864/39916800) = 1.71957e\text{–}4$

$[(4/11) \cdot (11{\cdot}12{\cdot}13{\cdot}14)]/12! = [(0.3636 \cdot 24024)/479001600] = (8736/479001600) = 1.82379e\text{–}5$

$[(4/12) \cdot (12{\cdot}13{\cdot}14{\cdot}15)]/13! = [(0.333 \cdot 32760)/6227020800] = (10920/6227020800) = 1.75365e\text{–}6$

$\ldots$

Then the *difference* between $[(a \cdot b \cdot c \cdot d)_1 - (a \cdot b \cdot c \cdot d)_2]$ and $n!$, increases with each increase in n, and for all $n > 7$, no product of $(a{\cdot}b{\cdot}c{\cdot}d)$ incorporating n can ever again equal $n!$.

Concomitant with the inability of the products of $(a{\cdot}b{\cdot}c{\cdot}d)$ to equal $n!$ for all $n > 7$, we also note (focusing on only the first-instance disparities) that for $6! = (1{\cdot}2{\cdot}3{\cdot}4{\cdot}5{\cdot}6)$,

$(a{\cdot}b{\cdot}c{\cdot}d) = (3{\cdot}4{\cdot}5{\cdot}6)$ is lacking a factor of 2.

$(a{\cdot}b{\cdot}c{\cdot}d) = (4{\cdot}5{\cdot}6{\cdot}7)$ is lacking a factor of 2.

$(a{\cdot}b{\cdot}c{\cdot}d) = (5{\cdot}6{\cdot}7{\cdot}8)$ is lacking a factor of 3.

$(a{\cdot}b{\cdot}c{\cdot}d) = (6{\cdot}7{\cdot}8{\cdot}9)$ is lacking a factor of 5.

Of these, it is the prime 2 deficiencies which appear worthy of further exploration.

Given $(a{\cdot}b{\cdot}c{\cdot}d)$, comprised of only four consecutive integer factors, we can have within any $(a{\cdot}b{\cdot}c{\cdot}d)$ only two products of the prime 2, with one a product of only 2^1 and the other the product of a power of 2 never greater in magnitude than the terminating integer of the final $(a \cdot b \cdot c \cdot d)$ product, $d = (a+3)$. Examining the factorials and $(a{\cdot}b{\cdot}c{\cdot}d)$ products of the first three values of $n > 7$, we have:

With $8! = (1{\cdot}2{\cdot}3{\cdot}4{\cdot}5{\cdot}6{\cdot}7{\cdot}8)$ a product of 2^7, and $(5{\cdot}6{\cdot}7{\cdot}8), (6{\cdot}7{\cdot}8{\cdot}9), (7{\cdot}8{\cdot}9{\cdot}10)$, and $(8{\cdot}9{\cdot}10{\cdot}11)$, all products of 2^4, then the power of 2 in $(a{\cdot}b{\cdot}c{\cdot}d)$ is insufficient to satisfy the power of 2 requirements of 8!, and no product of $(a{\cdot}b{\cdot}c{\cdot}d)$ can equal 8!.

For $9! = (1{\cdot}2{\cdot}3{\cdot}4{\cdot}5{\cdot}6{\cdot}7{\cdot}8{\cdot}9)$, again a product of 2^7 (with the power of 2 in $n!$ only increasing with each subsequent even value of n), and $(6{\cdot}7{\cdot}8{\cdot}9), (7{\cdot}8{\cdot}9{\cdot}10)$, and $(8{\cdot}9{\cdot}10{\cdot}11)$ all products of 2^4, and $(9{\cdot}10{\cdot}11{\cdot}12)$ a product of 2^3, then the power of 2 in $(a{\cdot}b{\cdot}c{\cdot}d)$ is insufficient to satisfy the power of 2 requirements of 9!.

For $10! = (1{\cdot}2{\cdot}3{\cdot}4{\cdot}5{\cdot}6{\cdot}7{\cdot}8{\cdot}9{\cdot}10)$, a product of 2^8; and $(7{\cdot}8{\cdot}9{\cdot}10)$ and $(8{\cdot}9{\cdot}10{\cdot}11)$, both products of 2^4, and $(9{\cdot}10{\cdot}11{\cdot}12)$ and $(10{\cdot}11{\cdot}12{\cdot}13)$ products of 2^3, then the power of 2 in $(a{\cdot}b{\cdot}c{\cdot}d)$ cannot satisfy the power of 2 requirements of 10!, and no product of $(a{\cdot}b{\cdot}c{\cdot}d)$ can equal 10!.

Since for any n expressible as a power of 2, there is an n count of integers —that we shall denote "*the realm of n*"— before arrival at the next power of 2,

n	**Integers Within the Realm of n**			
$2^0 = 1$	1			
$2^1 = 2$	2, 3			
$2^2 = 4$	4, 5, 6, 7			
$2^3 = 8$	8, 9, 10, 11	12, 13, 14, 15		
$2^4 = 16$	16, 17, 18, 19	20, 21, 22, 23	24, 25, 26, 27	28, 29, 30, 31
$2^5 = 32$	32, 33, 34, 35	36, 37, 38, 39	40, 41, 42, 43	44, 45, 46, 47
	48, 49, 50, 51	52, 53, 54, 55	56, 57, 58, 59	60, 61, 62, 63
$\dots,$				

then where d, the terminating integer of an $(a \cdot b \cdot c \cdot d)$ factor sequence, falls within the realm of n, a power of 2 —with the greatest power of 2 within the realm of n being that of n— then the power of 2 in $(a{\cdot}b{\cdot}c{\cdot}d)$ can never be greater than that of the power of 2 in n, *times* 2^1 (see Table 3)[1].

Table 3. **$(a{\cdot}b{\cdot}c{\cdot}d)$ Factor Sequences within the Realms of n**

n	$(a{\cdot}b{\cdot}c{\cdot}d)$	$(a{\cdot}b{\cdot}c{\cdot}d)$	$(a{\cdot}b{\cdot}c{\cdot}d)$	$(a{\cdot}b{\cdot}c{\cdot}d)$	**Max Power of 2 in $(a{\cdot}b{\cdot}c{\cdot}d)$**
$2^3 = 8$	(5·6·7·**8**)	(6·7·8·**9**)	(7·8·9·**10**)	(8·9·10·**11**)	
	(9·10·11·**12**)	(10·11·12·**13**)	(11·12·13·**14**)	(12·13·14·**15**)	$\mathbf{(2^3{\cdot}2^1) = 2^4}$
$2^4 = 16$	(13·14·15·**16**)	(14·15·16·**17**)	(15·16·17·**18**)	(16·17·18·**19**)	
	(17·18·19·**20**)	(18·19·20·**21**)	(19·20·21·**22**)	(20·21·22·**23**)	
	(21·22·23·**24**)	(22·23·24·**25**)	(23·24·25·**26**)	(24·25·26·**27**)	
	(25·26·27·**28**)	(26·27·28·**29**)	(27·28·29·**30**)	(28·29·30·**31**)	$\mathbf{(2^4{\cdot}2^1) = 2^5}$
$2^5 = 32$	(29·30·31·**32**)	(30·31·32·**33**)	(31·32·33·**34**)	(32·33·34·**35**)	
	(33·34·35·**36**)	(34·35·36·**37**)	(35·36·37·**38**)	(36·37·38·**39**)	
	...				
					$\mathbf{(2^5{\cdot}2^1) = 2^6}$

Then for all $n > 7$, where d of $(a \cdot b \cdot c \cdot d)$ falls within the realm of any n, a power of 2, with the maximum possible power of 2 in $(a{\cdot}b{\cdot}c{\cdot}d)$ being that of the power of 2 in n, times 2^1; the power of 2 in $n!$ is equal to the power of 2 in n, times 2^1, *times* the power of 2 in every even integer in $n!$ greater than 2 and less than n.

[1]Note that in Table 3, each $(a \cdot b \cdot c \cdot d)$ factor sequence corresponds to that of the fourth and final $(a{\cdot}b{\cdot}c{\cdot}d)$ sequence of an n value where $a = n$ and $d = (n + 3)$.

Let the symbol, "|^2|", be read as the "*power of 2*":

$n=8$: $|\hat{}2|$ in $8! = [(2^3 \cdot 2^1)\cdot(2^2\cdot 2^1)] = 2^7$; $|\hat{}2|$ in $(8\cdot 9\cdot 10\cdot 11) = (2^3\cdot 2^1) = 2^4$

$n=16$: $|\hat{}2|$ in $16! = [(2^4 \cdot 2^1)\cdot(2^2\cdot 2^1\cdot 2^3\cdot 2^1\cdot 2^2\cdot 2^1)] = 2^{15}$; $|\hat{}2|$ in $(16\cdot 17\cdot 18\cdot 19) = (2^4\cdot 2^1) = 2^5$

$n=32$: $|\hat{}2|$ in $32! = [(2^5 \cdot 2^1)\cdot(2^2\cdot 2^1\cdot \ \ldots\ \cdot 2^2\cdot 2^1)] = 2^{31}$; $|\hat{}2|$ in $(32\cdot 33\cdot 34\cdot 35) = (2^5\cdot 2^1) = 2^6$

$n=64$: $|\hat{}2|$ in $64! = [(2^6 \cdot 2^1)\cdot(2^2\cdot 2^1\cdot \ \ldots\ \cdot 2^2\cdot 2^1)] = 2^{63}$; $|\hat{}2|$ in $(64\cdot 65\cdot 66\cdot 67) = (2^6\cdot 2^1) = 2^7$

...

and for all $n > 7$, the power of 2 in $(a\cdot b\cdot c\cdot d)$ can never satisfy the power of 2 requirements[2] of any factorial of n, and no $(a\cdot b\cdot c\cdot d)$ product can ever equal $n!$.

■

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[2]Observe *in the equations immediately above* that for n, a power of 2, the power-of-2 in $n!$ is equal to $2^{(n-1)}$, e.g., $|\hat{}2|$ in $(2^7! = 128!) = 2^{127}$. With, for $n = 8$, $|\hat{}2|$ in $8! = [(2^3\cdot 2^1)\cdot(2^2\cdot 2^1)]$; and the even factors of $n!$ increasing with each increase in n, a power of 2, a proof of this observation is not essential to this paper and is not addressed here.