Algebraic Invariants of Gravity

Hans Detlef Hüttenbach

Abstract. Newton's mechanics is simple. His equivalence principle is simple, as is the inverse square law of gravitational force. A simple theory should have simple solutions to simple models. A system of n particles, given their initial speed and positions along with their masses, is such a simple model. Yet, solving for $n > 2$ is not simple.
This paper discusses what could be done to get around that problem.

1. Problem Statement

Classical mechanics is essentially a linear, "first order" theory in which the dynamical quantities describe properties of the particles themselves, such as the law of inertia, $F = ma$, as well as energy and momentum conservation etc.
The graviational force, $F = (const)\nabla\frac{m_1 m_2}{|x_1 - x_2|}$, is the exception to that theory: it is a product of quantities, namely the mutual interaction the masses, disguised as a linear first order quantity F. That makes it complicated to even deal with a gravitational interaction of two particles, necessitating elliptic integrals, Legendre polynoms, Bessel functions, and all that, in order to derive its solutions. But it can be done, and it involves some beautiful mathematics and calculations, which explains, why it's done in physics first hand up to this day. The result is that the particles move (with their reduced masses) around the center of mass in all curves given by the intersection of a plane with a cone.
That is mathematically interesting, as it allows to describe the set of solutions through a hyperbolic, quadratic equation, namely that of the cone itself. And it straight leads to the question, if not a quadratic approach to the dynamics would be simpler.

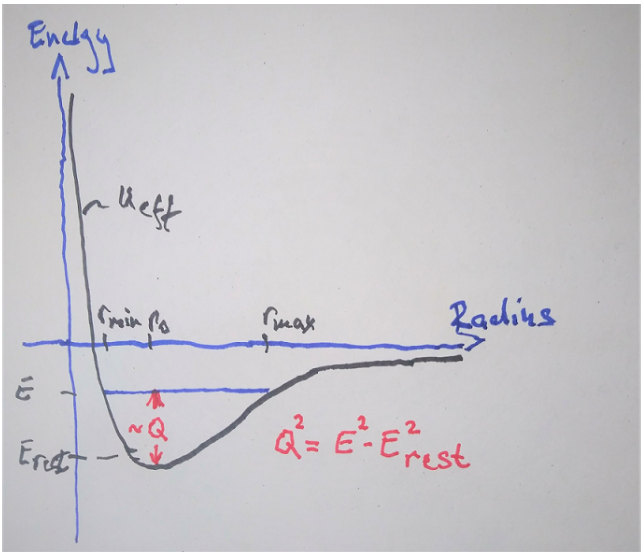
2. The Cone

The picture of that cone is always that of a two-dimensional surface in three dimensions: because it is easy to visualize; but, the angular coordinate is

cyclic, so can and should be eliminated: this reduces the problem to the two diagonals of a 2-dimensional coordinate system, intersecting at right angles at the origin. Let $(a, b) \in \mathbb{R}^2$ denote these pairs of coordinates. Then the intersection of planes with the cones reduces to the intersection of lines with the diagonals, i.e.: these intersections follow the quadratic equation: $a^2 - b^2 = 0$: an algebraic invariant!
We now may pick any dimensional variable for a or b, and as always in conservative mechanical systems, it is favourable to select the total energy E as one of these, a, say. Because the energy (as any conserved quantity) is unique up to an additive constant, the resulting equation is: $E^2 - b^2 = c^2$, where b and c are to be conserved quantities of dimension of energy. The other evident conserved energy quantity is the system's rest energy E_{rest}, which will represent b. And with that we know $Q^2 := E^2 - E_{rest}^2$ must be an invariant. Let's give Q a good name: I'll call Q the heat ot the (2-particle) system.

3. The 2-Particle System

Let's investigate Q in more detail: The gravitational 2-particle system reduces to the problem of a reduced mass μ in a gravitational potential field $U(r) = g\frac{M\mu}{r}$ of a total mass $M = m_1 + m_2$ in the origin $r = 0$, where r denotes the distance of μ to the origin. (When m_1 is small w.r.t. the other mass m_2, say, then μ is approximately m_1.) The gravitational potential has to be added the potential energy of the distractive centrifugal force, which gives $U_{eff}(r) = \frac{L^2}{2\mu r^2} + U(r)$. This is called the effective potential, sketched as follows:



In here, E is the total energy of the reduced mass μ in a bounded state, and r_{min} and r_{max} its return points. The motion of μ is periodic. At the turning points the radial kinetic energy vanishes, and it is maximal at r_0, which denotes the radial point, for which U_{eff} is minimal. At r_0 then μ is not subjected to any forces, so μ is free and therefore $E^2 = \mu^2 c^4 + \mu^2 v^2 c^2$ holds, where v denotes the radial speed and $p = \mu v$ is the momentum (in the radial direction). So we get $E - E_{rest} = \sqrt{\mu^2 c^4 + p^2 c^2} - \mu c^2$, which in the non-relativistic limit $v << c$ reduces to the maximal kinetic energy $T := p^2/2\mu$. As good as this estimate looks, as fundamentally misleading it becomes:
The 2-particle system describes exactly the motion of a free particle of mass μ in a static gravitational field U of mass M centered at the origin. And as such the square energy E^2 of that particle must be constant at all times, even at its turning points r_{min} and r_{max}. As μ the radial velocity of μ is zero in the turning points, its entire square of kinetic energy of μ in r_0 is converted into the square $V^2(r)$ of potential energy, so $Q^2 = p^2 c^2 = V^2(r_{min}) = V^2(r_m ax)$, and we do expect $Q^2 = V^2(r(t)) + p^2(r(t))c^2 \equiv const$ for all times t. On the other hand, with $Q = V + T$, we never end up in a periodical of V into T, such that Q^2 and therefore Q stay constant. As is well-known from statistical physics, the best guess is the retreat to the average values $\bar{V} = \bar{T} = (1/2)Q$; this just marks the point of deviation from determinism to probability in statistical physics: the underlying problem with $H = T + V$ as not being conserved over time - even in a calculable 2-particle model - is mitigated through probability.

Let's remove that probabilitistic layer again and accept that the free particle μ in the central graviational field satisfies $Q^2 = p^2(r(t)) + V^2(r(t)) = const$ at all times. That means that the energy of μ is given by

$$E^2 = \mu_0^2 c^4 + Q^2 = \mu_0^2 c^4 + p^2(t)c^2 + V^2(t), \tag{3.1}$$

where $Q^2 = p^2(t)c^2 + V^2(t)$ and μ_0 are conserved over time.

And again we are back to the fundamental cone equation

$$X^2 - Y^2 = const!$$

Remark 3.1. In case there is some scepticism as to the hyperbolic equation as to allow $X^2 - Y^2$ to become zero or even negative: let's help: $X^2 = Y^2 + C^2$ is the equivalent, which is an elliptic equation.

4. Gravity of Closed n-Particle Systems

Let's consider a closed, gravitational system of n particles that are by fortune not colliding with each other as for instance our own solar system. As we know their positions at a time, we can reduce the problem to n particles of mass $m1_1, \ldots m_n$ moving in the mutual central gravitational field of some total mass M at the origin $r = 0$. Each of the particles m_k is a free particle solution within the mutual gravitational potential: i.e. $E_k^2 = m_{0,k}^2 c^4 + Q_k^2$,

($1 \leq k \leq n$), where Q_k^2 and $m_{0,k}^2$ are constants (in time): $Q^2 := \sum_k Q_k^2$ is constant: it's an invariant!

And now, let's allow the collision of the particles: There are two distictions to be made: inelasic and elastic collisions: In case of elastic collisions, the exchange of kinetic energy during collision is zero, pluse the total momentum is maintained, and the overall dynamics is equivalent to the particles passing through each other without any impact. That is the ultimate secret of ideal gas theory: it is nothing but a free theory, and it is the major part of statistical physics (to which I refer). If collisions are inelastic, then this will result in a deformation of the compound system over time, namely energy increasing at one region, while decreasing in other regions over time. At low temperatures, we normally don't see these deformations, but certainly expect them at high temperatues.

In all, we get:

Proposition 4.1 (Invariance of Heat). *Heat is a conserved quantity in a closed system of particles colliding only elastically. In particular, heat can move freely from one region of that system to another.*

In a time-invariant conserved system, the postulate of Thermodynamics, that heat is to flow only from hot to cold can only be satisfied, if one orders time such that this statement becomes true. And certainly, one may always use symmetry of time-inversion to order time such that heat only flows that way.

Remark 4.2. What was done, was to diagonalize the quadratic form: Any quadratic form is mathematically defined through a linear operator, that is itself the square of a normal operator, and this square operator rewrites into the sum of three linear symmetric operators a positive, a negative, and a zero-operator, where the zero-operator delivers the invariants, and the negative operator can be inverted to a positive operator by taking its absolute values (see e.g. [3]).

5. Parity

The next step is to from a system of n disrete particles to a continuum of particle densities, which will replace the mass squares m_k^2 with a square mass densitiy $\rho_k^2(t, \vec{x})$, and the Q_k^2 would become a density, representing the square of heat, and the sum over the k particles will be replaced by the integration over the spatial volume $4dx^3$. But there is a technical problem in the flow of particles $j(t, \vec{x}) := \rho(t, \vec{x})$, which is to replace the momenta $\vec{p}_k$: While in the discrete finite n-particle model collisions occur only sporadically, more exactly: at each time t on a set of measure zero, and the momenta are observable to the outside almost everywhere for all $\vec{x} \in \mathbb{R}^3$, in the continuous model, the fluxes superimpose destructively: in fact, within a solid body or a liquid in a container, the internal flow of particled cancels out completely to the outside, and macroscopically, all particles appear to stay at rest: the

total momentum of the particles cancels out, even locally.
To resolve that problem, notice that locally, both energy and momentum are constant right before and after an elastic collision, which - as is well-known from the theory of an ideal gas - means that the particles behave equivalently to ones that pass through freely instead of colliding and bouncing back. That way, we can model the situation by splitting the flux $\vec{j}$ into two component $\vec{j}_+$ and $\vec{j}_-$ of opposite parity. When both are equal, then their superposition will cancel completely, but that would not mean that the system was without motion, because the Euclidean square, of the components, $\left|j^2\right| := |j_-|^2 + |j_+|^2$ would be greater zero.
That leads straight to set $j = j_1\sigma_1 + \sigma_2 j_2 + \sigma_3 j_3$, where the σ_k are Hamilton's quaternions or Pauli's sigma matrices (both differ from each other by a factor i): if we would go in with a unit vector $2^{-1/2}(1,1)$ on the r.h.s., as equal pairing of positive and negative parity, then that would lead to destructive superposition. The other extremes are the unit vectors $(1,0)$ and $(0,1)$, in which case all motion will be synchronous, either to the left or to the right, like waves in a water glass that shake seemingly synchronously to either side: Evidently, $j(t,\vec{x}) \neq 0 \implies \rho(t,\vec{x}) \neq 0$, in other words: in the absence of masses the heat is zero. Other contraints for the scalar components j_1, j_2, j_3 then obviously are that their absolute sqares need to be well-defined and finite, and that they all vanish on the system's boundary. That again gives two further extremes: j_1, j_2, j_3 could be constant within the confined body's region, in which case the whole motional energy within the body was kinetical (although at the boundary this energy will have to be converted instantly into potential energy and to be released as kinetical energy thereafter), and it also can be totally potential energy, in case of which nothing would move at all.

6. Laplace and the Inverse Square Law

Consider a mass m_1 located in the origin, say. Then the square of its gravitational potential per unit mass is $V^2(r) = G^2 \frac{m_1^2}{r^2}$, and if we integrate over any sphere around the origin of radius $r > 0$ we trivially get $4\pi G^2 m_1^2$, and that then holds for any (decently smooth) boundary over a stars-haped region containing the origin. Hence. integrating over such a region containing n masses $m1, \ldots, m_n$ we get out $4\pi G^2 \sum_{1\leq k\leq n} m_k^2$, and if we'd want to take its root, then that would be the square root of that. Simple. The only problem is that we conceive mass to be additive, rather than its square, so the common standpoint is that the elliptic Euclidean geometry has to be flat, disregarding the fact that two masses m_1 and m_2 at locations $\vec{x}_1$ and $\vec{x}_2$ both have their own, linearly independent 3-dimensional location coordinates.
The demanded additivity can be enforced by introducing an additional attractive potential between masses:
Given a closed system of n masses $m_1, \ldots, m_n$ with their center of mass at

the origin with total mass $M = m_1 + \cdots + m_n$, the square of its gravitational potential is given by
$V^2(r) = 4\pi G^2 \frac{\sum_k m_k^2}{r^2} = 4\pi G^2 \frac{M^2}{r^2} - 4\pi G^2 \frac{2\sum_{k<l} m_k m_l}{r^2}$.
For $k = 1, \ldots, n$ let $\mu_k := \frac{\sum_{j,j\neq k} m_j m_k}{M}$ be the reduced k-th mass. Then $\sum_{k<l} m_k m_l = M \sum_k \mu_k$, and therefore

$$V(r) = \sqrt{4\pi} G \frac{M}{r} \sqrt{1 - 2M\frac{\sum_k \mu_k}{M^2}} \approx \sqrt{4\pi} G \big(\frac{M}{r} - \frac{\sum_k \mu_k}{r}\big).$$

So the difference field $\Phi(r) := \sqrt{4\pi} G \big(-\frac{\sum_k \mu_k}{r}\big)$ is (up to perhaps a factor $\sqrt{4\pi}$) just the graviational potential of n reduced masses μ_k of the total mass $M = m_1 + \cdots + m_n$ in a stationary gravitational field of mass 1 at the origin, or equivalently, the sum of the reduced masses for a partition of the unit mass $1 = m_1 + \cdots + m_n$ in a stationary field of mass M at the origin. Hence, that potential Φ would be zero, if we would have taken the the total mass of the n masses $m_1, \ldots, m_n$ to be the Euclidean norm of the vector $(m_1, \ldots, m_n)$, rather than its scalar sum $m_1 + \cdots + m_n$.

Remark 6.1. The extra factor $\sqrt{4\pi}$ can of course be included into G. Note also that $V^2(r) = G^2 \frac{m^2}{r^2}$ represents the intensity of a signal that is emitted continually from the mass source m at the origin and spreads radially at a constant velocity c. To get to the relativistics, all it needs is to restrict time to the local time of the mass source, which means that $V^2(r)$ will be replaced by $V^2(t, r = ct) = G^2 \frac{m^2}{t^2 - r^2}$.

Remark 6.2. Note the difference between $V(r)$ and $V_{grav}(r)$:
While $V(r) = \sqrt{4\pi} G \frac{m}{r}$ gravitational field equivalent of the inert mass source, itself, mass source m at origin itself, $V_{grav}(r) = G\frac{m}{r}$ is defined as potential interaction energy between a mass $m_1 = m$ and a unit mass $m_2 = 1$ at a distance of r apart of each other.

We can get at another interpretation of gravitational potential by re-calculating the above with just two masses m_1 and m_2 without the help of reduced masses:
Again, integration $V^2(r) = \frac{m^2}{r^2}$ over the r-sphere gives m, so $\frac{m}{\sqrt{4\pi}} V(r)$ can be said to have m as self-energy (where $c \equiv 1$ has been set). If m is the sum $m = m_1 + m_2$ of two masses $m_1, m_2 > 0$, then $m^2 = (m_1^2 + m_2^2)(1 + \frac{2m_1 m_2}{m_1^2 + m_2^2})$, so $m \approx= 1 + \frac{m_1 m_2}{(m_1^2 + m_2^2)^{1/2}}$. Other than a product of masses per unit mass, the term $m_1 m_2$ in $V_{grav} = G\frac{2m_1 m_2}{r}$ should be interpreted as product of masses per per Euclidean unit mass $m_1^2 + m_2^2 = 1$!

Let's explain what happened above: Given the gravitational field U of a mass source m, square that, and integrate over a containing surface: What we get - up to a constant factor, is the square of the total inert energy of that source, scaleless, i.e.: disregarding the size of the enclosing surface! That is evidently the square of self-energy of that field, and in particular, in line with the postulates of equivalence of the gravitational fields with the inert masses (by Newton and reformulated by Laplace, [6]): field and particle are not complementary, but equivalent, dual images of the same thing! But more: In a composed system of masses, the square fields U_k^2, $(1 \leq k \leq n)$ superimpose, i.e. add freely, and so do the square of the masses m_k^2, $(1 \leq k \leq n)$! That is: the replacement of the masses m_k with the squares m_k^2, makes it a free theory, scale-less at all lengths! Even more: the gravitational field between the particles turns out to be at a very high precision to be that gauge field needed to correct the error by taking $M^2 = \left(\sum_k m_k\right)^2$ as square of the total mass, instead of just $M^2 = \sum_k m_k^2$! Plus, that gauge field Φ must be attractive, as long as all the masses m_k are non-negative!

Frankly said, that puts General Relativity upside down: While in General Relativity the preferred free geometry is plane, and it needs energy (of the masses) to bend it into an elliptic shape, it now appears that the opposite is true: the geometry of space time is elliptic, and it needs the support of a virtual gravitational field to bend it from a flat space into its free elliptic Euclidean manifold: In other words: This makes solar systems and galaxies become stable systems due to the minimal deviation of total energy from $\sqrt{\sum_k m_k^2} c^2$ missing or low gravitational forces, while it is commonly held that these are to be meta-stabile due to their large common gravitational field.

Let's have a closer look: Given n atomic free masses $m_1, \ldots, m_n$ in the plane Euclidean space-time, the total energy is Mc^2, where $M = \sum_k m_k$. On the other hand, the minimal possible energy of these n masses is (the Euclidean norm) $mc^2 = \sqrt{\sum_k m_k^2} c^2$, for which the squares of masses add freely. And the deviation of the two energies calculated above was (in 2^{nd} order approximation) $\sum_k \mu_k c^2$, where the μ_k are the reduced masses. That makes the deviation of the masses to be identified as heat

Proposition 6.3. *A bounded system of n atomic particles of mass $m_1, \ldots, m_n$ can maximally contain the heat $Q = (\sum_k \mu_k)c^2$. This is the maximal mechanical work that can be withdrawn from that system.*

As was seen in section 5, the description of heat by vector fluxes needs the matrix-valued flux $j = j_1\sigma_1 + \sigma_2 j_2 + \sigma_3 j_3$ which operates on component pairs of positive and negative parity. According to the correspondence of the gravitational field U with the mass sources m, one will expect a pseudeovector field $A(\vec{x}) = A_1(\vec{x})\sigma_1 + A_2(\vec{x})\sigma_2 + A_3(\vec{x})\sigma_3$ that corresponds to the source flux j. Not surprisingly, such a field would transmit heat over large distances. But surprisingly, that field is the electromagnetic field, which is based on electrical charges rather than masses. That leads straight to:

7. Action and Charges

The next step will be to notice the similarity of the relation for E^2 with the energy equation of a free system in special relativity: Let's set $E_{kin} := cp_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3$, and likewise $\mathcal{E} = jc = j_1\sigma_1 + j_2\sigma_2 + j_3\sigma_3)$, with c being

the speed of light, and replace c by r/t. If now we multiply the energy by the time t, we get out the action, the two coordinates x and y of the cone section above transform up to constants into time and space, the invariance of the cone equation w.r.t. the inversion of x and y becomes the invariance of time and space inversion, and the derived energy equation turns into the constancy of the absolute square of the action, i.e.: the action becomes the unitarity of action in space and time.
We already dealt with parity, the space inversion, and we saw that fluxes with positive and negative parity can superimpose destructively. On the same line, time inversion leads to energy inversion, but that means equivalently mass inversion. We therefore should have negative and positive masses, to which the parity rules then will carry over. And we do have, namely electrical charges: With the exception of neutrinos, of which is currently not known whether thse have a mass greater zero, down to the very quarks every particle of mass greater zero is charged, plus: it is known that the negative charges match with the positive ones. So, this strongly points to mass as being the absolute values of charges, i.e.: we may conceive the mass m as being equivalent - up to a factoring coupling constant ϵ - to the absolute value $\sqrt{q_+^2 + q_-^2}$ of a charge pair (q_+, q_-) of a positive charge q_+ and a negative charge q_-.
Therefore, fields V, A_1, A_2, A_3 should follow the covariant Maxwell equations,

$$\Box A^\mu = \frac{4\pi}{c} j^\mu, (0 \leq \mu \leq 3),$$

where $A^0 = V, j^0 = \rho$ is the charge density, j_1, j_2, j_3 are the (relativistic) charge fluxes, A_1, A_2, A_3 are their vector field components as above, $\Box := \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_3^2}$ is the d'Alembert operator, and $(x_0 := ct, x_1, \ldots, x_3) \in \mathbb{R}^3$ are the components in 4-dimensional time and space.
Because we have two components for either parity and two components for either charge, it follows that we can and should state these equations as one equation in terms of the four Hermititian Dirac matrices $\alpha_1, \ldots, \alpha_3$ as

$$\Box A = \frac{4\pi}{c} j,$$

where $A := A_0\alpha_0 + \cdots + A_3\alpha_3$ and $j := j_0\alpha_0 + \cdots + j_3\alpha_3$ ([1, Ch. IX]). This matrix equation operates on a complex space $\mathbb{C}^4$, in which - up to $SU(4)$-equivalence - the first two components represent positive and negative parity of positive charges and the last two the positive and negative parity of negative charges. (Up to the fact that we track charge and parity of either sign, instead of destructively superimpose their values, the last equation is nothing but an equivalent rewrite of the Maxwell equations.)
These equations say that charge sources and their motion can equivalently be rewritten as radial waves of energy that are steadily emitted from these sources and spread at the speed of light. Up to an additional coupling constant, that is exactly what we would expect for the first energy component $V = A_0$ for masses, and it boils down to the Laplace equation for the gravitational field in the non-relativistic limit. The other three components are

analogous extensions to the fluxes. And, if there was no other energy than gravitational interaction, then the three other components were sheer invariants, as by principle, gravity does not depend on the speed of its mass sources. So, as to gravity alone, we would demand the vanishing of the vector components $A_1 = A_2 = A_3 \equiv 0$ and would not need to care about parity and mechanical heat.
The point now is that in electrodynamics, it's just the other way round: Note that the 1^{st} component A_0 always is a longitudinal energy wave (i.e. in the direction of the spreading wave). But according to electrodynamics, the waves have to be transversal. That can only be fulfilled, if and only if $A_0 \equiv 0$. And it is clear, why: over a distance larger than the atomic scale, particles are neutral, and a superposition of positive and negative charge densities will destructively cancel itself, while the sum of absolute squares of positive and negative charges still is non-zero.

So, the Maxwell equations suffice to describe both gravitation and electromagnetism. Note that a 4-dimensional vector field $(A_0, \ldots, A_3)$ is needed for this, so the symmetry (Lie) group of this is $U(2) = SU(2) \times U(1)$ (instead of just $U(1)$). In here, $U(1)$ comes in by the the association of the absolute square of charge pairs with the absolute square of mass, in particular, the coupling constant ϵ attaches to that.

It was also shown that a mathematically satisfactory formulation needs the distinction of either sign of parity, which leads to map the quadrupel $(j_0, \ldots, j_3)$ to $j := j_0\alpha_0, \ldots, j_3\alpha_3)$. And these 4-vector fluxes now have $U(4)$ as their symmetry group, which is defined as the group of all unitary mappings on the parity-charge quadrupels $(\lambda_1, \cdots, \lambda_4) \in \mathbb{C}^4$. $U(4)$ is a super group of $U(2)$, more exactly, we have: $U(4) = SU(3) \times SU(2) \times U(1) \times U(2)$. Now $SU(3)$ is known to be the symmetry group of strong interaction, $SU(2)$ that for weak interaction, while the group $U(2)$ is our symmetry group of long ranged photonic interaction of charge and mass. So, what's that additional group $U(1)$ about? It couples the square of baryons and leptons with the square of masses, in other words: we have another coupling constant (aside of ϵ), through which baryons and leptons get assigned a mass. This explains on mathematical grounds, why protons are thousand time heavier than the electron. Keeping with contemporary physics, we might equvalently associate this group $U(1)$ as the symmetry group for the Higgs particle.

8. The Mass Dilemma

According to Special Relativity, the total energy for a free particle system is given by $E^2 = m^2c^4 + p^2c^2$, where E is its total energy, m the total inertial mass of that system, p the absolute value of total momentum, and c is the speed of light. When, in particular the mass system is at rest (which means that the total momentum vanishes), then $E = mc^2$ is termed rest energy and m the rest mass. Still, that rest mass can consist of any number of not massless particles, and we may add a heat Q to that resting system. Then

$E^2_{heated} = m^2c^4 + Q^2 = m^2_{heated}c4$. Because the principle of equivalence of gravitational and inert mass mandates the ratio of inert and gravitational mass to be constant $G = m/m_{grav}$, we run into the conflict with the principle of velocity independence of masses in the gravitational field: the square m^2_{grav} has to be proportional to its square of heat, either, in order to maintain mass equivalence! If that was the case, gravity and termodynamics as well as mechanics and electrodynamics would only decouple from each other for the temperature limit $T \to 0^0K$.

Simple experiments could test for that: if the (mathematical) pendula of two masses of different temperature, suspended at a rod of same length (measured from suspension point to the center of mass) have the same period, then the weight of the masses have to increase with temperature. On the same line, one could prepare three identical metallic spheres with valves to pump air in or out; while the first one is unchanged, the second one gets some of its air pumped out, which is weighed, say 10 (gravitational) grams of weight (at normal pressure), and the missing 10 grams replaced by spraying 10 grams of lacquer to its surface. The third sphere then is brushed off 10 grams of weight, which are replaced by 10 grams of air pumped into the shell. Then measure the weight of the three shells as well as their ratio of inert and gravitational mass (via mathematical pendulum). In case the mass equivalent proves to be correct, the 2^{nd} and 3^{rd} spheres are to be of the same weight and weigh more than the first shell.

9. Lorentz Transformation of Mass and Charge

As is well-known, the relativistic energy of a free particle of mass m_0 and momentum $\vec{p} \in \mathbb{R}^3$ is given by $E^2 = m_0^2c^4 + p^2c^2$. Setting $\vec{p} = m\vec{v}$, Einstein concluded: $m^2c^4 = m_0^2c^4 + m^2v^2c^2$, and therefore: $m^2 = \frac{m_0^2}{(1-(v/c))^2}$, so $m = +\frac{m_0}{\sqrt{1-(v/c)^2}}$.

The last step is a simplification of the complete algebraic solution: The square root of E is $E = \alpha_0 m_c^2 + mc\sum_k \alpha_k v_k$, where the $\alpha_0, \dots \alpha_3$ are the 4×4 α-matrices that - as already discussed above - operate on a vector space $\mathbb{C}^4$, for which each and every unit vector $\lambda \in \mathbb{C}^4$ is a solution of the square equation. In the standard Dirac representation, α_0 is the diagonal matrix with eigenvalues $+1, +1, -1, -1$, and α_1 is the interchange operator $\alpha_1 : (\lambda_1 \dots, \lambda_4) \mapsto (\lambda_4, \lambda_3, \lambda_2, \lambda_1)$. It is this the reason, why for $(\lambda_1, \dots, \lambda_4)$ the first two components could be associated with positive charge or mass of positive and negative parity and the last two of a negative mass or charge with positive and negative parity.

Another implication of $E = \alpha_0 m_0 c^2 + mc\sum_k \alpha_k v_k$ is that the operator $\mathcal{C}$ which interchanges the first two components with the last ones, namely $\mathcal{C} : (\lambda_1, \dots, \lambda_4) \mapsto (\lambda_3, \lambda_4, \lambda_1, \lambda_2)$, is the charge inversion, while $\mathcal{P} : (\lambda_1, \dots, \lambda_4) \mapsto (\lambda_2, \lambda_1, \lambda_4, \lambda_3)$ is the parity inversion, and $\mathcal{T} = \mathcal{CP} : \lambda \mapsto -\lambda$ is the energy (or time) inversion. In particular, $\mathcal{TPC} = \mathbb{I}_4$ is the identity.

Given all that, let's see what the Lorentz transformation of charges rewrites in terms of alpha matrices: We have analogously $m = m_0(1 - \vec{\gamma} \cdot (\vec{v}/c))$, where $\gamma_0 = \alpha_0$ and $\gamma_k = \gamma_0 \alpha_j$, $(1 \leq k \leq 3)$ are the Dirac matrices. And because $m\alpha_0 \vec{v} \cdot \vec{\alpha}/c = (-m)\alpha_0(-\vec{v}) \cdot \vec{\alpha}/c = \mathcal{CP} m\alpha_0 \vec{v} \cdot \vec{\alpha}/c$, although the absolute value $|m|$ of the charge increases (according to Einstein's relation), the net charge stays constant, in other words, the additional charge is neutral. This additional charge is then the source of magnetic fields, and of course it can be interpreted as being made of circular currents, because any flux of particles in a confined body or box must composed of circular motions.
Moreover, because of that neutrality of that magnetic field source, heat would increase the gravity of the mass sources without affecting the electrically sensitive atomic light spectra of their contained, heated particles.

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Hans Detlef Hüttenbach