**Refutation of Heyting algebra**



**Abstract:** Using the Lindenbaum method, we show pseudo-complementation is *not* tautologous along with its eight properties. This refutes Heyting algebra. Based thereon, what follows is the Gödel n-valued matrix logic is refuted and the derivative intuitionistic propositional logic.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated *proof* value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). For results, the 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET p, q, r, s: x, y, z, s;
~ Not, ¬ ; + Or, ∨, ∪; - Not Or; & And, ∧, ∩; \ Not And;
> Imply, greater than, →, ⊢; < Not Imply, less than, ∈;
= Equivalent, ≡, ⊨, :=, ⟺, ↔; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(s=s) T as tautology; (s@s) **F** as contradiction;
(%s<#s) C as contingency, Δ, ordinal 1;
(%s>#s) N as non-contingency, ∇, ordinal 2;
~( y < x) ( x ≤ y), ( x ⊆ y).

From: Alex Citkin, A.; Muravitsky, A. (2018). Lindenbaum method tutorial. UNILOG. arxiv.org/pdf/1901.05411.pdf acitkin@gmail.com alexeim@nsula.edu

A Heyting algebra is an algebra H = hH; ∧, ∨, →, ¬, 1i of type h∧, ∨, →, ¬, 1i, where ∧ (meet) and ∨ (join), → (relative pseudo-complementation) are binary operations, ¬ (pseudo-complementation) is a unary operation and 1 (unit) is a 0-ary operation, if, besides the equalities (l1)–(l2) and (b1) (Section 1.2.1), the following equalities are satisfied for arbitrary elements x, y, z of H:

**Remark 1.6.0:** The equalities below assume that 1 is Tautology as (s=s). However if 1 is taken as ordinal one, N non-contingency, as (%s>#s), then (h5), (h6) are *not* tautologous but rather truth table mixtures of T and N values.

(h1) $x \wedge (x \rightarrow y) = x \wedge y$, (h2) $(x \rightarrow y) \wedge y = y$,
(h3) $(x \rightarrow y) \wedge (x \rightarrow z) = x \rightarrow (y \wedge z)$, (h4) $x \wedge (y \rightarrow y) = x$,
(h5) $\neg 1 \vee y = y$, (h6) $\neg x = x \rightarrow \neg 1$.

[More trivial equalities elided.]

The following property characterizes pseudo-complementation:

$$x \leq y \rightarrow z \Longleftrightarrow x \wedge y \leq z. \quad (1.8.0.1)$$

(~(q<p)>r)=(p&~(r<q)) ; TFFF FFFT TFFF FFFT (1.8.0.2)

**Remark 1.8:** Eq. 1.8.0.2 as rendered is *not* tautologous, meaning pseudo-complementation is refuted.

Indeed, assume first that

$x \leq y \rightarrow z.$ (1.8.1.1.1)

~(q<p)>r ; FFTF TTTT FFTF TTTT (1.8.1.1.2)

Then, in view of (l1–i), the monotonicity of $\wedge$ w.r.t. $\leq$ and (h1), we have:

$x \wedge y \leq y \wedge (y \rightarrow z) = y \wedge z \leq z.$ (1.8.1.2.1)

(~(q<(p&q))&(q>r))=(q&(~r<r)) ; FFFF FFTF FFFF FFTF (1.8.1.2.2)

If Eq. 1.8.1.1, then 1.8.1.2.1: (1.8.1.3.1)

(~(q<p)>r)>((~(q<(p&q))&(q>r))=(q&(~r<r))) ;
TTFT FFTF TTFT FFTF (1.8.1.3.2)

Conversely, suppose that

$x \wedge y \leq z,$ (1.8.2.1.1)

p&~(r<q) ; FTFT FFFT FTFT FFFT (1.8.2.1.2)

that is $x \wedge y = x \wedge y \wedge z.$ (1.8.2.2.1)

(p&q)=((p&q)&r) ; TTTF TTTT TTTF TTTT (1.8.2.2.2)

**Remark 1.8.2:** Eq. 1.8.2.1.2 and 1.8.2.2.2 are *not* equivalent as claimed.

Then, in virtue of (h2), (h4), (h3), we obtain:

$x \leq y \rightarrow x$ (1.8.4.1)

~(q<p)>p ; FTTT FTTT FTTT FTTT (1.8.4.2)

$= (y \rightarrow x) \wedge (y \rightarrow y)$ (1.8.5.1)

(q>p)&(q>q) ; TTFT TTFT TTFT TTFT (1.8.5.2)

$= y \rightarrow (x \wedge y)$ (1.8.6.1)

q>(p&q) ; TTFT TTFT TTFT TTFT (1.8.6.2)

$= y \rightarrow (x \wedge y \wedge z)$ (1.8.7.1) q>((p&q)&r) ; TT**FF** TT**F**T TT**FF** TT**F**T (1.8.7.2)

$= (y \to (x \wedge y)) \wedge (y \to z) \leq y \to z.$ (1.8.8.1)

(q>(p&q))&(~(q<(q>r))>r) ; **FFF**T TT**F**T **FFF**T TT**F**T (1.8.8.2)

**Remark 1.8.9:** Eqs. 1.8.4.1-1.8.8.1 are supposed to be equivalent as a group, but is *not*. (1.8.9.1)

(((~(q<p)>p)=((q>p)&(q>q)))=(q>(p&q)))=((q>((p&q)&r))=
((q>(p&q))&(~(q<(q>r))>r))) ; T**F**T**F** **F**TTT T**F**T**F** FTTT (1.8.9.2)

Using (1.8.0.1), we receive immediately:

$x \leq y \Longleftrightarrow x \to y = 1;$ (1.9.0.1)

~(q<p)=((p>q)=(s=s)) ; T**FF**T T**FF**T T**FF**T T**F**FT (1.9.0.2)

[More trivial equalities elided.]

Proposition 1.2.3. Let H = hH;∧,∨,→,¬, 1i be a Heyting algebra. For arbitrary elements x, y and z of H the following properties hold:

(a) $x \leq y \to x,$ (1.2.3.a.1)

~(q<p)>p ; **F**TTT **F**TTT **F**TTT **F**TTT (1.2.3.a.2)

(b) $x \to y \leq (x \to (y \to z)) \to (x \to z),$ (1.2.3.b.1)

(p>~((p>(q>r))<q))>(p>r) ; TTT**F** TTTT TTT**F** TTTT (1.2.3.b.2)

(c) $x \leq y \to (x \wedge y),$ (1.2.3.c.1)

~(q>p)>(p&q) ; TT**F**T TT**F**T TT**F**T TT**F**T (1.2.3.c.2)

(d) $x \wedge y \leq x,$ (1.2.3.d.1)

p&~(p<q) ; **FFF**T **FFF**T **FFF**T FF**F**T (1.2.3.d.2)

(e) $x \leq x \vee y,$ (1.2.3.e.1)

~((p+q)<p)=(p=p) ; TT**F**T TT**F**T TT**F**T TT**F**T (1.2.3.e.2)

(f) $x \to z \leq (y \to z) \to ((x \vee y) \to z),$ (1.2.3.f.1)

(p>~((q>r)<r))>((p+q)>r) ; TT**FF** TTTT TT**FF** TTT (1.2.3.f.2)

(g) x → y ≤ (x → ¬y) → ¬x, (1.2.3.g.1)

(p>~((p>~q)<q))>~p ; TTT**F** TTT**F** TTT**F** TTT**F** (1.2.3.g.2)

(h) x ≤ ¬x → y. (1.2.3.h.1)

~((~p>q)<p)=(p=p) ; TT**F**T TT**F**T TT**F**T TT**F**T (1.2.3.h.2)

**Remark 1.2.3:** Because Eqs. 1.2.3.a-h (and 1.8.01 farther above) are *not* tautologous we abandon our evaluation here.
We also note it a mistake that "properties ... in Proposition 1.2.3 for Heyting algebras can be applied to Boolean algebras".

We use the method of Lindenbaum to show pseudo-complementation is *not* tautologous along with its eight properties. This refutes Heyting algebra. Based thereon, what follows is the Gödel n-valued matrix logic is refuted and the derivative intuitionistic propositional logic.