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The ABC Conjecture: A Proof of $C < rad^2(ABC)$

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Abstract In this paper, we consider the ABC conjecture then we give a proof that $C < rad^2(ABC)$ that it will be the key of the proof of the ABC conjecture.

Keywords Elementary number theory · real functions of one variable.

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To the memory of my Father who taught me arithmetic
To the memory of Jean Bourgain (1954-2018) for his mathematical work notably in the field of Number Theory

1 Introduction and notations

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) ([1]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

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Conjecture 1 (***ABC*** **Conjecture**): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{3}$$

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.616751$ ([2]). Here we will give a proof that:

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{4}$$

This result, I think is the key to obtain a proof of the veracity of the ABC conjecture.

2 A Proof of the condition (4)

Let a, b, c positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$.

If $c \leq rad(ab)$ then we obtain:

$$c \leq rad(ab) < rad^2(abc) \tag{5}$$

and the condition (4) is verified.

In the following, we suppose that $c > rad(ab)$.

2.1 Case $c = a + 1$

$$c = a + 1 = \mu_a rad(a) + 1 \overset{?}{<} rad^2(ac) \tag{6}$$

2.1.1 $\mu_a = 1$

In this case, $a = rad(a)$, it is immediately truth that :

$$c = a + 1 < 2a < rad(a)rad(c) < rad^2(ac) \tag{7}$$

Then (6) is verified.

2.1.2 $\mu_a \neq 1\,, \mu_a < rad(a)$

we obtain :

$$c = a + 1 < 2\mu_a.rad(a) \Rightarrow c < 2rad^2(a) \Rightarrow c < rad^2(ac) \tag{8}$$

Then (6) is verified.

2.1.3 $\mu_a \geq rad(a)$

We have $c = a + 1 = \mu_a.rad(a) + 1 \leq \mu_a^2 + 1 \stackrel{?}{<} rad^2(ac)$. We suppose that $\mu_a^2 + 1 \geq rad^2(ac) \Longrightarrow \mu_a^2 > rad^2(a).rad(c) > rad^2(a)$ as $rad(c) > 1$, then $\mu_a > rad(a)$, that is the contradiction with $\mu_a \geq rad(a)$. We deduce that $c < \mu_a^2 + 1 < rad^2(ac)$ and the condition (6) is verified.

2.2 $c = a + b$

We can write that c verifies:

$$c = a + b = rad(a).\mu_a + rad(b).\mu_b = rad(a).rad(b)\left(\frac{\mu_a}{rad(b)} + \frac{\mu_b}{rad(a)}\right)$$
$$\Longrightarrow c = rad(a).rad(b).rad(c)\left(\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)}\right) \quad (9)$$

We can write also:

$$c = rad(abc)\left(\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)}\right) \quad (10)$$

To obtain a proof of (4), one method is to prove that :

$$\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)} < rad(abc) \quad (11)$$

2.2.1 $\mu_a = \mu_b = 1$

In this case, it is immediately truth that :

$$\frac{1}{rad(a_i} + \frac{1}{rad(b_j} \leq \frac{5}{6} < rad(c).rad(abc) \quad (12)$$

Then (4) is verified.

2.2.2 $\mu_a = 1$ *and* $\mu_b > 1$

As $b < a \Longrightarrow \mu_b rad(b) < rad(a) \Longrightarrow \dfrac{\mu_b}{rad(a)} < \dfrac{1}{rad(b)}$, then we deduce that:

$$\frac{1}{rad(b)} + \frac{\mu_b}{rad(a)} < \frac{2}{rad(b)} < rad(c).rad(abc) \quad (13)$$

Then (4) is verified.

2.2.3 $\mu_b = 1$ *and* $\mu_a \leq (b = rad(b))$

In this case we obtain:

$$\frac{1}{rad(a)} + \frac{\mu_a}{rad(b)} \leq \frac{1}{rad(a)} + 1 < rad(c).rad(abc) \tag{14}$$

Then (4) is verified.

2.2.4 $\mu_b = 1$ *and* $\mu_a > (b = rad(b))$

As $\mu_a > rad(b)$, we can write $\mu_a = rad(b) + n$ where $n \geq 1$. We obtain:

$$c = \mu_a rad(a)+rad(b) = (rad(b)+n)rad(a)+rad(b) = rad(ab)+nrad(a)+rad(b) \tag{15}$$

We verify that $n < b$, then:

$$c < 2rad(ab)+rad(b) \Longrightarrow c < rad(abc)+rad(abc) < rad^2(abc) \Longrightarrow c < rad^2(abc) \tag{16}$$

2.2.5 $\mu_a.\mu_b \neq 1$, $\mu_a < rad(a)$ *and* $\mu_b < rad(b)$

we obtain :

$$c = \mu_c rad(c) = \mu_a.rad(a) + \mu_b.rad(b) < rad^2(a) + rad^2(b) < rad^2(abc) \tag{17}$$

2.2.6 $\mu_a.\mu_b \neq 1$, $\mu_a \leq rad(a)$ *and* $\mu_b \geq rad(b)$

We have:

$$c = \mu_a.rad(a) + \mu_b.rad(b) < \mu_a\mu_b rad(a)rad(b) \leq \mu_b rad^2(a)rad(b) \tag{18}$$

Then if we give a proof that $\mu_b < rad(b)rad^2(c)$, we obtain $c < rad^2(abc)$. As $\mu_b \geq rad(b) \Longrightarrow \mu_b = rad(b)+\alpha$ with α a positive integer ≥ 0. Supposing that $\mu_b \geq rad(b)rad^2(c) \Longrightarrow \mu_b = rad(b)rad^2(c) + \beta$ with $\beta \geq 0$ a positive integer. We can write:

$$\begin{aligned} rad(b)rad^2(c) + \beta = rad(b) + \alpha \Longrightarrow \beta < \alpha \\ \alpha - \beta = rad(b)(rad^2(c) - 1) > 3rad(b) \Longrightarrow \mu_b = rad(b) + \alpha > 4rad(b) \end{aligned} \tag{19}$$

Finally, we obtain:

$$\begin{cases} \mu_b \geq rad(b) \\ \mu_b > 4rad(b) \end{cases} \tag{20}$$

Then the contradiction and the hypothesis $\mu_b \geq rad(b)rad^2(c)$ is false. Hence:

$$\mu_b < rad(b)rad^2(c) \Longrightarrow c < rad^2(abc) \tag{21}$$

2.2.7 $\mu_a.\mu_b \neq 1\,, \mu_a \geq rad(a)$ *and* $\mu_b \leq rad(b)$

The proof is identical to the case above.

2.2.8 $\mu_a.\mu_b \neq 1\,, \mu_a \geq rad(a)$ *and* $\mu_b \geq rad(b)$

We write:

$$c = \mu_a rad(a) + \mu_b rad(b) \leq \mu_a^2 + \mu_b^2 < \mu_a^2.\mu_b^2 \stackrel{?}{<} rad^2(a).rad^2(b).rad^2(c) = rad^2(abc) \tag{22}$$

As $\mu_a \geq rad(a)$ and $\mu_b \geq rad(b)$, we can write that :

$$\mu_a = rad(a) + m$$
$$\mu_b = rad(b) + n$$

with $m, n \geq 0$ two positive integers. Let $F(x, y)$ the function :

$$F(x,y) = (x+rad(a))(y+rad(b)) - rad(abc),\ \ (x,y) \in I =]-rad(a), +\infty[\times]-rad(b), +\infty[\tag{23}$$

The set of points $M(x, y) \in I$ verifying $F(x, y) = 0$ is the hyperbola $\mathcal{C}$ given by :

$$y = \frac{-rad(b).x + rad(abc) - rad(ab)}{x + rad(a)} \tag{24}$$

The curve $\mathcal{C}$ intersects the axis $x = 0$ and $y = 0$ at the two points $M_1(0, y_1 = rad(b)(rad(c) - 1))$ and $M_2(x_2 = rad(a)(rad(c) - 1), 0)$. The region below the curve $\mathcal{C}$ verifies $F(x, y) < 0$. $F(m, n) = \mu_a.\mu_b - rad(abc) < 0$ if we have $m < x_2 \Rightarrow m < rad(a)(rad(c) - 1))$ and $n < y_1 \Rightarrow n < rad(b)(rad(c) - 1))$. We suppose now that:

$$m \geq rad(a)(rad(c) - 1)) \Longrightarrow m > rad(a) \Longrightarrow \mu_a > 2rad(a) \Longrightarrow a > 2rad^2(a)$$
$$n \geq rad(b)(rad(c) - 1)) \Longrightarrow n > rad(b) \Longrightarrow \mu_b > 2rad(b) \Longrightarrow b > 2rad^2(b)$$
$$then \quad c > 2(rad^2(a) + rad^2(b)) > 4rad(ab) \Longrightarrow c > 4rad(ab) \tag{25}$$

The last inequality $c > 4rad(ab)$ gives the contradiction with the condition $c > rad(ab)$ supposed above. Then we obtain $F(m, n) < 0 \Longrightarrow \mu_a.\mu_b - rad(abc) < 0 \Longrightarrow c < rad^2(abc)$.

We announce the theorem:

Theorem 1 ***(Abdelmajid Ben Hadj Salem, 2019)*** *Let a, b, c positive integers relatively prime with $c = a + b$ and $b < a$, then $c < rad^2(abc)$.*

References

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