A Note About the *ABC* Conjecture - A Proof of The Conjecture: $C < rad^2(ABC)$

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Abstract: In this paper, we consider the *ABC* conjecture, then we give a proof of the conjecture $C < rad^2(ABC)$ that it will be the key of the proof of the *ABC* conjecture.

A Note About the *ABC* Conjecture - A Proof of The Conjecture: $C < rad^2(ABC)$

To the memory of my Father who taught me arithmetic

To the memory of Jean Bourgain (1954-2018) for his mathematical work notably in the field of Number Theory

1. Introduction and notations

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1.1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{1.2}$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) ([1]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

Conjecture 1.3. *(**ABC Conjecture**): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\varepsilon > 0$, there exists $K(\varepsilon)$ such that :*

$$c < K(\varepsilon).rad(abc)^{1+\varepsilon} \tag{1.4}$$

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.616751$ ([2]). A conjecture was proposed that $c < rad^2(abc)$ ([3]). Here we will give a proof of it.

Conjecture 1.5. *Let a, b, c positive integers relatively prime with $c = a + b$, then:*

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{1.6}$$

This result, I think is the key to obtain a proof of the veracity of the *ABC* conjecture.

2. A Proof of the conjecture (1.5)

Let a, b, c positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$.
If $c < rad(ab)$ then we obtain:

$$c < rad(ab) < rad^2(abc) \tag{2.1}$$

and the condition (1.6) is verified.

In the following, we suppose that $c \geq rad(ab)$.

2.1 Case $c = a+1$

$$c = a+1 = \mu_a rad(a) + 1 \overset{?}{<} rad^2(ac) \tag{2.2}$$

2.1.1 $\mu_a = 1$

In this case, $a = rad(a)$, it is immediately truth that :

$$c = a+1 < 2a < rad(a)rad(c) < rad^2(ac) \tag{2.3}$$

Then (2.2) is verified.

2.1.2 $\mu_a \neq 1, \mu_a < rad(a)$

we obtain :

$$c = a+1 < 2\mu_a.rad(a) \Rightarrow c < 2rad^2(a) \Rightarrow c < rad^2(ac) \tag{2.4}$$

Then (2.2) is verified.

2.1.3 $\mu_a \geq rad(a)$

We have $c = a+1 = \mu_a.rad(a)+1 \leq \mu_a^2+1 \overset{?}{<} rad^2(ac)$. We suppose that $\mu_a^2+1 \geq rad^2(ac) \Longrightarrow \mu_a^2 > rad^2(a).rad(c) > rad^2(a)$ as $rad(c) > 1$, then $\mu_a > rad(a)$, that is the contradiction with $\mu_a \geq rad(a)$. We deduce that $c < \mu_a^2+1 < rad^2(ac)$ and the condition (2.2) is verified.

2.2 $c = a+b$

We can write that c verifies:

$$c = a+b = rad(a).\mu_a + rad(b).\mu_b = rad(a).rad(b)\left(\frac{\mu_a}{rad(b)} + \frac{\mu_b}{rad(a)}\right)$$
$$\Longrightarrow c = rad(a).rad(b).rad(c)\left(\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)}\right) \tag{2.5}$$

We can write also:

$$c = rad(abc)\left(\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)}\right) \tag{2.6}$$

To obtain a proof of (1.6), one method is to prove that :

$$\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)} < rad(abc) \tag{2.7}$$

2.2.1 $\mu_a = \mu_b = 1$

In this case, it is immediately truth that :

$$\frac{1}{rad(a)} + \frac{1}{rad(b)} \leq \frac{5}{6} < rad(c).rad(abc) \tag{2.8}$$

Then (1.6) is verified.

2.2.2 $\mu_a = 1$ and $\mu_b > 1$

As $b < a \Longrightarrow \mu_b rad(b) < rad(a) \Longrightarrow \dfrac{\mu_b}{rad(a)} < \dfrac{1}{rad(b)}$, then we deduce that:

$$\frac{1}{rad(b)} + \frac{\mu_b}{rad(a)} < \frac{2}{rad(b)} < rad(c).rad(abc) \tag{2.9}$$

Then (1.6) is verified.

2.2.3 $\mu_b = 1$ and $\mu_a \leq (b = rad(b))$

In this case we obtain:

$$\frac{1}{rad(a)} + \frac{\mu_a}{rad(b)} \leq \frac{1}{rad(a)} + 1 < rad(c).rad(abc) \tag{2.10}$$

Then (1.6) is verified.

2.2.4 $\mu_b = 1$ and $\mu_a > (b = rad(b))$

As $\mu_a > rad(b)$, we can write $\mu_a = rad(b) + n$ where $n \geq 1$. We obtain:

$$c = \mu_a rad(a) + rad(b) = (rad(b) + n)rad(a) + rad(b) = rad(ab) + nrad(a) + rad(b) \tag{2.11}$$

We have $n < b$, if not $n \geq b \Longrightarrow \mu_a \geq 2b \Longrightarrow a \geq 2brad(a) \Longrightarrow a \geq 3b \Longrightarrow c > 3b$, then the contradiction with $c > 2b$. We can write:

$$c < 2rad(ab) + rad(b) \Longrightarrow c < rad(abc) + rad(abc) < rad^2(abc) \Longrightarrow c < rad^2(abc) \tag{2.12}$$

2.2.5 $\mu_a.\mu_b \neq 1, \mu_a < rad(a)$ and $\mu_b < rad(b)$

we obtain :

$$c = \mu_c rad(c) = \mu_a.rad(a) + \mu_b.rad(b) < rad^2(a) + rad^2(b) < rad^2(abc) \tag{2.13}$$

2.2.6 $\mu_a.\mu_b \neq 1, \mu_a \leq rad(a)$ and $\mu_b \geq rad(b)$

We have:

$$c = \mu_a.rad(a) + \mu_b.rad(b) < \mu_a\mu_b rad(a)rad(b) \leq \mu_b rad^2(a)rad(b) \tag{2.14}$$

Then if we give a proof that $\mu_b < rad(b)rad^2(c)$, we obtain $c < rad^2(abc)$. As $\mu_b \geq rad(b) \Longrightarrow \mu_b = rad(b) + \alpha$ with α a positive integer ≥ 0. Supposing that $\mu_b \geq rad(b)rad^2(c) \Longrightarrow \mu_b = rad(b)rad^2(c) + \beta$ with $\beta \geq 0$ a positive integer. We can write:

$$rad(b)rad^2(c) + \beta = rad(b) + \alpha \Longrightarrow \beta < \alpha$$
$$\alpha - \beta = rad(b)(rad^2(c) - 1) > 3rad(b) \Longrightarrow \mu_b = rad(b) + \alpha > 4rad(b) \tag{2.15}$$

Finally, we obtain:

$$\begin{cases} \mu_b \geq rad(b) \\ \mu_b > 4rad(b) \end{cases} \tag{2.16}$$

Then the contradiction and the hypothesis $\mu_b \geq rad(b)rad^2(c)$ is false. Hence:

$$\mu_b < rad(b)rad^2(c) \Longrightarrow c < rad^2(abc) \tag{2.17}$$

2.2.7 $\mu_a.\mu_b \neq 1, \mu_a \geq rad(a)$ and $\mu_b \leq rad(b)$

The proof is identical to the case above.

2.2.8 $\mu_a.\mu_b \neq 1, \mu_a \geq rad(a)$ and $\mu_b \geq rad(b)$

We write:

$$c = \mu_a rad(a) + \mu_b rad(b) \leq \mu_a^2 + \mu_b^2 < \mu_a^2.\mu_b^2 \stackrel{?}{<} rad^2(a).rad^2(b).rad^2(c) = rad^2(abc) \quad (2.18)$$

Supposing that $\mu_a.\mu_b \geq rad(abc)$, we obtain:

$$\begin{gathered}\mu_a.\mu_b \geq rad(abc) \Rightarrow rad(a).rad(b).\mu_a.\mu_b \geq rad^2(ab)rad(c) \Longrightarrow \\ ab \geq rad^2(ab).rad(c) \Rightarrow a^2 > ab \geq rad^2(ab).rad(c) \\ \Rightarrow a > rad(ab)\sqrt{rad(c)} \geq rad(ab)\sqrt{7} \Rightarrow \\ \begin{cases} c > \sqrt{7}rad(ab) \geq 3rad(ab) \\ \\ c \geq rad(ab) \end{cases}\end{gathered} \quad (2.19)$$

The inequality $c \geq 3rad(ab)$ gives the contradiction with the condition $c \geq rad(ab)$ supposed at the beginning of this section. Then we obtain $\mu_a.\mu_b - rad(abc) < 0 \Longrightarrow c < rad^2(abc)$.

We announce the theorem:

Theorem 1. ***(Abdelmajid Ben Hadj Salem, 2019)*** *Let a, b, c positive integers relatively prime with $c = a + b$ and $1 \leq b < a$, then $c < rad^2(abc)$.*

3. About The Proof of The *ABC* Conjecture

3.1 Case: $\varepsilon \geq 1$

Using the result of the theorem above, we have $\forall \varepsilon \geq 1$:

$$c < rad^2(abc) \leq rad(abc)^{1+\varepsilon} = K(\varepsilon).rad(abc)^{1+\varepsilon}, \quad K(\varepsilon) = 1,\ \varepsilon \geq 1 \quad (3.1)$$

It still open the case $\varepsilon < 1$.

References

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