Definitive Prove of Legendre's conjecture

By Kenneth Watanabe, PhD

Abstract

Legendre's conjecture, states that there is a prime number between n^2 and $(n + 1)^2$ for every positive integer n. In this paper, an equation was derived that determines the number of prime numbers less than n for large values of n. Then by mathematical induction, it is proven that there is at least 1 prime number between n^2 and $(n + 1)^2$ for all positive integers n thus proving Legendre's conjecture.

Introduction

Legendre's conjecture, proposed by Adrien-Marie Legendre, states that there is a prime number between n^2 and $(n + 1)^2$ for every positive integer n. The conjecture is one of Landau's problems (1912) on prime numbers. In this paper, an equation is derived to determine the number of prime numbers less than n^2. Then by mathematical induction, it is shown that there is at least 1 prime between n^2 and $(n+1)^2$ thus proving the Legendre conjecture is true.

Functions

Before we get into the proof, let me define a few functions that are necessary.

Let the function l(x) represent the largest prime number less than x. For example, l(10.5) = 7, l(20) = 19 and l(19) = 17.

Let the function λ(x) represent the largest prime number less than or equal to x. For example, λ(10.5) = 7, λ(20) = 19 and λ(23) = 23.

Let the function κ(n) represent the number of composite numbers in the set of odd numbers less than or equal to n excluding 1. For example, κ(15) = 2 since there are two composite numbers 9 and 15 that are less than ot equal to 15.

Let the function π(n) represent the number of prime numbers in the set of odd numbers less than or equal to n. For example, for n = 15, π(n) = 5 since there are 5 prime numbers {3,5,7,11,13} less than 15.

Let capital P represent all the odd integers less than n excluding 1.

Methodology for Proving Legendre's Conjecture

To prove Legendre's conjecture, we will start out with the set of odd integers greater than 1 and less than or equal to n. Then we will eliminate the composite (non-prime) numbers leaving just the prime numbers. We start by identifying all numbers divisible by 3 but not equal to 3. Then we identify all numbers divisible by 5 but not divisible by 3. Then we identify all numbers divisible by 7 but not divisible by 5 or 3, etc. This is continued up to λ(√n), the largest prime number less than or equal to √n, since there are no prime numbers greater than √n that will evenly divide n that are not already divisible by a lower prime. The remaining numbers in the set will be the prime numbers less than or equal to n.

Let us start with the list all odd numbers less than n excluding 1 as shown below.

{3,5,7,9,11,13,15,17,19,21,23,25,27,29,31,33,35,37,…n}

There are P = (n-1)/2 numbers in the list.

Excluding 3, every third number (highlighted in yellow) beginning with 9 is divisible by 3.

{3,5,7,9,11,13,15,17,19,21,23,25,27,29,31,33,35,37,…n}

Number of numbers divisible by 3 limit n-> ∞ = P/3

Excluding 5, every fifth number beginning with 15 is divisible by 5.

{3,5,7,9,11,13,15,17,19,21,23,25,27,29,31,33,35,37,…,n}

But notice that, of the set of numbers divisible by 5, every third number is also divisible by 3.

{15,25,35,45,55,65,75,85,95,105,…,n}

So to avoid double counting, we must multiply by (2/3) giving the following:

Number of numbers divisible by 5 and not 3 limit n-> ∞ = P(2/3)(1/5)

Excluding 7, every seventh number beginning with 21 is divisible by 7.

But notice that every 3rd number (yellow) is also divisible by 3 and every 5th number (green) is divisible by 5.

{21,35,49,63,77,91,105,119,133,147,161,175…n}

{21,35,49,63,77,91,105,119,133,147,161,175…n}

So to avoid double counting, we must multiply by (2/3) and (4/5) giving the following:

The number of numbers divisible by 7 and not 5 or 3 limit n-> ∞ = P(2/3)(4/5)(1/7)

The general formula for the number of numbers divisible by prime number p but not equal to p as n-> ∞ is as follows:

Number of numbers divisible only by p limit n-> ∞ = P(2/3)(4/5)(6/7)(10/11)…((l(p) – 1)/l(p))(1/p)

Number of numbers divisible only by p limit n-> ∞ = P$(1/p)\prod_{q=3}^{l(p)}(q-1)/q$

The total number of composite numbers in the set of odd numbers less than or equal to n, defined as κ(n), is thus defined as follows:

κ(n) = P{1/3 + (2/3)(1/5) + (2/3)(4/5)(1/7) + (2/3)(4/5)(6/7)(1/11) + … + (2/3)(4/5)(6/7)(10/11)…((l(λ(√n)) – 1)/l(λ(√n)))(1/ λ(√n))}

This can be written as

κ(n) = P$\sum_{p=3}^{\lambda(\sqrt{n})}(\frac{1}{p})\prod_{q=3}^{l(p)}(q-1)/q$

Let us define the function W(x) = $\sum_{p=3}^{x}(\frac{1}{p})\prod_{q=3}^{l(p)}(q-1)/q$

where x is a prime number and the sum and products are over prime numbers

Then the equation for κ(n) simplifies to the following:

κ(n) = PW(λ(√n))

The number of primes less than or equal to n limit n-> ∞ is

π(n) = P - κ(n)

$= P – PW(\lambda(\sqrt{n}))$

$= P(1-W(\lambda(\sqrt{n})))$

As n approaches ∞, the value of P approaches (n/2). Substituting P with (n/2) in the above equation gives the following equation for the number of primes less than n as n approaches ∞.

$\pi(n) = (n/2)(1-W(\lambda(\sqrt{n})))$ Equation 1

To verify that no mistakes were made in the derivation of equation 1, I plotted the actual number of primes less than n (blue) against equation 1 (orange) in Figure 1. Equation 1 slightly underestimated the actual number of primes for n <= 5,000, but for n <= 50,000, the curves were virtually indistinguishable.

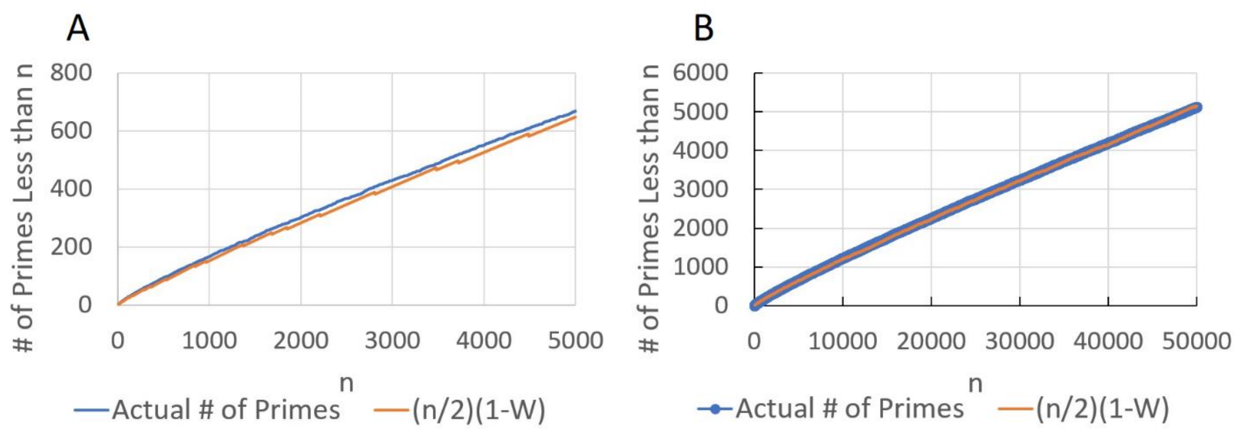

Figure 1. The actual number of primes less than n (blue) is slightly underestimated by equation 1 (orange) for values of n up to 5,000 (A). But for values of n up to 50,000, (B) the curves are virtually indistinguishable.

Since I will be using mathematical induction, I must get the expression $1-W(p_{i+1})$ in terms of $W(p_i)$. To do this, we must look at the actual values of W(p).

$1-W(3) = 1 – (1/3) = 2/3$

$1\text{-}W(5) = 1 – (1/3) – (2/3)(1/5) = (2/3)(4/5)$

$1\text{-}W(7) = 1 – (1/3) – (2/3)(1/5) – (2/3)(4/5)(1/7) = (2/3)(4/5)(6/7)$

Notice the value of $1 - W(p_i)$ (yellow) can be substituted into the green part of $1 - W(p_{i+1})$.

Therefore, these equations can be simplified to:

$1\text{-}W(p_{i+1}) = (1\text{-}W(p_i))(p_{i+1} - 1)/p_{i+1}$ Equation 2

Now that we have a formula for number of primes less than n, we can calculate the number of primes between n^2 and $(n+1)^2$.

$\pi(n^2) = (n^2/2)(1\text{-}W(\lambda(n)))$

$\pi((n+1)^2) = ((n+1)^2/2)(1\text{-}W(\lambda(n+1)))$

There are two cases. The first case is $\lambda(n) = \lambda(n+1)$. This is the case where $n \neq p_i – 1$. The second case is where $n = p_i – 1$ in which case $\lambda(n) = p_{i-1}$ and $\lambda(n+1) = p_i$.

Case 1: Let us look at the first case where $n \neq p - 1$.

Let us prove for all $n \neq p - 1$, there is at least 1 prime number between n^2 and $(n+1)^2$. That means the difference between $\pi((n+1)^2)$ and $\pi(n^2)$ must be greater than 1.

$\pi(n^2) = (n^2/2)(1\text{-}W(\lambda(n)))$

$\pi((n+1)^2) = ((n+1)^2/2)(1\text{-}W(\lambda(n+1))) = ((n+1)^2/2)(1\text{-}W(\lambda(n)))$

Let $\Delta\pi(n^2)$ be the difference between $\pi((n+1)^2)$ and $\pi(n^2)$.

$\Delta\pi(n^2) = \pi((n+1)^2) - \pi(n^2)$

$\Delta\pi(n^2) = ((n+1)^2/2)(1\text{-}W(\lambda(n))) - (n^2/2)(1\text{-}W(\lambda(n)))$

$\Delta\pi(n^2) = \{((n+1)^2/2) - (n^2/2)\}(1\text{-}W(\lambda(n)))$

$\Delta\pi(n^2) = \{((n+1)^2 - n^2)/2)\}(1\text{-}W(\lambda(n)))$

$\Delta\pi(n^2) = \{((n^2 +2n + 1) - n^2)/2)\}(1\text{-}W(\lambda(n)))$

$\Delta\pi(n^2) = \{((2n + 1)/2)\}(1\text{-}W(\lambda(n)))$ Equation 3

To prove $\Delta\pi(n^2) > 1$ for all $n \neq p - 1$, we will use mathematical induction.

Base case n=3. Plugging values into equation 3 gives us the following:

$\Delta\pi(n^2) = \{((6 + 1)/2)\}(1\text{-}W(\lambda(3)))$

$\Delta\pi(n^2) = (7/2)(1\text{-}(1/3))$

$\Delta\pi(n^2) = (7/2)(2/3)$

$\Delta\pi(n^2) = (7/3) > 1$

Let's assume $\Delta\pi(n^2) = \{((2n + 1)/2)\}(1\text{-}W(\lambda(n))) > 1$ for all $n \neq p - 1$

Prove that $\Delta\pi((n+1)^2) > 1$

Plugging n+1 for n in equation 3 gives the following:

$\Delta\pi((n+1)^2) = \{((2(n+1) + 1)/2)\}(1\text{-}W(\lambda(n+1)))$

$\Delta\pi((n+1)^2) = \{((2n + 3)/2)\}(1\text{-}W(\lambda(n)))$

Taking the ratio of $\Delta\pi((n+1)^2)/\Delta\pi(n^2)$ gives

$\Delta\pi((n+1)^2)/\Delta\pi(n^2) = \{((2n + 3)/2)\}(1\text{-}W(\lambda(n))) / \{((2n + 1)/2)\}(1\text{-}W(\lambda(n)))$

$\Delta\pi((n+1)^2)/\Delta\pi(n^2) = \{((2n + 3)/2)\} / \{((2n + 1)/2)\}$

$\Delta\pi((n+1)^2)/\Delta\pi(n^2) = (2n + 3)/(2n + 1) > 1$

This proves that for all $n \neq p - 1$ where p is a prime number, there is at least 1 prime number between n^2 and $(n+1)^2$.

Case 2: Let us look at the first case where n = p - 1.

$\pi(n^2) = (n^2/2)(1\text{-}W(\lambda(n)))$

$\pi((n+1)^2) = ((n+1)^2/2)(1\text{-}W(\lambda(n+1)))$

Suppose $n = p_{i+1} - 1$, then $\lambda(n) = p_i$ and λ $(n+1) = p_{i+1}$.

Substituting p_i for $\lambda(n)$ and substituting p_{i+1} for $\lambda(n+1)$ gives the following:

$\pi(n^2) = (n^2/2)(1\text{-}W(p_i))$

$\pi((n+1)^2) = ((n+1)^2/2)(1\text{-}W(p_{i+1}))$

$\pi((n+1)^2) = ((n+1)^2/2)(1\text{-}W(p_i))(p_{i+1} - 1)/p_{i+1}$ using equation 2

The difference between $\pi(n^2)$ and $\pi((n+1)^2)$ gives

$\Delta\pi(n^2) = \pi((n+1)^2) - \pi(n^2)$

$\Delta\pi(n^2) = ((n + 1)^2/2)(1\text{-}W(p_i))(p_{i+1} - 1)/p_{i+1} - [n^2/2] (1\text{-}W(p_i))$

$= \{((n + 1)^2)(p_{i+1} - 1)/p_{i+1} - n^2\} (1\text{-}W(p_i))/2$

$= \{p_{i+1}^2(p_{i+1} - 1)/p_{i+1} - (p_{i+1} - 1)^2\} (1\text{-}W(p_i))$ /2 substituting n with $p_{i+1} - 1$

$= \{p_{i+1}^2 - p_{i+1} - (p_{i+1}^2 – 2p_{i+1} + 1)\} (1-W(p_i))/2$

$= \{p_{i+1}^2 - p_{i+1} - p_{i+1}^2 + 2p_{i+1} - 1)\} (1-W(p_i))/2$

$= \{ p_{i+1} - 1\} (1-W(p_i))/2$

To prove $\Delta\pi(n^2) > 1$ for all $n = p_{i+1} - 1$, we will use mathematical induction.

Base case $p_{i+1} = 5$, $p_i = 3$ and $n = p_{i+1} - 1 = 4$.

Plugging 4 for n, and 5 for p_{i+1} and 3 for p_i gives:

$\Delta\pi(4^2) = (5-1)(1-W(3))/2$

$\Delta\pi(4^2) = 4(1- (1/3))/2$

$\Delta\pi(4^2) = 4(2/3)/2$

$\Delta\pi(4^2) = 4/3 > 1$

Assume $\Delta\pi(n^2) > 1$ for all $n = p_{i+1} - 1$

Prove $\Delta\pi(n^2) > 1$ for all $n = p_{i+2} - 1$

$\Delta\pi(n^2) = (p_{i+1} – 1) (1-W(p_i))/2$

$\Delta\pi(n^2) = \{(p_{i+2} – 1)(1-W(p_i))(p_{i+1} - 1)/p_{i+1}\}/2$ Using equation 2

$\Delta\pi(n^2) = \{(p_{i+2} – 1)/ p_{i+1}\}\{(p_{i+1} – 1) (1-W(p_i))/2\}$

Since we know $(p_{i+2} – 1)/ p_{i+1} > 1$ and we assumed $(p_{i+1} – 1) (1-W(p_i))/2 > 1$, the product must be greater than 1.

This proves that for all n = p – 1 where p is a prime number, there is at least 1 prime number between n^2 and $(n+1)^2$.

Summary

In summary, I derived the following equation for the number of prime numbers less than n for large values of n.

$$\pi(n) = (n/2)(1-W(\lambda(\sqrt{n})))$$

where $\lambda(\sqrt{n})$ is the largest prime number less than or equal to $\sqrt{n}$ and W(x) is defined as follows:

$$W(x) = \sum_{p=3}^{x}(\frac{1}{p}) \prod_{q=3}^{l(p)}(q-1)/q$$

where x is a prime number and the sum and products are over prime numbers.

I have proven by mathematical induction, that the number of prime numbers between n^2 and $(n+1)^2$ is greater than 1 for all positive integers n, thus confirming the Legendre Conjecture.