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A Complete Proof of the ABC Conjecture: The End of The Mystery

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Abstract In this paper, we consider the ABC conjecture. Firstly, we give a proof of a the first conjecture that $C < rad^2(ABC)$. It is the key of the proof of the ABC conjecture. Secondly, a proof of the ABC is given for $\epsilon \geq 1$, then for $\epsilon \in]0,1[$ for the two cases: $c \leq rad(abc)$ and $c > rad(abc)$. We choose the constant $K(\epsilon)$ as $K(\epsilon) = 6^{1+\epsilon} e^{\left(\dfrac{1}{\epsilon^2} - \epsilon\right)}$. Five numerical examples are presented.
It is the end of the mystery of the ABC conjecture!

Keywords Elementary number theory · real functions of one variable.

Mathematics Subject Classification (2010) 11AXX · 26AXX

To the memory of my Father who taught me arithmetic
To the memory of Jean Bourgain (1954-2018) for his mathematical work notably in the field of Number Theory

1 Introduction and notations

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \qquad (1)$$

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We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) ([1]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

Conjecture 1 (***ABC* Conjecture**): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{3}$$

$K(\epsilon)$ depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ ([2]). A conjecture was proposed that $c < rad^2(abc)$ ([3]). Here we will give a proof of it.

Conjecture 2 Let a, b, c positive integers relatively prime with $c = a + b$, then:

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{4}$$

This result, I think is the key to obtain a proof of the veracity of the ABC conjecture.

2 A Proof of the conjecture (2)

Let a, b, c positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$.

If $c < rad(ab)$ then we obtain:

$$c < rad(ab) < rad^2(abc) \tag{5}$$

and the condition (4) is verified.

In the following, we suppose that $c \geq rad(ab)$.

2.1 Case $c = a + 1$

$$c = a + 1 = \mu_a rad(a) + 1 \overset{?}{<} rad^2(ac) \tag{6}$$

2.1.1 $\mu_a = 1$

In this case, $a = rad(a)$, it is immediately truth that :

$$c = a + 1 < 2a < rad(a)rad(c) < rad^2(ac) \tag{7}$$

Then (6) is verified.

2.1.2 $\mu_a \neq 1\,, \mu_a < rad(a)$

we obtain :

$$c = a + 1 < 2\mu_a.rad(a) \Rightarrow c < 2rad^2(a) \Rightarrow c < rad^2(ac) \tag{8}$$

Then (6) is verified.

2.1.3 $\mu_a \geq rad(a)$

We have $c = a + 1 = \mu_a.rad(a) + 1 \leq \mu_a^2 + 1 \overset{?}{<} rad^2(ac)$. We suppose that $\mu_a^2 + 1 \geq rad^2(ac) \Longrightarrow \mu_a^2 > rad^2(a).rad(c) > rad^2(a)$ as $rad(c) > 1$, then $\mu_a > rad(a)$, that is the contradiction with $\mu_a \geq rad(a)$. We deduce that $c < \mu_a^2 + 1 < rad^2(ac)$ and the condition (6) is verified.

2.2 $c = a + b$

We can write that c verifies:

$$c = a + b = rad(a).\mu_a + rad(b).\mu_b = rad(a).rad(b)\left(\frac{\mu_a}{rad(b)} + \frac{\mu_b}{rad(a)}\right)$$
$$\Longrightarrow c = rad(a).rad(b).rad(c)\left(\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)}\right) \tag{9}$$

We can write also:

$$c = rad(abc)\left(\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)}\right) \tag{10}$$

To obtain a proof of (4), one method is to prove that :

$$\frac{\mu_a}{rad(b).rad(c)} + \frac{\mu_b}{rad(a).rad(c)} < rad(abc) \tag{11}$$

2.2.1 $\mu_a = \mu_b = 1$

In this case, it is immediately truth that :

$$\frac{1}{rad(a)} + \frac{1}{rad(b)} \leq \frac{5}{6} < rad(c).rad(abc) \tag{12}$$

Then (4) is verified.

2.2.2 $\mu_a = 1$ *and* $\mu_b > 1$

As $b < a \Longrightarrow \mu_b rad(b) < rad(a) \Longrightarrow \dfrac{\mu_b}{rad(a)} < \dfrac{1}{rad(b)}$, then we deduce that:

$$\frac{1}{rad(b)} + \frac{\mu_b}{rad(a)} < \frac{2}{rad(b)} < rad(c).rad(abc) \tag{13}$$

Then (4) is verified.

2.2.3 $\mu_b = 1$ *and* $\mu_a \leq (b = rad(b))$

In this case we obtain:

$$\frac{1}{rad(a)} + \frac{\mu_a}{rad(b)} \leq \frac{1}{rad(a)} + 1 < rad(c).rad(abc) \tag{14}$$

Then (4) is verified.

2.2.4 $\mu_b = 1$ *and* $\mu_a > (b = rad(b))$

As $\mu_a > rad(b)$, we can write $\mu_a = rad(b) + n$ where $n \geq 1$. We obtain:

$$c = \mu_a rad(a) + rad(b) = (rad(b)+n)rad(a) + rad(b) = rad(ab) + nrad(a) + rad(b) \tag{15}$$

We have $n < b$, if not $n \geq b \Longrightarrow \mu_a \geq 2b \Longrightarrow a \geq 2brad(a) \Longrightarrow a \geq 3b \Longrightarrow c > 3b$, then the contradiction with $c > 2b$. We can write:

$$c < 2rad(ab) + rad(b) \Longrightarrow c < rad(abc) + rad(abc) < rad^2(abc) \Longrightarrow c < rad^2(abc) \tag{16}$$

2.2.5 $\mu_a.\mu_b \neq 1$, $\mu_a < rad(a)$ *and* $\mu_b < rad(b)$

we obtain :

$$c = \mu_c rad(c) = \mu_a.rad(a) + \mu_b.rad(b) < rad^2(a) + rad^2(b) < rad^2(abc) \tag{17}$$

2.2.6 $\mu_a.\mu_b \neq 1$, $\mu_a \leq rad(a)$ *and* $\mu_b \geq rad(b)$

We have:

$$c = \mu_a.rad(a) + \mu_b.rad(b) < \mu_a\mu_b rad(a)rad(b) \leq \mu_b rad^2(a)rad(b) \tag{18}$$

Then if we give a proof that $\mu_b < rad(b)rad^2(c)$, we obtain $c < rad^2(abc)$. As $\mu_b \geq rad(b) \Longrightarrow \mu_b = rad(b) + \alpha$ with α a positive integer ≥ 0. Supposing that $\mu_b \geq rad(b)rad^2(c) \Longrightarrow \mu_b = rad(b)rad^2(c) + \beta$ with $\beta \geq 0$ a positive integer. We can write:

$$rad(b)rad^2(c) + \beta = rad(b) + \alpha \Longrightarrow \beta < \alpha$$
$$\alpha - \beta = rad(b)(rad^2(c) - 1) > 3rad(b) \Longrightarrow \mu_b = rad(b) + \alpha > 4rad(b) \tag{19}$$

Finally, we obtain:

$$\begin{cases} \mu_b \geq rad(b) \\ \mu_b > 4rad(b) \end{cases} \tag{20}$$

Then the contradiction and the hypothesis $\mu_b \geq rad(b)rad^2(c)$ is false. Hence:

$$\mu_b < rad(b)rad^2(c) \Longrightarrow c < rad^2(abc) \tag{21}$$

2.2.7 $\mu_a.\mu_b \neq 1\,,\mu_a \geq rad(a)$ *and* $\mu_b \leq rad(b)$

The proof is identical to the case above.

2.2.8 $\mu_a.\mu_b \neq 1\,,\mu_a \geq rad(a)$ *and* $\mu_b \geq rad(b)$

We write:

$$c=\mu_a rad(a)+\mu_b rad(b)\leq \mu_a^2+\mu_b^2<\mu_a^2.\mu_b^2\overset{?}{<}rad^2(a).rad^2(b).rad^2(c)=rad^2(abc) \tag{22}$$

Supposing that $\mu_a.\mu_b \geq rad(abc)$, we obtain:

$$\begin{gathered}\mu_a.\mu_b \geq rad(abc) \Rightarrow rad(a).rad(b).\mu_a.\mu_b \geq rad^2(ab)rad(c) \Longrightarrow \\ ab \geq rad^2(ab).rad(c) \Rightarrow a^2 > ab \geq rad^2(ab).rad(c) \\ \Rightarrow a > rad(ab)\sqrt{rad(c)} \geq rad(ab)\sqrt{7} \Rightarrow \\ \begin{cases} c > \sqrt{7}rad(ab) \geq 3rad(ab) \\ \\ c \geq rad(ab) \end{cases}\end{gathered} \tag{23}$$

The inequality $c \geq 3rad(ab)$ gives the contradiction with the condition $c \geq rad(ab)$ supposed at the beginning of this section. Then we obtain $\mu_a.\mu_b - rad(abc) < 0 \Longrightarrow c < rad^2(abc)$.

We announce the theorem:

Theorem 1 ***(Abdelmajid Ben Hadj Salem, 2019)*** *Let* a,b,c *positive integers relatively prime with* $c=a+b$ *and* $1\leq b<a$*, then* $c<rad^2(abc)$*.*

3 The Proof of The ABC Conjecture (1)

We denote $R = rad(abc)$.

3.1 Case: $\epsilon \geq 1$

Using the result of the theorem above, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon}e^{\left(\frac{1}{\epsilon^2}-\epsilon\right)},\ \epsilon \geq 1 \tag{24}$$

3.2 Case: $\epsilon < 1$

3.2.1 Case: $c \leq R$

In this case, we can write :

$$c \leq R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon} e^{\left(\dfrac{1}{\epsilon^2} - \epsilon\right)}, \ \epsilon < 1 \tag{25}$$

and the ABC conjecture is true.

3.2.2 Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon} e^{\left(\dfrac{1}{\epsilon^2} - \epsilon\right)}, 0 < \epsilon < 1 \tag{26}$$

If not, then $\exists \epsilon_0 \in]0, 1[$, so that the triplets (a, b, c) checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \tag{27}$$

are in finite number. We have:

$$\begin{aligned} c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow \\ R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > c.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} > K(\epsilon_0) \end{aligned} \tag{28}$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow c > K(\epsilon_0)^{\left(\dfrac{1}{1-\epsilon_0}\right)} \tag{29}$$

We deduce that it exists an infinity of triples (a, b, c) verifying (27), hence the contradiction. Then the proof of the ABC conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with a, b, c relatively coprime:

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \quad with \quad K(\epsilon) = 6^{1+\epsilon} e^{\left(\dfrac{1}{\epsilon^2} - \epsilon\right)} \tag{30}$$

Q.E.D

4 Examples

In this section, we are going to verify some numerical examples.

4.1 Example of Eric Reyssat

We give here the example of Eric Reyssat [1], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{31}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$.
For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = 6^{1.01} e^{9999.99} = 1.8884880155640644914779227374022e + 4343 \tag{32}$$

Let us verify (30):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \overset{?}{<} K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow$$
$$6436343 \ll K(0.01) \times 15042 \tag{33}$$

Hence (30) is verified.

4.2 Example of A. Nitaj

4.2.1 Case 1

The example of Nitaj about the ABC conjecture [1] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \tag{34}$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \tag{35}$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \tag{36}$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \tag{37}$$

we take $\epsilon = 100$ we have:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 6^{101} e^{-99.9999}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 8.2558649305610435609546415285004e + 48$$

then (30) is verified.

4.2.2 Case 2

We take $\epsilon = 0.5$, then:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \tag{38}$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 6^{1.5}.e^{3.5}.(2.3.5.7.11.13.41.79.311.953)^{1.5} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 75\,333\,109\,597\,556\,257\,182\,261.66 \tag{39}$$

We obtain that (30) is verified.

4.2.3 Case 3

We take $\epsilon = 1$, then

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 6^2.(2.3.5.7.11.13.41.79.311.953)^2 \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 29\,918\,625\,700\,491\,952\,961\,692\,755\,600 \quad (40)$$

We obtain that (30) is verified.

4.3 Example of Ralf Bonse

The example of Ralf Bonse about the ABC conjecture [2] is:

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \quad (41)$$
$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983$$
$$rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$
$$rad(abc) = 1.5683959920004546031461002610848e+33 \quad (42)$$

For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = 6^{1.01}.e^{9999.99} = 5.2903884296336672264108948608106e+4343$$

Let us verify (30):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<}$$
$$6^{1.01}.e^{9999.99}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.01}$$
$$\Longrightarrow 3.4136998783296235160378273576498e+44 <$$
$$1.7819595478010681971905561514574e+4377 \quad (43)$$

The equation (30) is verified.

Ouf, end of the mystery!

5 Conclusion

This is an elementary proof of the *ABC* conjecture, confirmed by four numerical examples. We can announce the important theorem:

Theorem 2 *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2019) Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \quad (44)$$

where $K(\epsilon)$ is a constant depending of ϵ equal to $6^{1+\epsilon}.e^{\left(\frac{1}{\epsilon^2} - \epsilon\right)}$.

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