A Possible Alternative Model of Probability Theory (Part II)

What follows provides more details on the outline offered in "A Possible Alternative Model of Probability?" (download from: *vixra.org/author/d_williams*) which could be considered as Part I. Please download it and read before tackling Part II.

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1. The Substitution Method

Consider the following substitutions:

$$1.\ \sum f(ran\#) \quad \Rightarrow \quad \int_{0}^{1} f(x) \quad x \in \mathbb{Q}\left(\frac{odd}{even}\right)$$

$$2.\ \prod f(ran\#) \quad \Rightarrow \quad \prod_{0}^{1} f(x) \quad x \in \mathbb{Q}\left(\frac{odd}{even}\right)$$

$$3.\quad 1/n \quad \Rightarrow \quad dx$$

$$4.\quad f(ran\#) \quad \Rightarrow \quad f(x)$$

where

a) ran# = a random number between 0 and 1

b) the integral and product integral in 1) and 2) on the RHS are "dx-less" (see "Dx-less Integrals" at *vixra.org/author/d_williams*)

c) Q(odd/even) = the set of rationals with odd numerators and even denominators

By making the above substitutions for a stochastic expression with elements from the left, you can produce what appear to be decent long-term (large n) estimates of likely values using the corresponding elements on the right. These estimates are often better than those offered by expectations – E(x) – the apparent "goto" tool in most (all?) introductory texts on Stochastic Processes.

For instance, say you wanted the estimate of the value of

$$\prod_{n} e * ran\#$$

for large n.

You could take the expectation

$$E(e * ran\#) = e/2$$

then

$$E(\prod_{n} e*ran\#) = (e/2)^n \to \infty \quad as \quad n \to \infty$$

Or, you could use the above substitutions to get

$$E(\prod_{n} e*ran\#) \to \prod_{x \in \mathbb{Q}\left(\frac{odd}{even}\right)}{}_0^1 e*x = \sqrt{2}$$

A big difference. Simulations with programs, calculators, spreadsheets for large n suggest the estimate via substitutions is more accurate than by expectations.

2. Wild Integrals as Potential Estimators

More interestingly, it appears you may be able to extend the range of calculus to include seriously wild integrals. For instance, the following unconventional "integrals"

$$\int_0^1 x^{1/dx} = \frac{\sqrt{e}}{(e-1)} = 0.9505...$$

$$\int_0^1 \ln(1+(dx/x)^2) = \ln(\cosh(\pi))$$

$$\int_0^1 (dx/x)^2 = \pi^2/6$$

$$\int_0^1 (dx^{1+x})\ln(1/dx) = 1$$

may be 'half decent' estimators of the stochastic expressions

$$\sum_{i=1}^{n} ran\#^n \ as\ n \to \infty$$

$$\sum_{i=1}^{n} \ln(1+\frac{1}{n*ran\#})\ as\ n \to \infty$$

$$\sum_{i=1}^{n} (\frac{1}{n*ran\#})^2\ as\ n \to \infty$$

$$\sum_{i=1}^{n} ((1/n)^{1+ran\#} * \ln(n))\ as\ n \to \infty$$

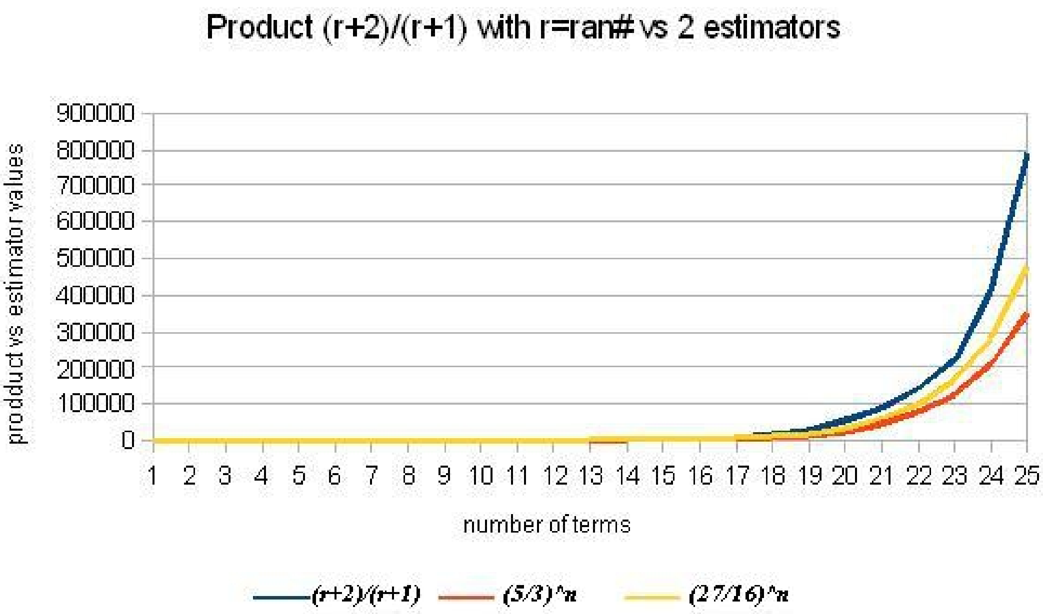
(Note: To save time and space, from now on I'll omit the Q(odd/even) part. Make it the convention that the standard partition applies – that is, use mid-points of equal sized subintervals over the domain – unless otherwise specified. Things like Q(odd/even) are needed as dx-less integrals can be partition dependent).

In any case, I'm willing to bet they are better estimators than expectations.

For example:

$$from \prod_0^1 4e(x+2)/27 = 1$$
$$and \prod_0^1 e(x+1)/4 = 1$$
$$make \prod_0^1 (16/27)(\frac{x+2}{x+1}) = 1$$
$$then$$
$$\prod_1^n \frac{r+2}{r+1} \approx (\frac{27}{16})^n$$
$$is\ a\ better\ estimator\ than$$
$$E(\prod_1^n \frac{r+2}{r+1}) = (5/3)^n$$

Simulations show the dx-less estimator is better than using expectations.



Graph: product (r+2)/r+1) vs 2 estimators
(zoom in to improve quality of graph)

Every stochastic product (and corresponding series) I've simulated (several dozen) has always been better approximated by a dx-less expression than with expectations. Is this always the case? I don't know. More investigation (by many more people) is needed.

3. Building an Alternative Model of Probability Theory

More generally, you can build an alternative model of probability theory using these substitutions.

For instance, the population mean would be

$$E(f(ran\#)) = \int_0^1 f(x)dx$$

(note: the integral on the right this time is standard not dx-less)

Similarly, an alternative version of variance can be produced

$$\sigma^2 = Vr(f(ran\#)) = \int_0^1 (f(x) - E(f(x)))^2 dx$$
$$= \int_0^1 (f(x))^2 dx - [E(f(x))]^2$$
$$= \int_0^1 (f(x))^2 dx - (\int_0^1 f(x) dx)^2$$

Compare with the standard version of variance

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 pr(x) dx = E(x^2) - [E(x)]^2$$

Example: consider pr(x)=x/2 (0<x<1)

With standard probability theory:

$$E(x) = \int_0^2 x(x/2) dx = \left. x^3/6 \right|_0^2 = 4/3$$
$$E(x^2) = \int_0^2 x^2(x/2) dx = \left. x^4/8 \right|_0^2 = 2$$
$$\therefore \sigma^2 = E(x^2) - [E(x)]^2 = 2 - (4/3)^2 = 2/9$$

With alternative probability theory:

$$pr(x) \Rightarrow f(x) = 2\sqrt{x} \ (for\ 0 < x < 1)$$
$$\sigma^2 = \int_0^1 (f(x))^2 dx - (\int_0^1 f(x) dx)^2$$
$$= \int_0^1 (2\sqrt{x})^2 dx - (\int_0^1 (2\sqrt{x}) dx)^2$$
$$= \left. 2x^2 \right|_0^1 - \left. (4/3)x^{3/2} \right|_0^1 = 2/9$$

So everything looks good. We have a way to switch between pr(x) and f(x) and back– see Part I – and can calculate population means and variance.

But when you look at the Law of Large Numbers and Central Limit Theorem you get a slight difference for some (but not all) pr(x) and f(x).

Law of Large Numbers

Old Prob Theory | New Prob Theory

$for\ S_n = x_1 + x_2 + ... + x_n$ $(x_i = \text{ random variables})$ $S_n \to n\mu \text{ as } n \to \infty$	$\sum_1^n f(ran\#_i) \to \int_0^1 f(x)\ as\ n \to \infty$ $for\ suitable\ f(x)$

Central Limit Theory

Old Prob Theory | New Prob Theory

$\Pr\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} < \beta\right) \to \Phi(\beta)\ as\ n \to \infty$	$\Pr\left(\frac{\sum_1^n f(ran\#_i) - \int_0^1 f(x)}{\sigma\sqrt{n}} < \beta\right) \to \Phi(\beta)?\ as\ n \to \infty$

Now here's the problem: For certain distributions, the normal curves are slightly shifted when using the new technique compared to the old. For example, f(r)=1+ln(ran#) has the normal curve centred on ln(sqr(2))=0.344... not 0 as per standard probability theory.

As I stated in Part I (please read) I tried to reconcile these two different results without success until wondering if they could both be “right” (or at least not inconsistent) as per alternative models of geometry, logic, analysis and so on. Currently I am still in this limbo land.

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4. Transforming dx-less integrals

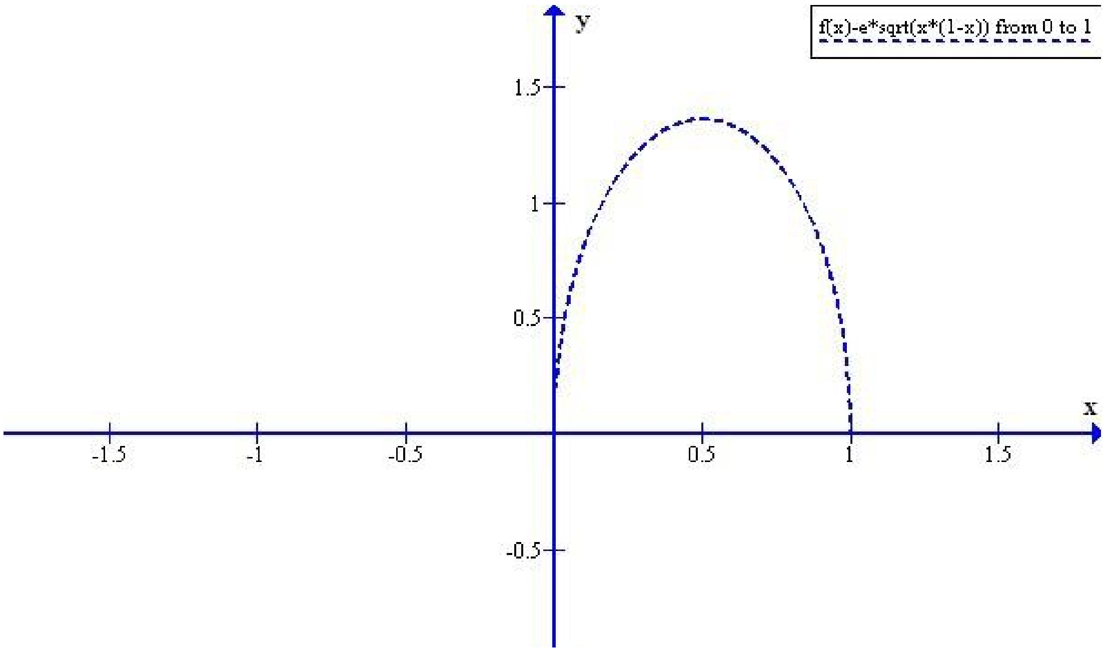
The great thing about ***dx-less product integrals*** is that (with a bit of care) you can transform them in numerous ways. For example you can multiply them together, divide them (with obvious restrictions), exponentiate, transpose parts of the function in the interval (with care), stretch and compress sections (with compensating exponentiation to retain convergence), “merge” sections with common ranges, and so on.

For instance with

$$\prod_0^1 e * x = \sqrt{2}$$

you can replace e*x with e*(1-x) – that is rotate around the line x=1/2 - to get the same product over (0,1). You then multiply e*x with e*(1-x) and take the square root to give e*sqr(x(1-x)), a different looking function (see below) that gives the same product value.

$$\prod_0^1 e * \sqrt{x(1-x)} = \sqrt{2}$$



*<u>Graph</u>: New function from flipping f(x)=e*x over (0,1) around x=1/2 to give e*(1-x), multiplying them together then taking the square root .*

The variety of dx-less product integrals that can be created using these techniques seems endless. Questions arise as to whether all functions (of significant classes) can be approximated by transforming just a few "seed" dx-less product integrals. There could be many more ways to transform such products awaiting discovery.

<u>5. What needs to be done</u>

I encourage others to investigate this area. There's a lot to do, like:

a) Understand dx-less integrals and product integrals (tests for convergence, any more surprises out there?). Classify them. Find all transformations.

b) Find all differences between the 2 models of probability. Are there other models?

c) Apply the new model to "real" world problems. For instance, can a version of Quantum Mechanics be devised using functions of random numbers? Does it differ from the standard model? What problems can be more easily solved using f(r) functions?

d) Understand "wild integrals". Which converge? Which satisfy the Law of Large Numbers and CLT?

e) Test all results in Stochastic Processes texts using simulations. An overemphasis on theory has probably resulted in mathematically correct but misleading (inappropriate/unsuitable, etc) results. See "The Fair bet Paradox" and "Betting on a Tossed Fair Coin" for simple alternative results (downloadable at: *vixra.org/author/d_williams).*

We are still in the exploratory phase regarding these strange critters. Why not help figure out *WTF* is going on?

End Notes

Some finite product approximations of certain f(x)

The simple BASIC program

```
label start
e=exp(1)
input n
p=1
d=2*n
for i=1 to n
x=(2*i-1)/d
p=e*x*p
next i
print p
goto start
```

can be used to calculate partial product approximations for f(x)=e*x, giving the output (for various input n) of:

n=	p=
1	1.3591409...
10	1.40873667...
100	1.4136244...
1000	1.4141544...
10,000	1.414207669...

(I used the free Small Basic program available on the web for these calculations)

For f(x)=(e/4)*(x+1), replace “p=e*x*p” in the above program with “p=(e/4)*(x+1)*p” to get output of:

n=	p=
1	1.019355...
10	1.002083381...
100	1.000208352...
1000	1.0000208335...
10,000	1.0000020833...

For f(x)=(4e/27)*(x+2), make the appropriate swap to get output of:

n=	p=
1	1.006771...
10	1.0006944...
100	1.00006944...
1000	1.000006944...
10,000	1.0000006944...

For f(x)=(16/27)*(x+2)/(x+1), make the appropriate swap to get output of:

n=	p=
1	0.987654...
10	0.99861397...
100	0.99986112...
1000	0.999986111...
10,000	0.9999986111...

For

$$\frac{\prod_{0}^{1/3}(x+1)}{\sqrt{\prod_{1/3}^{2/3}(x+2/3)\prod_{2/3}^{1}(x+1/3)}} = 0.866...?$$

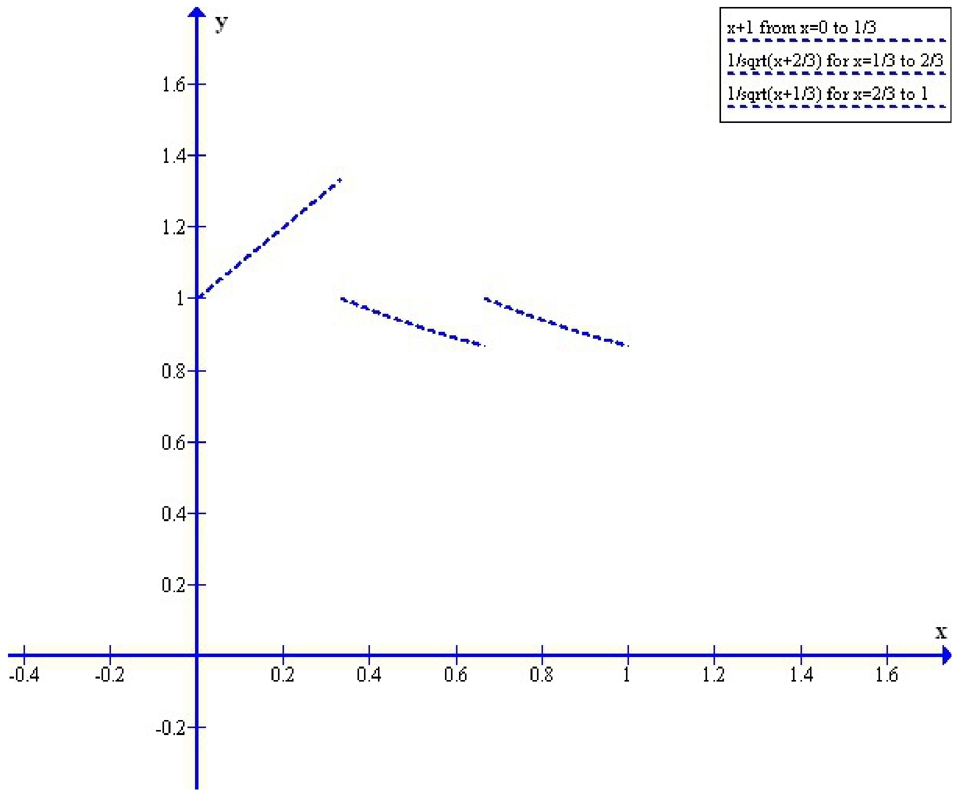
Use

```
label start
input n
p=1
d=2*n
for i=1 to n
x=(2*i-1)/d
if x<1/3 then p=(x+1)*p
if x>1/3 and x<2/3 then p=(1/sqr(x+2/3))*p
if x>2/3 then p=(1/sqr(x+1/3))*p
next i
print p
goto start
```

Giving output of

n=	p=
1	0.925320...
10	0.8726...
100	0.86668...
1000	0.86609155...

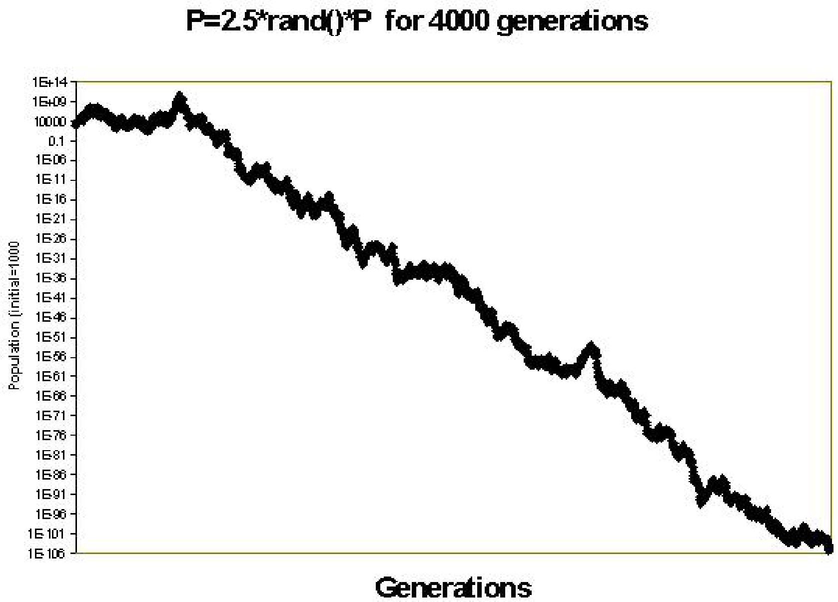
10,000	0.8660320...



Graph of function above

And so on (but what the hell is 0.866...?).

It is also relatively easy to make approximations using **spreadsheets** which can then be graphed. The ***rand()*** function can also be used to make stochastic products and series as well. Comparisons can then be made between the stochastic output and the dx-less estimators.



Here are some simple dx-less product integrals to start playing with:

$$\prod_0^1 \frac{f(x)}{f(1-x)} = 1$$
$$\prod_0^1 e^{\alpha(x-1/2)} = 1 \; for \; \alpha \in \mathbb{R}$$
$$\prod_0^{1/2} f(x) \prod_{1/2}^1 1/f(1-x) = 1$$
$$\prod_0^{1/3} f(x) \sqrt{\prod_{1/3}^{2/3} 1/f(x-1/3) \prod_{1/3}^{2/3} 1/f(1-x)} = 1$$
$$\prod_0^1 e*(\frac{a^a}{(a+1)^{a+1}})(x+a) = 1 \; for \; a \in \mathbb{R}_{>0}$$
$$\ldots \; and \; so \; on$$

And some more general dx-less product integrals (none proved, just significant numerical evidence for):

$$\prod_0^1 (\sqrt{e} * x)^x = 1$$
$$\prod_0^1 e^{-\pi^2/12} * (1+x)^{1/x} = 1$$
$$\prod_0^1 2*(\sin(\pi x/2)) = \sqrt{2}$$
$$\prod_0^1 e^{\gamma} \ln(1/x) = \sqrt{2} \quad where \; \gamma = 0.5772\ldots$$
$$\prod_0^1 \sqrt{2} * (\sin(\pi x))^x = \sqrt{2}$$
$$\prod_0^1 e^{-\pi/2} * (1+\cos(\pi x))^{1/(\cos(\pi x))} = 1/2$$
$$\prod_0^1 x\left[(2-x)^\wedge(12/\pi^2(1-x)\right] = \sqrt{2}$$
$$\prod_0^1 (\frac{e^{\sqrt{2}}}{1+\sqrt{2}}) x(x+\sqrt{x^2+1}) = \sqrt{2}$$
$$\prod_0^1 2\sin(\pi x) = 2$$
$$\prod_0^1 (x+0.54331\ldots) = 1$$
$$\prod_0^1 (2x+0.17696\ldots) = 1$$
$$\prod_0^1 \frac{2}{3} * e^\wedge(\frac{x^2-x}{\ln(x)}) = 1$$
$$\prod_0^1 x * e^\wedge(\sin^2(2\pi x)) = \sqrt{2}$$

- where all the above products are over Q(odd/even).

Find other products/series and manipulate to your heart's content. Then work out some general theorems of convergence (and so on) and tell others. Any help and feedback would be appreciated.

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