A demonstration of the Titius–Bode law and the number of Saturn's rings by Newtonian methods using the Kerr-Newman solution of the general relativity theory

Fumitaka Inuyama. Senior Power Engineer (e-mail: inusanin@yahoo.co.jp)
Permanent address: Jonan-ku Torikai 5-17-33, Fukuoka City, 814-0103, Japan
Thermal Power Dept., Kyushu Electric Power Co., Inc.(www.kyuden.co.jp) (Retired)

Abstract

The beautiful Titius–Bode law ($\xi = 0.4 + 0.3 \times 2^{\mathrm{n}}$) discovered 250 years ago, is considered to be a mathematical coincidence rather than an "exact" law, because it has not yet been proved physically. However, considering the disturbance reparation and stability of the asteroid belt orbit, there must be some underlying logical necessity.

Planetary orbits are often computed by Newtonian mechanics calculating the kinetic energy and the universal gravitation energy. But applying the principle of energy-minimum to the Newtonian mechanics leads that the stable orbital radius is only one value, and this result disagrees perfectly with actual phenomena. The cause of this difference must be an extraction shortage of elements which rule over the planetary orbits. Other elements are the electric charge energy and the rotation energy which are guided by the Kerr-Newman solution discovered in 1965 of the general relativity theory. That is, I applied the principle of energy-minimum and Newtonian methods to the complicated energy equation which adopts mass, electric charge and rotation elements of the central core star as the Sun. Herewith, the Titius–Bode law is demonstrated mathematically and the number of Saturn's rings, maximum 31 is calculated.

Keywords

Demonstration; Titius–Bode law; Saturn's rings number; energy stable orbits; Kerr-Newman solution; relativity theory.

1. Introduction

The Titius–Bode law, discovered 250 years ago, is considered to be a mathematical coincidence rather than an "exact" law [1], because it has not yet been proved physically. However, considering the disturbance reparation and stability of the asteroid belt orbit, there must be some underlying logical necessity. Here by Newtonian methods using the Kerr-Newman solution of the general relativity theory, I demonstrate the Titius-Bode law and apply its solution method to Saturn's rings. This is a mathematical equation calculation. The detailed analysis processes are provided in a separate paper [2].

2. Methods

The following is an outline of the solution method used and the key equations in the text.

1) The equation for energy in the space-time field is obtained from the Kerr-Newman solution, a strict solution of the Einstein equations of general relativity.

$$f_1(\rho, \theta, d\rho/dt, d\theta/dt, d\varphi/dt, \varepsilon) = 0 \qquad \text{(eq. 3)}$$

2) This energy equation is partially differentiated by θ to the minimum energy. The result is $\theta=\pi/2$, so that, the calculation below proceeds at $\theta=\pi/2$, i.e., in the equatorial plane.

$$f_2(\rho, \pi/2, d\rho/dt, 0, d\varphi/dt, \varepsilon) = 0$$

3) Applying the variational principle to the Kerr-Newman solution to calculate $d\varphi/dt$ leads to the angular momentum equivalent J.

$\xi(\rho, d\varphi/dt, J) = 0$ (eq. 6)

4) Because of an additional radius $d\rho = 0$ at the aphelion and perihelion distance R, the calculation below is performed at distance R.

$f_3(R, \pi/2, 0, 0, d\varphi/dt, \varepsilon) = 0$ (eq. 7)

5) Substituting $d\varphi/dt$ from $\xi = 0$ into $f_3 = 0$ results in a relation for the radius, the angular momentum equivalent, and the energy.

$f_4(R, \pi/2, 0, 0, J, \varepsilon) = 0$ (eq. 9)

6) The orbital distance R is determined by the energy and the angular momentum equivalent, i.e., $R = R(\varepsilon, J)$. R is partially differentiated by ε , that is, f_4 is partially differentiated by ε.

$g(R, J, \varepsilon, \partial R/\partial\varepsilon) = 0$ (eq. 10)

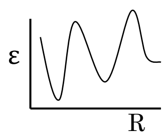
7) Taking the angular momentum equivalent J from $f_4(R, \pi/2, 0, 0, J, \varepsilon) = 0$ and substituting it into $g(R, J, \varepsilon, \partial R/\partial\varepsilon) = 0$ gives an important differential equation composed of the radius and the energy.

$h(R, \varepsilon, d\varepsilon/dR) = 0$ (eq. 11)

8) Solving the differential equation h results in a complicated set of arctan, log, and power functions, plus an integration constant K.

$H(R, \varepsilon, K) = 0$ (eq. 14) (eq. 15)

9) Using that the minimum energy is $d\varepsilon/dR = 0$ in $h(R, \varepsilon, d\varepsilon/dR) = 0$, the following simultaneous equations are obtained and solved.



$h(r, \varepsilon_{min}, 0) = 0$ ① $H(r, \varepsilon_{min}, K) = 0$ ② (eq. 16)

10) Because the integration constant K is common to all orbits, the Titius–Bode law is demonstrated and the number of Saturn's rings is calculated.

$I(r, K) = 0$ (eq. 22) (eq. 23)

2.1. The Energy Equation

2.1.1. Introduction to the Energy Equation

There are two preconditions for the following analysis.

1) The analysis object must be sufficiently far from the center of mass.
2) The rotation speed of the center of mass must not be too fast. The characteristic Boyer-Lindquist coordinates in the Kerr solution are equal to general polar coordinates in the first-order term a/ρ[3].

The strict Boyer-Lindquist metric of the Kerr-Newman geometry [4] is

$$ds^2 = -\frac{R^2\,\Delta}{\rho^2}\left(dt - a\sin^2\theta d\,\phi\right)^2 + \frac{\rho^2}{R^2\,\Delta}dr^2 + \rho^2 d\theta^2 + \frac{R^4\sin^2\theta}{\rho^2}\left(d\,\phi - \frac{a}{R^2}dt\right)^2$$

At large radius r, the Boyer-Lindquist metric is

$$ds^2 \to -\left(1 - \frac{2M}{r}\right)dt^2 - \frac{4aM\sin^2\theta}{r}dtd\,\phi + \left(1 + \frac{2M}{r}\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\,\phi\right)$$

Symbols are changed from the Boyer-Lindquist metric to the general polar coordinates below.

The Kerr-Newman solution of general relativity is given by (eq.1). In this expression, m, a and e are the mass, rotation and electric charge elements respectively.

$$(1)\quad ds^2 = \left(1 - \frac{2m\rho - e^2}{\rho^2 + a^2\cos^2\theta}\right)(cdt)^2 - \frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 - 2m\rho + e^2}d\rho^2 - (\rho^2 + a^2\cos^2\theta)d\theta^2$$
$$-\left[(\rho^2 + a^2) + \frac{(2m\rho - e^2)a^2\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\right]\sin^2\theta\, d\varphi^2 - \frac{2(2m\rho - e^2)a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}cdt\cdot d\varphi$$

Dividing ds by the time elements ($c\,dt$), Γ gives the form:

$$\frac{1}{\Gamma^2} = \left(\frac{ds}{cdt}\right)^2$$

The Lorentz transformation factor $\gamma\,(=c\,dt/ds)$ in the Minkowski space-time of special relativity is an important component to the energy $E = Mc^2 = M_0\,\gamma\,c^2$. $\Gamma\,(=c\,dt/ds)$ in the Kerr-Newman solution of general relativity is analogous to γ.

On this occasion, following the principle of minimum energy, the sign of m is changed to $-m$, the sign of a is $+a$, and the sign of e is $+e$, and therefore, the energy equation is $E = \Gamma(\rho, \theta, \varphi, t, -m, a, e)$.

$$(2)\quad \frac{1}{E^2} = \left(1 + \frac{2m\rho + e^2}{\rho^2 + a^2\cos^2\theta}\right) - \frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 + 2m\rho + e^2}\left(\frac{d\rho}{cdt}\right)^2 - (\rho^2 + a^2\cos^2\theta)\left(\frac{d\theta}{cdt}\right)^2$$
$$-\left[(\rho^2 + a^2) - \frac{(2m\rho + e^2)a^2\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\right]\sin^2\theta\left(\frac{d\varphi}{cdt}\right)^2 + \frac{2(2m\rho + e^2)a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\left(\frac{d\varphi}{cdt}\right)$$

Since E has a decisive massive energy M_0c^2, it is converted into ε $1/E^2 = 1 - 2\varepsilon$ in (eq.3).

$$(3)\quad -2\varepsilon = \frac{2m\rho + e^2}{\rho^2 + a^2\cos^2\theta} - \frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 + 2m\rho + e^2}\left(\frac{d\rho}{cdt}\right)^2 - (\rho^2 + a^2\cos^2\theta)\left(\frac{d\theta}{cdt}\right)^2$$
$$-\left[(\rho^2 + a^2) - \frac{(2m\rho + e^2)a^2\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\right]\sin^2\theta\left(\frac{d\varphi}{cdt}\right)^2 + \frac{2(2m\rho + e^2)a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\left(\frac{d\varphi}{cdt}\right)$$

Partial differentiation is used to minimize the energy $\varepsilon(\rho, \theta, \varphi, t)$ using $\partial\varepsilon/\partial\theta = 0$.

$$\left\{\begin{aligned} &\frac{(2m\rho + e^2)a^2}{(\rho^2 + a^2\cos^2\theta)^2} + \frac{a^2}{\rho^2 + a^2 + 2m\rho + e^2}\left(\frac{d\rho}{cdt}\right)^2 + a^2\left(\frac{d\theta}{cdt}\right)^2 \\ &-\left[(\rho^2 + a^2) - \frac{(2m\rho + e^2)2a^2\sin^2\theta}{\rho^2 + a^2\cos^2\theta} - \frac{(2m\rho + e^2)a^4\sin^4\theta}{(\rho^2 + a^2\cos^2\theta)^2}\right]\left(\frac{d\varphi}{cdt}\right)^2 \\ &+\left[\frac{2(2m\rho + e^2)a}{\rho^2 + a^2\cos^2\theta} + \frac{2(2m\rho + e^2)a^3\sin^2\theta}{(\rho^2 + a^2\cos^2\theta)^2}\right]\left(\frac{d\varphi}{cdt}\right) \end{aligned}\right\}\cdot\sin 2\theta = 0$$

That is, the energy E and ε are minimized at $\theta = \pi/2$ and the planets gather on the equatorial plane where the energy is steady.

2.1.2. Time component from the variational principle

When the rotation speed of the center of mass is not too fast, the Kerr-Newman solution expanded in the first order of a/ρ takes the form given in (eq.4):

$$(4)\quad \left(\frac{ds}{ds}\right)^2 = 1 = \left(1 - \frac{2m}{\rho} + \frac{e^2}{\rho^2}\right)\left(\frac{cdt}{ds}\right)^2 - \frac{1}{1 - \frac{2m}{\rho} + \frac{e^2}{\rho^2}}\left(\frac{d\rho}{ds}\right)^2 - \rho^2\left(\frac{d\theta}{ds}\right)^2 - \rho^2\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2$$
$$-\frac{2a}{\rho}\left(2m - \frac{e^2}{\rho}\right)\sin^2\theta\left(\frac{cdt}{ds}\right)\left(\frac{d\varphi}{ds}\right)$$

Applying the variational principle to the Kerr-Newman solution, the Euler–Lagrange equation [5] is adopted.

$$\delta\int\Bigg[\left(1-\frac{2m}{\rho}+\frac{e^2}{\rho^2}\right)\left(\frac{cdt}{ds}\right)^2-\frac{1}{1-\frac{2m}{\rho}+\frac{e^2}{\rho^2}}\left(\frac{d\rho}{ds}\right)^2-\rho^2\left\{\left(\frac{d\theta}{ds}\right)^2+\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2\right\}$$
$$-\frac{2a}{\rho}\left(2m-\frac{e^2}{\rho}\right)\sin^2\theta\left(\frac{cdt}{ds}\right)\left(\frac{d\varphi}{ds}\right)\Bigg]ds=0$$

Eventually, (eq.5) is obtained at the equatorial plane of the rotating center of mass where the energy is stable. Hereafter, I perform the calculation at the equatorial plane ($\theta=\pi/2$) of the rotating center of mass.

(5)
$$\begin{cases}\dfrac{d}{ds}\left[\left(1-\dfrac{2m}{\rho}+\dfrac{e^2}{\rho^2}\right)\left(\dfrac{cdt}{ds}\right)-\dfrac{a}{\rho}\left(2m-\dfrac{e^2}{\rho}\right)\left(\dfrac{d\varphi}{ds}\right)\right]=0 & \text{time component}\\ \dfrac{d}{ds}\left[\rho^2\left(\dfrac{d\varphi}{ds}\right)+\dfrac{a}{\rho}\left(2m-\dfrac{e^2}{\rho}\right)\left(\dfrac{cdt}{ds}\right)\right]=0 & \varphi\text{ component}\end{cases}$$

The two equations in (eq.5) are integrated over ds. Using the resulting pair of simultaneous equations, $d\varphi/dt$ (eq. 6) with an integration variable J is obtained.

(6)
$$\frac{d\varphi}{dt}=\frac{\left(\frac{d\varphi}{ds}\right)}{\left(\frac{dt}{ds}\right)}=\frac{J\left(\rho-2m+\frac{e^2}{\rho}\right)+a\left(\frac{e^2}{\rho}-2m\right)}{\rho^3+Ja\left(2m-\frac{e^2}{\rho}\right)}\cdot c$$

J : the angular momentum equivalent
(a kind of Carter constant in relativity theory)

The distance variables are defined as follows:

ρ : An arbitrary orbital distance in two- or three-dimensional coordinates.

R : An aphelion and perihelion distance at the equatorial plane of the rotating center of mass.

r : An aphelion and perihelion distance, energetically stable at the equatorial plane.

2.1.3. Introduction of the angular momentum equivalent

Because of an additional ρ at the aphelion and perihelion distance, $d\rho=0$, the energy equation is given by (eq.7).

(7)
$$0=2\varepsilon+\frac{2m}{R}+\frac{e^2}{R^2}-R^2\left(\frac{d\varphi}{cdt}\right)^2+\frac{4a}{R}\left(m+\frac{e^2}{2R}\right)\left(\frac{d\varphi}{cdt}\right)$$

$d\varphi/cdt$ (eq. 6) is composed of the angular momentum equivalent and is substituted into (eq.7). J is obtained as in (eq.8) by adopting the secondary order R.

(8)
$$J=\frac{4am+R\delta\sqrt{R(2\varepsilon R+2m+C)}}{R^2(R-2m+C)-a(2m-C)\delta\sqrt{R(2\varepsilon R+2m+C)}}R^2$$

Here $\delta=\pm1$ and $C=e^2/R$. δ is related to the orbital rotation direction.

2.2. The Space Fantasy Differential Equation

2.2.1. Introduction of the Space Fantasy differential equation

Changing the angular momentum equivalent J (eq. 8), the relation of R, ε, and J is as given in (eq.9) at the aphelion and perihelion distances. (eq.9) is more complicated than the Kepler-Newton equation $2\varepsilon R^2+2mR-J^2=0$

$$(9)\quad 0 = 2\varepsilon + \frac{2m}{R} + \frac{e^2}{R^2} - R^2\left[\frac{J\left(R - 2m + \frac{e^2}{R}\right) + a\left(\frac{e^2}{R} - 2m\right)}{R^3 + Ja\left(2m - \frac{e^2}{R}\right)}\right]^2$$

$$+\frac{4a}{R}\left(m + \frac{e^2}{2R}\right)\left[\frac{J\left(R - 2m + \frac{e^2}{R}\right) + a\left(\frac{e^2}{R} - 2m\right)}{R^3 + Ja\left(2m - \frac{e^2}{R}\right)}\right]$$

Since the orbital distance R is determined by the energy ε and the angular momentum equivalent J, i.e. $R = R(\varepsilon, J)$. Partially differentiating R by ε, then substituting in J, and adopting the reciprocal of $\partial R/\partial\varepsilon$, a new differential equation is given by (eq.10).

$$(10)\quad \frac{\partial\varepsilon}{\partial R}[R^3 + Ja(2m - C)]^2$$

$$= \frac{(m + C)[R^3 + Ja(2m - C)]^2}{R^2} + \frac{[J(R - 2m + C) - 2am + aC]\,[J(R - 2m + C) + 3aC]\cdot R}{1}$$

$$+\frac{2R^2[J(R - 2m + C) - 4am]\,[J^2a(m - C) - JR^2(R - 3m + 2C) + aR^2(3m - 2C)]}{R^3 + Ja(2m - C)}$$

Here, substituting (eq. 8) for J into (eq.10), the second order R is obtained.
By way of extensive calculations, the relation between ε and R is obtained as in (eq.11).

$$(11)\quad \boldsymbol{\frac{d\varepsilon}{dR}R^4(R^2 - 4mR + 2CR + 4m^2)}$$

$$\boldsymbol{= mR^2(-R^2 + 8mR - 4CR - 12m^2) + \varepsilon\cdot 2R^3(-R^2 + 6mR - 4CR - 8m^2)}$$

$$\boldsymbol{+2am(2R^2 + 2mR - CR - 12m^2)\delta\sqrt{R(2\varepsilon R + 2m + C)}}$$

$$\boldsymbol{+\varepsilon\cdot 4aR(3mR - 2CR - 6m^2 + 7Cm)\delta\sqrt{R(2\varepsilon R + 2m + C)}} \qquad C = e^2/R \quad (R \;\text{2ry order})$$

This second order equation (eq. 11) is called the Space Fantasy differential equation.
Solving the SF differential equation for S, a change of variables is performed. The result is (eq.12).

$$S = R\sqrt{R(2\varepsilon R + 2m + C)}$$

$$(12)\quad \boldsymbol{\frac{dS}{dR} = \frac{2e^2(e^2 + 2m^2)}{SR} + \frac{4a\delta m + S}{R} + \frac{6a\delta mS^2}{R^5}} \qquad (R \;\text{0 order})$$

The form of the differential equation in (eq.12) is more complicated than Riccati's differential equation, which never has an exact general solution [6]. Since $6a\delta mS^2/R^5$ is smaller than S/R , $4a\delta m/R$, and is treated as a constant θ, an approximate differential equation is obtained as in (eq.13).

$$\frac{dS}{dR} = \frac{1}{S}\left[\frac{2E^4}{R} + \frac{4a\delta mS}{R}\left(1 + \frac{6S^2}{4R^4}\right) + \frac{S^2}{R}\right] \qquad E^4 = e^2(e^2 + 2m^2)$$

$$\fallingdotseq \frac{1}{S}\left[\frac{2E^4}{R} + \frac{4a\delta mS}{R}(1 + \theta) + \frac{S^2}{R}\right] \qquad \theta = \frac{3S_0^2}{2R_0^4} \quad (S_0^2, R_0^4 \text{ are centroids } S^2/3, R^4/5)$$

$$(13)\quad \frac{SdS}{S^2 + 4a\delta mS(1 + \theta) + 2E^4} = \frac{dR}{R}$$

The quadrature formulae [7] solve (eq.13) and give in (eq.14) and (eq.15).

In the case that the discriminant $\Delta = E^4 - 2a^2m^2(1+\theta)^2 > 0$:

$$\frac{1}{2}\log[S^2 + 4a\delta m(1+\theta)S + 2E^4] - \frac{4a\delta m(1+\theta)}{2\sqrt{2E^4 - 4a^2m^2(1+\theta)^2}}\arctan\left(\frac{2S + 4a\delta m(1+\theta)}{2\sqrt{2E^4 - 4a^2m^2(1+\theta)^2}}\right)$$

$$= \log R + K$$

$$(14) \quad K = \frac{S^2 + 4a\delta m(1+\theta)S + 2E^4}{R^2} \cdot \mathrm{EXP}\left[\frac{-4a\delta m(1+\theta)}{\sqrt{2E^4 - 4a^2m^2(1+\theta)^2}}\arctan\left(\frac{S + 2a\delta m(1+\theta)}{\sqrt{2E^4 - 4a^2m^2(1+\theta)^2}}\right)\right]$$

In the case that the discriminant $\Delta = E^4 - 2a^2m^2(1+\theta)^2 < 0$:

$$\log[S^2 + 4a\delta mS(1+\theta) + 2E^4]$$

$$-\frac{2a\delta m(1+\theta)}{\sqrt{4a^2m^2(1+\theta)^2 - 2E^4}} \cdot \log\left[\frac{S + 2a\delta m(1+\theta) - \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}{S + 2a\delta m(1+\theta) + \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}\right] = 2\log R + K$$

$$(15) \quad K = \log\left[\frac{\dfrac{S^2 + 4a\delta mS(1+\theta) + 2E^4}{R^2}}{\left[\dfrac{S + 2a\delta m(1+\theta) - \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}{S + 2a\delta m(1+\theta) + \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}\right]^{\frac{2a\delta m(1+\theta)}{\sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}}}\right]$$

2.2.2. Conditions of the energy minimum orbit

Since the minimum energy is $d\varepsilon/dR = 0$ in the SF differential equation (eq. 11), it is a cubic equation in ε.

$$\begin{aligned}
0 = {} & \varepsilon^3 \cdot 32a^2r^3(3mr - 2Cr - 6m^2 + 7Cm)^2 \\
& + \varepsilon^2 \cdot r^2 \begin{bmatrix} 16a^2(3mr - 2Cr - 6m^2 + 7Cm)^2(2m + C) \\ +32a^2m(2r^2 + 2mr - Cr - 12m^2)(3mr - 2Cr - 6m^2 + 7Cm) \\ -4r^3(-r^2 + 6mr - 4Cr - 8m^2)^2 \end{bmatrix} \\
& + \varepsilon \cdot 4mr \begin{bmatrix} 2a^2m(2r^2 + 2mr - Cr - 12m^2)^2 \\ +4a^2(2r^2 + 2mr - Cr - 12m^2)(3mr - 2Cr - 6m^2 + 7Cm)(2m + C) \\ -r^3(-r^2 + 8mr - 4Cr - 12m^2)(-r^2 + 6mr - 4Cr - 8m^2) \end{bmatrix} \\
& + m^2[4a^2(2r^2 + 2mr - Cr - 12m^2)^2(2m + C) - r^3(-r^2 + 8mr - 4Cr - 12m^2)^2]
\end{aligned}$$

Solving this cubic equation, a solution ε_{min} (eq. 16) very close to 0 is adopted in accordance with the principle of the energy minimum.

$$(16) \quad \varepsilon_{min} = \frac{-m}{4r} \cdot \frac{r^3(r^2 - 8mr + 4Cr + 12m^2)^2 - 4a^2(2m + C)(2r^2 + 2mr - Cr - 12m^2)^2}{\begin{bmatrix} r^3(r^2 - 8mr + 4Cr + 12m^2)(r^2 - 6mr + 4Cr + 8m^2) \\ -4a^2(2m + C)(2r^2 + 2mr - Cr - 12m^2)(3mr - 2Cr - 6m^2 + 7Cm) \\ -2a^2m(2r^2 + 2mr - Cr - 12m^2)^2 \end{bmatrix}}$$

$$\fallingdotseq \frac{-m}{4r} \qquad (r\ \ 0 \text{ order})$$

ε_{min} (eq. 16) is substituted to the change of variables $S = r\sqrt{r(2\varepsilon r + 2m + C)}$ (eq. 12).

$$S^2 = \frac{-mr^4}{2} \cdot \frac{r^4(r^2 - 8mr + 4e^2 + 12m^2)^2 - 4a^2(2mr + e^2)(2r^2 + 2mr - e^2 - 12m^2)^2}{\begin{bmatrix} r^5(r^2 - 8mr + 4e^2 + 12m^2)(r^2 - 6mr + 4e^2 + 8m^2) \\ -4a^2(2mr + e^2)(2r^2 + 2mr - e^2 - 12m^2)(3mr^2 - 2e^2r - 6m^2r + 7me^2) \\ -2a^2mr^2(2r^2 + 2mr - e^2 - 12m^2)^2 \end{bmatrix}}$$

$$+ r^2(2mr + e^2)$$

$$= \frac{r^4 \times [r^8 \text{ polynomial}] + r^2(2mr+e^2)\times[r^9 \text{ polynomial}]}{[r^9 \text{polynomial}]} = \frac{r^2 \times P}{Q}$$

$$\fallingdotseq \frac{3m}{2} r^3 \qquad (r \ \ 0 \text{ order})$$

Here, P and Q are given by (eq.17) and (eq.18).

$$(17) \quad P = -mr^2/2\,[\,r^4(r^2-8mr+4e^2+12m^2)^2 - 4a^2(2mr+e^2)(2r^2+2mr-e^2-12m^2)^2\,] + (2mr+e^2)\times Q \qquad [r^{10} \text{ polynomial}]$$

$$(18) \quad Q = r^5(r^2-8mr+4e^2+12m^2)(r^2-6mr+4e^2+8m^2) - 4a^2(2mr+e^2)(2r^2+2mr-e^2-12m^2)(3mr^2-2e^2r-6m^2r+7me^2) - 2a^2mr^2(2r^2+2mr-e^2-12m^2)^2 \qquad [r^9 \text{ polynomial}]$$

And for θ,

$$\theta = \frac{3S_0^2}{2R_0^4} = \frac{5S^2}{2r^4} = \frac{5P}{2Qr^2} \fallingdotseq \frac{15m}{4r} \qquad (r \text{ at } 0 \text{ order})$$

2.3. The Titius–Bode Law

In the case of the discriminant $\Delta = E^4 - 2a^2m^2(1+\theta)^2 > 0$ of the SF differential equation, the function $f(\theta)$ is given in (eq.14) and is subjected to a Maclaurin series expansion. Terms above θ^2 are neglected. The result is given in (eq.19).

$$f(\theta) = \frac{S^2 + 4a\delta m(1+\theta)S + 2E^4}{R^2} \,\mathrm{EXP}\left[\frac{-4a\delta m(1+\theta)}{\sqrt{2E^4-4a^2m^2(1+\theta)^2}} \arctan\left(\frac{S+2a\delta m(1+\theta)}{\sqrt{2E^4-4a^2m^2(1+\theta)^2}}\right)\right] - K = 0$$

$$f(\theta) = f(0) + \frac{1}{1!}\cdot\frac{\partial f(0)}{\partial\theta}\theta + \frac{1}{2!}\cdot\frac{\partial^2 f(0)}{(\partial\theta)^2}\theta^2 + \cdots = 0$$

$$(19) \quad f(\theta) = \frac{3mr}{2}\,\mathrm{EXP}\left[\frac{-4a\delta m}{\sqrt{2E^4-4a^2m^2}} \arctan\left(\frac{r\sqrt{3mr}}{2\sqrt{E^4-2a^2m^2}}\right)\right] \times \left[1 - \frac{30a\delta m^2E^4}{r[2E^4-4a^2m^2]^{\frac{3}{2}}} \times \arctan\left(\frac{r\sqrt{3mr}}{2\sqrt{E^4-2a^2m^2}}\right)\right] - K = 0$$

Since r is very large, it is given as $\arctan\left(\frac{r\sqrt{3mr}}{2\sqrt{E^4-2a^2m^2}}\right) = \pi/2 + \pi N$. This is substituted into (eq.19).

$$K = \frac{3mr}{2}\,\mathrm{EXP}\left[\frac{-2a\delta m\pi(1+2N)}{\sqrt{2E^4-4a^2m^2}}\right]\cdot\left[1 - \frac{30a\delta m^2E^4}{r[2E^4-4a^2m^2]^{\frac{3}{2}}}\cdot\frac{\pi(1+2N)}{2}\right]$$

The integration constant K is common to all planets that orbit the center of mass. For the base planet they are r_1, N_1, $N - N_1 = n-1$, and the distance ratio $\xi = r/r_1$. The result is given in (eq.20).

$$(20) \quad n-1 = \frac{\sqrt{2E^4-4a^2m^2}}{4a\delta m\pi}\cdot\log\left[\frac{\xi - \dfrac{15a\delta m^2E^4\pi(2N_1+2n-1)}{r_1[2E^4-4a^2m^2]^{\frac{3}{2}}}}{1 - \dfrac{15a\delta m^2E^4\pi(2N_1+1)}{r_1[2E^4-4a^2m^2]^{\frac{3}{2}}}}\right]$$

For the other side, the Titius-Bode law is changed into (eq.21).

$$\xi_{Earth} = 0.4 + 0.3\times 2^n = 0.4 + 0.6\times 2^{n-1} \qquad (\xi_{Earth} : \text{the Earth basis } \xi)$$

$$(21) \quad n-1 = \frac{1}{\log 2}\cdot\log\frac{\xi_{Earth} - 0.4}{1-0.4}$$

The Titius-Bode law (eq. 21) is remarkably similar to the solution (eq. 20) of the approximate SF differential equation. If the two coefficients are same, the two equations are almost equal.

(The Earth is the base planet, $n=1$.)

$$\frac{1}{\log 2} = \frac{\sqrt{2E^4 - 4a^2m^2}}{4a\delta m\pi} \qquad 0.4 = \frac{15a\delta m^2E^4\pi(2N_1+1)}{r_1[2E^4 - 4a^2m^2]^{\frac{3}{2}}}$$

Since $r_1 = 1.5 \times 10^8 km$ for the Earth, $m=1.476km$ and $a=0.32km$ [8] for the Sun, we have $e = 2.1km$, $N_1 = 1.5 \times 10^7$. The 2n on the right side of (eq.20) is neglected because of the very large N_1. Thus

$$(22) \qquad \boldsymbol{\xi = \left[1 - \frac{30a\delta m^2E^4\pi N_1}{r_1[2E^4 - 4a^2m^2]^{\frac{3}{2}}}\right] \cdot \mathrm{EXP}\left[\frac{4am\pi(n-1)}{\sqrt{2E^4 - 4a^2m^2}}\right] + \frac{30a\delta m^2E^4\pi N_1}{r_1[2E^4 - 4a^2m^2]^{\frac{3}{2}}}}$$

$$\xi_{Earth} = \boldsymbol{(1 - 0.4) \cdot 2^{n-1} + 0.4}$$

$\delta = \pm 1$ is related to the orbital rotation direction.

(Eq.22) is now exactly equal to (eq.21). The Titius-Bode law has therefore been demonstrated.

2.4. The Saturn's Rings

Since the autorotation of Saturn is fast, the discriminant $\Delta = E^4 - 2a^2m^2(1+\theta)^2 > 0$ of the SF differential equation (eq. 15) is as follows.

$$K = \log\left[\frac{\frac{S^2 + 4a\delta mS(1+\theta) + 2E^4}{R^2}}{\left[\frac{S + 2a\delta m(1+\theta) - \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}{S + 2a\delta m(1+\theta) + \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}\right]^{\frac{2a\delta m(1+\theta)}{\sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}}}\right]$$

Since the power number $\left[\frac{2a\delta m(1+\theta)}{\sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}\right]$ is nearly 1, the denominator is expressed as $(1-\lambda)$. λ is very small, but not zero. The solution of the SF differential equation is (eq.23).

$$1 - \lambda = \left[\frac{S + 2a\delta m(1+\theta) - \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}{S + 2a\delta m(1+\theta) + \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}\right]^{\frac{2a\delta m(1+\theta)}{\sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}}$$

$$(23) \qquad K = \frac{S^2 + 4a\delta mS(1+\theta) + 2E^4}{r^2} \cdot \frac{1}{(1-\lambda)}$$

The integration constant K is common to all the rings that belong to Saturn. For the base ring, the variables are r_1 and $F=K$, and the polynomial of S is (eq.24).

$$(24) \qquad S^4 - 2S^2[F(1-\lambda)r^2 - 2E^4 + 8a^2m^2(1+\theta)^2] + [F(1-\lambda)r^2 - 2E^4]^2 = 0$$

P (eq. 17) and Q (eq. 18) are substituted into (eq.24) to give S and θ. Finally, the polynomial of r is (eq.25).

$$(25) \qquad \boldsymbol{P^2Qr^2 - 2P\left\{[F(1-\lambda)r^2 - 2E^4]Q^2 + 2a^2m^2\left(2Q + \frac{5P}{r^2}\right)^2\right\} + [F^2(1-\lambda)^2r^2 - 4F(1-\lambda)E^4]Q^3 = 0}$$

The degree of (eq.25) is the highest for the first term, P^2Qr^2, and is r to the 31st〔+10×2+9+2〕power. So (eq.25) is a polynomial of r^{31} with λ high degree coefficients. Thus, planets with rings such as Saturn have a maximum of 31 rings. The real number of rings decreases by roots of complex number, minus roots, equal roots and swelling of the center core.

3. Discussion

In summary, the Titius-Bode law, discovered 250 years ago, is considered to be a mathematical coincidence rather than an "exact" law. But I have proved the law and that Saturn can have a maximum of 31 rings physically. The 250-year-mystery of astronomy is resolved by not computer analysis but theoretical analysis.

About the Kerr-Newman solution of the Einstein's equation , the no-hair theorem postulates that all black hole solutions of the Einstein-Maxwell equations of gravitation and electromagnetism in general relativity can be completely characterized by only three externally observable classical parameters: mass, electric charge, and angular momentum. All other information (for which "hair" is a metaphor) about the matter which formed a black hole or is falling into it, "disappears" behind the black-hole event horizon and is therefore permanently inaccessible to external observers. [9], [10]

In this manner, on the ground of that this theory is based on the steady state Kerr-Newman solution in a mature space-time, it cannot be applied to a still young, unstable and transitional state space-time. Three important equations can be summarized as follows.

(eq.11) is a fundamental approximate differential equation, based on the steady state Kerr-Newman solution and can be applied to Solar system, other planets and rings in universe at stable state. There must be many solutions of (eq.11).

(eq.22) is one of approximate solutions of (eq.11). This is energy stable and applicable to Solar system planets and some of around 4000 extrasolar planets in universe. However, it is not applicable to still young, unstable and transitional state planets like comets.

(eq.25) is one of approximate solutions of (eq.11). This is energy stable and applicable to Saturn's rings and other some extrasolar planets' rings.

This theory is applied to planets, which belong to one center of mass in universe, but it is not available to a bulge space near center of mass. And it cannot be applied to galaxies, which should be considered to be under influences of dark matter and dark energy.

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Author Contributions

F. I. developed the theory and wrote the manuscript.

Competing Interests

The author declares no competing interests including financial and non-financial interests.

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