Refutation: constructive Zermelo-Fraenkel set theory (CZF) and extended Church's thesis (ECT)

Abstract: We evaluate constructive Zermelo-Fraenkel set theory (CZF) of intuitionistic logic. Of the eight CZF axioms, only the induction scheme is tautologous. From CZF axioms for infinity, set induction, and extensionality the deduction of one set denoted ω is *not* tautologous. An equation for extended Church's thesis (ECT) is *not* tautologous. This supports previous work that intuitionistic logic is *not* tautologous.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated proof value, **F** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪; - Not Or; & And, ∧, ∩; \ Not And;
> Imply, greater than, →, ↦, ≻, ⊃, ⊢, ⊨, ↠; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞;
= Equivalent, ≡, :=, ⟺, ↔, ≜, ≈; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) `T` as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, ∅, Null, ⊥, zero;
(%z<#z) `C` as contingency, Δ, ordinal 1; (%z>#z) `N` as non-contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y); (A=B) (A~B).
Note: For clarity we usually distribute quantifiers on each variable as designated.

From: Rathjen, M. (2005). Constructive Zermelo-Fraenkel Set Theory CZF.
jucs.org/jucs_11_12/constructive_set_theory_and/jucs_11_12_2008_2033_rathjen.pdf
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Definition: 2.1 (Axioms of CZF) The language of CZF is the first order language of Zermelo-Fraenkel set theory, LST, with the non logical primitive symbol ∈. CZF is based on intuitionistic predicate logic with equality. The set theoretic axioms of axioms of CZF are the following:

1. Extensionality: $\forall a\, \forall b\, (\forall y\, (y \in a \leftrightarrow y \in b) \rightarrow a = b)$ (2.1.1)

LET a, b, y, x: p, q, r, s

((#r<#p)=(#r<#q))>(#p=#q) ; `TCCT TTTT TCCT TTTT` (2.1.2)

2. Pair: $\forall a\, \forall b\, \exists x\, \forall y\, (y \in x \leftrightarrow y = a \vee y = b)$ (2.2.1)

LET p, q, r, s: a, x, y, z

((#r<%s)=#r)=((#p+%s)=q) ; N**F**CT N**F**CT **FF**TT NNCC (2.2.2)

3. Union: $\forall a\, \exists x\, \forall y\, (y \in x \leftrightarrow \exists z \in a\; y \in z)$ (2.3.1)

(#r<%q)=(%s<((#p&#r)<%s)) ; NNNN **FF**NN **FFFF** NN**FF** (2.3.2)

4. Restricted separation scheme: $\forall a\, \exists x\, \forall y\, (y \in x \leftrightarrow y \in a \wedge \phi(y))$, for every restricted formula $\phi(y)$, where a formula $\phi(x)$ is *restricted*, or $\Delta 0$, if all the quantifiers occurring in it are restricted, i.e. of the form $\forall x \in b$ or $\exists x \in b$. (2.4.1)

LET p, q, r, s: ϕ, a, x, y

```
((s<r)=(x<q))&(p&s) ;          FFFF FFFF FFFF FTFT(16),
                               FFFF FFFF FTFF FFFT(16)
```
(2.4.2)

5. Subset collection scheme: $\forall a\, \forall b\, \exists c\, \forall u\, \forall x \in a\, \exists y \in b\, \phi(x,y,u) \rightarrow \exists d \in c\, (\forall x \in a\, \exists y \in d\, \phi(x,y,u) \wedge \forall y \in d\, \exists x \in a\, \phi(x,y,u))$ for every formula $\phi(x,y,u)$ (2.5.1)

```
((#x<#q)&(%y<(#r&#(#p&((x&y)&u)))))>((%z<%s)&(((#x<#q)&((%y<
%z)&#(p&(x&(y&#u)))))&((#y<%z)&((%x<#q)&#(p&(x&(y&#u))))))) ;
                               TTTT TTTT TTTT TTTT(48),
                               CCTT CCTT CCTT CCTT(16)
```
(2.5.2)

6. Strong collection scheme: $\forall a\, \forall x \in a\, \exists y\, \phi(x,y) \rightarrow \exists b\, (\forall x \in a\, \exists y \in b\, \phi(x,y) \wedge \forall y \in b\, \exists x \in a\, \phi(x,y))$ for every formula $\phi(x,y)$ (2.6.1)

```
((#x<#q)&(%y&#(p&(x&y))))>(((#x<#q)&((%y<%r)&#(p&(x&y))))&
((#y<%r)&((%x<#q)&#(p&(x&y))))) ;
                               TTTT TTTT TTTT TTTT(48),
                               TTTT TCTT TTTT TCTT(16)
```
(2.6.2)

7. Infinity: $\exists x \forall u\, [\, u \in x \leftrightarrow 0 = u \vee \exists v \in x\, (u = v \cup \{v\})$ where y+1 is $y \cup \{y\}]$, and 0 is the empty set, defined in the obvious way

```
((y+(%y>#y))=(y&y))>((#u<%x)=((y@y)=(#u&(%v<(#x&(#u=(#u&#u)))))) ;
                               NNNN NNNN NNNN NNNN( 4),
                               FFFF FFFF FFFF FFFF(28)
```
(2.7.2)

8. Set induction scheme: (IND $\in$) $\forall a\, (\forall x \in a\, \phi(x) \rightarrow \phi(a)) \rightarrow \forall a\, \phi(a)$, for every formula $\phi(a)$ (2.8.1)

```
#(p&q)>(((#x<(#q&#(p&q)))>#(p&q))>(p&#q)) ;
                               TTTT TTTT TTTT TTTT
```
(2.8.2)

From Infinity, Set induction, and Extensionality one can deduce that there exists exactly one set x such that $\forall u[u \in x \leftrightarrow 0 = u \vee \exists v \in x(u = v \cup \{v\})]$; this set will be denoted by ω. (2.9.1)

```
((((#x<#q)&(%y&#(p&(x&y))))>(((#x<#q)&((%y<%r)&#(p&(x&y))))&
((#y<%r)&((%x<#q)&#(p&(x&y))))))&((#(p&q)>(((#x<(#q&#(p&q)))>
#(p&q))>(p&#q)))&(((#r<#p)=(#r<#q))>(#p=#q))))>(((#u<x)=(((y@y)=
(#u&%v))<(x&(#u=(%v&%v)))))>%x) ;
```

```
CCCC CCCC CCCC CCCC( 4),
TTTT TTTT TTTT TTTT(28)
```

(2.9.2)

Definition: 3.3 Extended Church's Thesis, ECT, asserts that

$$\forall n \in N\ \psi(n) \rightarrow \exists m \in N \phi(n,m) \text{ implies}$$
$$\exists e \in N \forall n \in N\ \psi(n) \rightarrow \exists m,p \in N\ [T(e,n,p) \wedge U(p,m) \wedge \phi(n,m)] \quad (3.3.1)$$

LET p, q, r, s, t, u, v, w, x:
p, ϕ, ψ, e, T, U, m, n, N.

```
((#w<(x&(r&w)))>(%v<((x&y)&(w&v)))) >
(((%s<x)&(#w<(x&(r&w))))>
(((%v&%p)<x)&(((t&s)&(w&p))&((u&(p&v))&(q&(w&v)))))) ;
TTTT TTTT CCCT CCCT( 1),
TTTT TTTT CCCC CCCC( 3),
TTTT TTTT TTTT TTTT(28)
```

(3.3.2)

Based on intuitionistic logic, CZF axioms as rendered in Eqs. 2.1-2.7 are *not* tautologous. Eq. 2.8 as the induction scheme is tautologous. Eq. 2.9 to derive one set named ω is *no*t tautologous. The definition for extended Church's thesis in Eq. 3 is *not* tautologous. These results support previous work that intuitionistic logic is *not* tautologous.