Refutation of the Banach-Tarski paradox

Abstract: We evaluate the crucial claim of the proof in Step 3, as a fleshed out detail. It is *not* tautologous, nor is it contradictory. This means the claim is a non tautologous fragment of the universal logic VŁ4 and constitutes the briefest known refutation of the Banach-Tarski paradox.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪; - Not Or; & And, ∧, ∩, · ; \ Not And;
> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ⊢, ⊨, → ;
< Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲;
= Equivalent, ≡, :=, ⟺, ↔, ≜, ≈, ≃; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, ∅, Null, ⊥, zero;
(%z<#z) C as contingency, Δ, ordinal 1; (%z>#z) N as non-contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆y); (A=B) (A~B).
Note: For clarity we usually distribute quantifiers on each variable as designated.

From: en.wikipedia.org/wiki/Banach–Tarski_paradox

Some details, fleshed out ... [for Step 3 of 4]
What remains to be shown is the Claim: $S^2 - D$ is equidecomposable with S^2.
Proof. Let λ be some line through the origin that does not intersect any point in D. This is possible since D is countable. Let J be the set of angles, α, such that for some natural number n, and some P in D, **r**(nα)P is also in D, where **r**(nα) is a rotation about λ of nα. Then J is countable. So there exists an angle θ not in J. Let ρ be the rotation about λ by θ. Then ρ acts on S^2 with no fixed points in D, i.e., $\rho^n(D)$ is disjoint from D, and for natural m<n, $\rho^n(D)$ is disjoint from $\rho^m(D)$. Let E be the disjoint of $\rho^n(D)$ over n = 0, 1, 2, ... Then

$$S^2 = E \cup (S^2 - E) \sim \rho(E) \cup (S^2 - E) = (E - D) \cup (S^2 - E) = S^2 - D, \quad (3.1)$$

LET p, q, r, s: E, D, ρ, S^2

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(s=((p+(s-p))=(  (r&p)+(s-p))))=(((p-q)+(s-p))=(s-q))  ;
                                FFTT FTTF FFTF FTTT                (3.2)
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where ~ denotes "is equidecomposable to".

Remark 3.2: We write "~" as "equivalent to". Eq. 3.2 as rendered is *not* tautologous. Because it is the crucial claim of the proof, the result is that the Banach-Tarski paradox is also not contradictory, and hence a non tautologous fragment of the universal logic VŁ4.