**Refutation of the Goldblatt-Thomason theorem**



**Abstract:** From the introduction, we evaluate the terms forth and back as a duality. Neither is negation of the other, hence refuting the core basis of the Goldblatt-Thomason theorem. The question posed by it is further answered by the universal logic VŁ4 that expresses by modal axioms *all* first-order definable properties of a binary relation, due to equivalences of the respective quantifier and modal operator.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated proof value, **F** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪; - Not Or; & And, ∧, ∩, · ; \ Not And; > Imply, greater than, →, ⇒, ↦, ≻, ⊃, ⊢, ⊨, → ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ←, ≲;
= Equivalent, ≡, :=, ⟺, ↔, ≜, ≈, ≃; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) `T` as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, ∅, Null, ⊥, zero;
(%z<#z) `C` as contingency, Δ, ordinal 1; (%z>#z) `N` as non-contingency, ∇, ordinal 2;
~( y < x) ( x ≤ y), ( x ⊆y); (A=B) (A~B).
Note: For clarity we usually distribute quantifiers on each variable as designated.

From: Goldblatt, R. (2019). Morphisms and duality for polarities and lattices with operators. arxiv.org/pdf/1902.09783.pdf rob.goldblatt@msor.vuw.ac.nz

1 Introduction

We develop here a new notion of 'bounded morphism' between certain structures that model propositional logics lacking the distributive law for conjunction and disjunction. Our theory adapts a well known semantic analysis of modal logic, which we now review.

There are two main types of semantical interpretation of propositional modal logics. In *algebraic* semantics, formulas of the modal language are interpreted as elements of a modal algebra (B, f), which comprises a Boolean algebra B with an additional operation f that interprets the modality ◊ and preserves finite joins. In *relational* semantics, formulas are interpreted as subsets of a Kripke frame (X, R), which comprises a binary relation R on a set X.

The relationship between these two approaches is explained by a *duality* that exists between algebras and frames. This is fundamentally category-theoretic in nature. The modal algebras are the objects of a category MA whose arrows are the standard algebraic homomorphisms. The Kripke frames are the objects of a category KF whose [*sic*] arrows are the *bounded morphisms*, α: (X, R) → (X′, R′), i.e. functions α: X → X′ satisfying the 'back and forth' conditions

(Forth): xRy implies α(x)R′α(y). (1.1.1.1)

**Remark 1.1.1.1:** The Forth label is later interchanged with the confusing name of preservation.

LET p, r, s, w, x, y, z: α, R, R', β, x, y, z

(x&(r&y))>((p&x)&(s&(p&y))) ; `TTTT TTTT TTTT TTTT(48),`
`TTTT` **`FFFF`** `TTTT` **`F`**`T`**`F`**`T(16)` (1.1.1.2)

(Back): α(x)R′z implies ∃y(xRy&α(y)=z). (1.1.2.1)

**Remark 1.1.2.1:** The Back label is later interchanged with the confusing name of reflection.

((p&x)&(s&z))>((x&(r&%y))&((p&%y)=z)) ;
`TTTT TTTT TTTT TTTT(80),`
`TTTT TTTT T`**`F`**`T`**`F`** `TCTC(16),`
`TTTT TTTT TTTT TTTT(16),`
`TTTT TTTT T`**`F`**`T`**`F`** `TTTT(16)` (1.2.2.2)

(Bounded morphisms are also known as p-morphisms. The adjective ‘bounded’ derives from the R-bounded existential quantification in (1.2.1).)

**Remark 1.n:** Eqs. 1.1.2.1 and 1.2.2.2 as rendered are not respective negations. This refutes Forth and Back as a *duality*.

## 12 Goldblatt-Thomason theorem

This theorem [pay-to-play reference, from 1975] was originally formulated as an answer to the question:

which first-order definable properties of a binary relation can be expressed by modal axioms? (12.1.1)

**Remark 12.1.1:** The universal logic VŁ4 answers Eq. 12.1.1 as:

"*all* first-order definable properties of a binary relation can be expressed by modal axioms" because in the universal logic VŁ4 the respective quantifiers are equivalent to the modal operators. (12.1.2)