In the name of the only God Allah, Allah the Lord of the worlds, the most Gracious the most Merciful.

The Number of the Primes Less than the Magnitude of P_n^2 by using the Primes 2,3,5,…,P_n

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Abstract

this paper carrying a method to calculate an approximation to the number of the prime numbers in the natural numbers interval $I = \{1,2,3,4, \dots \dots, P_n, P_n + 1, P_n + 2, \dots \dots, P_n^2\}$ by using the primes $(2,3,5, \dots, P_n)$ to specify the primes density in the sub intervals $I(P_n)$ as

$I(P_n) = \{P_n^2, P_n^2 + 1, P_n^2 + 2, P_n^2 + 3, \dots \dots, P_{n+1}^2 - 1\}$ has primes density of $(d(P_n))$= ($\prod_{i=1}^{i=n}(1 - \frac{1}{P_i}$).

Introduction

Actually to say the number of the prime numbers, or to say the prime counting function, or to say the primes distribution function, all are leading to the same meaning , the direct meaning, to find the prime's counter number and indirect to find the prime number with its own counter number.

Among the greatest works done on the prime numbers were the work of Bernhard Riemann on his paper (on the number of primes less than a given magnitude) [1] in his paper he studied the prime counting function using analytic method and discussed the relationship between zeta s and the distribution of the prime numbers, and showed that the prime counting function grows more slowly than the logarithmic integral, in this paper I have used to calculate the primes density in a specific intervals $I(P_n) = \{P_n^2, P_n^2 + 1, P_n^2 + 2, P_n^2 + 3, \dots \dots, P_{n+1}^2 - 1\}$, since all numbers in this interval are successive and may it prime or a composite as a product of some primes from 2,3, … , P_n which allowed me to calculate a specific composites density and specific primes density for this interval as shown bellow in the methodology.

Methodology

Defined the interval $I(P_n)$, as $I(P_n) = \{P_n^2, P_n^2+1, P_n^2+2, P_n^2+3, \dots\dots, P_{n+1}^2-1\}$

P_n represent the maximum prime number in the consecutive prime numbers 2,3,5,7,11, … …, P_n

$P_1 = 2, P_2 = 3, P_3 = 5, P_4 = 7, \dots\dots, P_n = P_n$

Any interval $I(P_n)$ of length L obey the relatively primality test as ½ of L is relatively prime to 2 and ½ of L is relatively factored by 2 , 2/3 of L is relatively prime to 3 and 1/3 of L is relatively factored by 3, 4/5 of L is relatively prime to 5 and 1/5 is relatively factored by 5, and so on.

The density of primes $d(P)$ for any length L of consecutive numbers of interval $I(P_n)$ obey that

The deducting of the composite numbers as follow:

deduction numbers factored by 2

$$L - \frac{1}{2}L = L(1-\frac{1}{2})$$

deduction the rest numbers factored by 3

$$L\left(1-\frac{1}{2}\right) - \frac{1}{3}L(1-\frac{1}{2}) = L(1-\frac{1}{2})(1-\frac{1}{3})$$

Deduction the rest numbers factored by 5

$$L(1-\frac{1}{2})(1-\frac{1}{3}) - \frac{1}{5}L(1-\frac{1}{2})(1-\frac{1}{3}) = L(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})$$

deduction the rest numbers factored By 7

$$L(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5}) - \frac{1}{7}L(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5}) = L(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})$$

And so on

.. … …. …..

Deduction the rest numbers factored by P_n

$$L\prod_{i=1}^{i=n-1}(1-\frac{1}{P_i}) - \frac{1}{P_n}L\prod_{i=1}^{i=n-1}(1-\frac{1}{P_i}) = L(\prod_{i=1}^{i=n}(1-\frac{1}{P_i})$$

Recollect it to get

$$L - \frac{1}{2}L - \frac{1}{3}L(1-\frac{1}{2}) - \frac{1}{5}L(1-\frac{1}{2})(1-\frac{1}{3}) - \dots - \frac{1}{P_n}L(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})..(1-\frac{1}{P_{n-1}}) = L\prod_{i=1}^{i=n}(1-\frac{1}{P_i})$$

so

$$L(1-(\frac{1}{2}+\frac{1}{3}\frac{1}{2}+\frac{1}{5}\frac{1}{2}\frac{2}{3}+\frac{1}{7}\frac{1}{2}\frac{2}{3}\frac{4}{5}+\dots+\frac{1}{P_n}\prod_{i=1}^{i=n-1}(1-\frac{1}{P_i}))) = L\prod_{i=1}^{i=n}(1-\frac{1}{P_i}) \qquad (1)$$

Will give the distribution function

$$(1-(\frac{1}{2}+\frac{1}{3}\frac{1}{2}+\frac{1}{5}\frac{1}{2}\frac{2}{3}+\frac{1}{7}\frac{1}{2}\frac{2}{3}\frac{4}{5}+\ldots+\frac{1}{P_n}\prod_{i=1}^{i=n-1}(1-\frac{1}{P_i}))) = \prod_{i=1}^{i=n}(1-\frac{1}{P_i}) \quad (2)$$

As the primes density $(d(P_n)) = \prod_{i=1}^{i=n}(1-\frac{1}{P_i})$ (3)

The composites density $(d(C_n)) = (\frac{1}{2}+\frac{1}{3}\frac{1}{2}+\frac{1}{5}\frac{1}{2}\frac{2}{3}+\frac{1}{7}\frac{1}{2}\frac{2}{3}\frac{4}{5}+\ldots+\frac{1}{P_n}\prod_{i=1}^{i=n-1}(1-\frac{1}{P_i}))$

$$(d(C_n)) = (1-\prod_{i=1}^{i=n}(1-\frac{1}{P_i})) \quad (4)$$

that is the distribution function for the primes density and composites density for L to $I(P_n)$

so the expectation value of the number of the primes in the interval of length $L \; for \; I(P_n)$ is $\langle \#P_n \rangle$

$$\langle \#P_n \rangle = L(d(P_n)) = L(\prod_{i=1}^{i=n}(1-\frac{1}{P_i})) \quad (5)$$

the expectation value of the number of the composites in the interval of length $L \; for \; I(P_n)$ is $\langle \#C_n \rangle$

$$\langle \#C_n \rangle = L(d(C_n) = L(1-\prod_{i=1}^{i=n}(1-\frac{1}{P_i})) \quad (6)$$

$$(d(P_n))+(d(C_n)) = 1$$

so to calculate the estimated value of the number of the primes under the value P_n^2 , since the length multiplied by density gives the estimated value, so the summation of the interval's length multiplied by the primes density in this sub interval will gives the estimated value of the primes around the upper bound of that intervals parallel with giving the primes distribution function, so

$$I(P_n) = \{P_n^2, P_n^2+1, P_n^2+2, \ldots\ldots, P_{n+1}^2-1\}$$

$$l_n = the\ length\ of\ the\ interval\ I(P_n)$$

$$l_n = P_{n+1}^2 - P_n^2 - 1$$

$$dP_n = the\ density\ of\ the\ primes\ in\ the\ interval\ I(P_n)$$

$$dP_n = \prod_{i=1}^{i=n}(1-\frac{1}{P_i})$$

$\varphi(P_n^2) = the\ primes\ distribution\ function$,or the number of the prime numbers under the magnitude P_n^2, (the primes counting function)under the magnitude P_n^2.

Since 2,3 are basic primes so we will add 2 at the beginning

So

$$\varphi(P_n^2) = 2 + \sum_{i=1}^{i=n-1} l_i\, dP_i = 2 + \{l_1\, dP_1 + l_2\, dP_2 + \cdots + l_{n-1} dP_{n-1}\} \quad (7)$$

$$= 2 + (P_2^2 - P_1^2 - 1)(1-\frac{1}{P_1}) + (P_3^2 - P_2^2 - 1)(1-\frac{1}{P_1})\left(1-\frac{1}{P_2}\right) + \cdots + (P_n^2 - P_{n-1}^2 - 1)\prod_{i=1}^{i=n-1}\left(1-\frac{1}{P_i}\right) \quad (8)$$

$$= 2 + (9-4-1)\frac{1}{2} + (25-9-1)\frac{1}{2}\frac{2}{3} + (49-25-1)\frac{1}{2}\frac{2}{3}\frac{4}{5} + \cdots + (P_n^2 - P_{n-1}^2 - 1)\frac{1}{2}\frac{2}{3}\frac{4}{5}\cdots\cdots\frac{P_{n-1}-1}{P_{n-1}}$$

$$= \{3\frac{1}{2} + 5\frac{1}{2}\frac{2}{3} + \cdots + P_{n-1}\prod_{i=1}^{i=n-2}\left(1-\frac{1}{P_i}\right) + P_n^2\prod_{i=1}^{i=n-1}\left(1-\frac{1}{P_i}\right)\} - \{\frac{1}{2} + \frac{1}{2}\frac{2}{3} + \frac{1}{2}\frac{2}{3}\frac{4}{5} + \cdots + \prod_{i=1}^{i=n-1}\left(1-\frac{1}{P_i}\right)\}$$

$$= \{P_n^2\prod_{i=1}^{i=n-1}(\frac{P_i-1}{P_i})\} + \{\sum_{i=1}^{i=n-2}(P_{i+1}\,(\prod_{j=1}^{j=i}(\frac{\mathrm{P_j}-1}{\mathrm{P_j}})))\} - \{\sum_{i=1}^{i=n-1}\prod_{j=1}^{j=i}(\frac{P_i-1}{P_i})\} \quad \text{for } n \geq 3$$

So for $n \geq 3$

$$\varphi(P_n^2) = \{P_n^2\prod_{i=1}^{i=n-1}(\frac{P_i-1}{P_i})\} + \{\sum_{i=1}^{i=n-2}(P_{i+1}\,(\prod_{j=1}^{j=i}(\frac{\mathrm{P_j}-1}{\mathrm{P_j}})))\} - \{\sum_{i=1}^{i=n-1}\prod_{j=1}^{j=i}(\frac{P_j-1}{P_j})\ \} \qquad (9)$$

Consider (7) and (8) as an expression to the primes distribution function, and (9) is the estimated value of the number of the primes less than the magnitude P_n^2 for $n \geq 3$.

So properly we can say

$$P_n = \sqrt{\frac{\varphi(P_n^2) - \sum_{i=1}^{i=n-2}(P_{i+1}\,(\prod_{j=1}^{j=i}(\frac{\mathrm{P_j}-1}{\mathrm{P_j}}))) + \{\sum_{i=1}^{i=n-1}\prod_{j=1}^{j=i}(\frac{P_j-1}{P_j})\ \}}{\prod_{i=1}^{i=n-1}(\frac{P_i-1}{P_i})}}$$

$$P_n = \left(\frac{1}{\prod_{i=1}^{i=n-1}(1-\frac{1}{P_i})}\right)^{\frac{1}{2}}\left(\varphi(P_n^2) - \sum_{i=1}^{i=n-2}(P_{i+1}\,(\prod_{j=1}^{j=i}(\frac{\mathrm{P_j}-1}{\mathrm{P_j}}))) + \{\sum_{i=1}^{i=n-1}\prod_{j=1}^{j=i}(\frac{P_j-1}{P_j})\ \}\right)^{\frac{1}{2}}$$

Maybe this is a useless defined to the prime number P_n but it carries some signs to the possible relations and variables to understand.

Among that a better useful defined to use the counter number $\varphi(P_n^2)$ to define the prime $P_{\varphi\left(P_n^2\right)}$ by comparison with the exact prime counting function $\pi(x)$ as $x = (P_n^2)$ then by recall the prime gap we can say.

$$P_{\pi(P_n^2)} = P_n^2 - r \qquad as \quad 2 \leq r \leq g_{\pi(P_n^2)}$$

Or

$$P_{\pi(P_n^2)+.5\pm.5} = P_n^2 \pm r_{\pm} \ \ as \ \ 2 \leq r_{\pm} \leq g_{\pi(P_n^2)} \ \ \text{with condition } r_+ + r_- = g_{\pi(P_n^2)}$$

so

$$P_{\varphi(P_n^2)} \cong P_n^2 - r \quad as \quad 0 \leq r \leq g_{\varphi(P_n^2)}$$

and refer to the preprint [2] the large gap could be within

$$g_{\varphi\left(P_n^2\right)} = \frac{2}{(\frac{1}{2}\prod_{i=2}^{i=n-1}(1-\frac{2}{P_i}))}$$

then

$$P_{\varphi(P_n^2)+.5\pm.5} \cong P_n^2 \pm r_\pm \;\; as \;\; 0 \le r_\pm \le g_{\varphi(P_n^2)} \qquad 0 \le r_\pm \le \frac{2}{(\frac{1}{2}\prod_{i=2}^{i=n-1}(1-\frac{2}{P_i}))}$$

Or we can say

$$P_{\varphi(P_n^2)+.5\pm.5} \cong P_n^2 \pm r_\pm \quad \text{as } r_\pm \text{ some relatively small value.}$$

Finally we can say for the magnitude $x \; as \; x \; is \; a \; positive \; real \; number$, and as P_n is the greatest prime less than or equal $\sqrt{x}$ then,

$$\varphi(x) = \varphi(P_n^2) + (x - P_n^2)dP_n$$

$$\varphi(x) = 2 + \sum_{i=1}^{i=n-1} l_i \, dP_i + (x - P_n^2)dP_n$$

$$= 2 + \{l_1 \, dP_1 + l_2 \, dP_2 + \cdots + l_{n-1}dP_{n-1}\} + (x - P_n^2)dP_n$$

$$= 2 + (P_2^2 - P_1^2 - 1)(1 - \frac{1}{P_1}) + (P_3^2 - P_2^2 - 1)(1 - \frac{1}{P_1})\left(1 - \frac{1}{P_2}\right) + \cdots + (x - P_n^2)\prod_{i=1}^{i=n}\left(1 - \frac{1}{P_i}\right)$$

$$= 2 + (9 - 4 - 1)\frac{1}{2} + (25 - 9 - 1)\frac{1}{2}\frac{2}{3} + (49 - 25 - 1)\frac{1}{2}\frac{2}{3}\frac{4}{5} + \cdots + (x - P_n^2)\frac{1}{2}\frac{2}{3}\frac{4}{5}\ldots\ldots\frac{P_n - 1}{P_n}$$

$$= \{3\frac{1}{2} + 5\frac{1}{2}\frac{2}{3} + \cdots + P_n\prod_{i=1}^{i=n-1}\left(1 - \frac{1}{P_i}\right) + x\prod_{i=1}^{i=n}\left(1 - \frac{1}{P_i}\right)\} - \{\frac{1}{2} + \frac{1}{2}\frac{2}{3} + \frac{1}{2}\frac{2}{3}\frac{4}{5} + \cdots + \prod_{i=1}^{i=n-1}\left(1 - \frac{1}{P_i}\right)\}$$

$$= \{x\prod_{i=1}^{i=n}(\frac{P_i - 1}{P_i})\} + \{\sum_{i=1}^{i=n-1}(P_{i+1}(\prod_{j=1}^{j=i}(\frac{P_j - 1}{P_j})))\} - \{\sum_{i=1}^{i=n-1}\prod_{j=1}^{j=i}(\frac{P_i - 1}{P_i})\} \quad \text{for } n \ge 3$$

So for $n \ge 3$

$$\varphi(x) = \{x\prod_{i=1}^{i=n}(\frac{P_i - 1}{P_i})\} + \{\sum_{i=1}^{i=n-1}(P_{i+1}(\prod_{j=1}^{j=i}(\frac{P_j - 1}{P_j})))\} - \{\sum_{i=1}^{i=n-1}\prod_{j=1}^{j=i}(\frac{P_j - 1}{P_j})\}$$

We can make some modifications on the primes distribution function to reach a better prime counting function, which may lead to a better defined to the prime number combination.

References:

[1] Riemann, Bernhard.(1859). The Number of Prime Numbers Less than a Given Quantity. Translated David R. Wilkins.(1998).

[2] Telfah, Ahmad. (2018).A proof of Legendre’s conjecture and Andrica’s conjecture. Preprint in Research Gate.