Interval Sieve Algorithm

Creating a Countable Set of Real Numbers from a Closed Interval

I. Introduction

I wrote a paper posted on viXra.org e-Print archive (**viXra:1806.0030**) titled The Function f(x) = C and the Continuum Hypothesis wherein I proposed a proof of the CH. In the paper I showed that by indexing (using a unique natural number for each index value) the calculation of the function's range values for each element of the domain I could establish a one-to-one correspondence between the domain and the index. It was pointed out that in order to present the domain of the function in the form of a list, which I needed to do in order to create the index, I first had to prove that the list contained every element in the interval from which the domain was defined.

I began working on the problem and came up with what I call the Interval Sieve Algorithm which uses a method of repeatedly dividing a closed interval into a series of closed sub-intervals and using numbers I call relative bounds of conjoined interval pairs to construct the list that will be used as the domain of $f(x) = C$.

II. Abstract

The Interval Sieve Algorithm is a method for generating a list of real numbers on any closed interval $[r_i, r_j]$ where $r_i < r_j$, which can then be defined as the domain of the function $f(x) = C$.

The purpose of this paper is to delineate the steps of the algorithm and show how it will generate a countable list from which the domain for the function $f(x) = C$ can be defined. Having constructed the list we will prove that the list is complete, that it contains all the numbers in the interval $[r_i, r_j]$.

Lastly we will demonstrate a restricted proof of the Continuum Hypothesis.

III. Given

1. The set of natural numbers

$$\mathbb{N}, \{n \in \mathbb{N} \mid 1 \leq n\}$$

2. The set of real numbers

$$\mathbb{R}, \{r \in \mathbb{R} \mid r \text{ is real}\}$$

3. The closed interval

$$[r_1, r_2] \text{ where } r_1 < r_2 \text{ and } r_1, r_2 \text{ are real numbers}$$

4. The list

$$L = \{r_1, r_2\}$$

IV. Definitions

1. The **lower bound** of a closed interval is the smaller of the two numbers comprising the interval. In the interval $[r_1, r_2]$ where $r_1 < r_2$, r_1 is the lower bound of the interval.

2. The **upper bound** of a closed interval is the larger of the two numbers comprising the interval. In the interval $[r_1, r_2]$, where $r_1 < r_2$, r_2 is the upper bound of the interval.

4. A **conjoined interval pair** is a pair of closed intervals where the upper bound of one and the lower bound of the other are the same number. $[r_i, [r_k,] r_j]$ is an example of a conjoined interval pair where r_k is both the upper bound of $[r_i, r_k]$ and the lower bound of $[r_k, r_j]$.

5. A **relative bound** is a number that is common to both intervals in a conjoined interval pair. In the conjoined interval pair $[r_1, [r_3,] r_2]$, where $r_1 < r_3 < r_2$, r_3 is a relative bound in both intervals $[r_1, r_3]$ and $[r_3, r_2]$.

The importance of the relative bound will become apparent when we get into the description of the Interval Sieve Algorithm.

6. The **immediate predecessor** of a number λ is a number β such that there exists no number δ where $\beta < \delta < \lambda$.

7. The **immediate successor** of a number λ is a number β such that there exists no number δ where $\lambda < \delta < \beta$.

From definitions 6 and 7, for any 2 real numbers λ and β, in the interval, we can always find another real number, δ, such that if $\lambda > \beta$ then $\beta < \delta < \lambda$ and if $\lambda < \beta$ then $\lambda < \delta < \beta$.

V. The Interval Sieve Algorithm

Procedure:

0. We begin the procedure with the interval

$$[r_1, r_2] \text{ where } r_1 < r_2 \text{ and } r_1, r_2 \text{ are real numbers}$$

and the list

$$L = \{r_1, r_2\}$$

1. Sub-divide each interval $[r_i, r_j]$ by selecting a number r_k such that $r_i < r_k < r_j$ to get a conjoined interval pair:

$$[r_i, \ [r_k,] \ r_j]$$

2. Insert the relative bound number, r_k, into the list L to get

$$L = \{r_i, r_k, r_j\}$$

3. Return to step 1.

The algorithm produces the following results:

Interval Sieve Algorithm

$[r_1$																$r_2]$
$[r_1$								$[r_3]$								$r_2]$
$[r_1$				$[r_4]$				$[r_3]$				$[r_5]$				$r_2]$
$[r_1$		$[r_6]$		$[r_4]$		$[r_7]$		$[r_3]$		$[r_8]$		$[r_5]$		$[r_9]$		$r_2]$
$[r_1$	$[r_{10}]$	$[r_6]$	$[r_{11}]$	$[r_4]$	$[r_{12}]$	$[r_7]$	$[r_{13}]$	$[r_3]$	$[r_{14}]$	$[r_8]$	$[r_{15}]$	$[r_5]$	$[r_{16}]$	$[r_9]$	$[r_{17}]$	$r_2]$

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Intervals Generated by the Algorithm

$[r_1, r_2]$

[r1, r3][r3, r2]

[r1, r4][r4, r3][r3, r5][r5, r2]

[r1, r6][r6, r4][r4, r7][r7, r3][r3, r8][r8, r5][r5, r9][r9, r2]

$[r_1, r_{10}][r_{10}, r_6][r_6, r_{11}][r_{11}, r_4][r_4, r_{12}][r_{12}, r_7][r_7, r_{13}][r_{13}, r_3][r_3, r_{14}][r_{14}, r_8][r_8, r_{15}][r_{15}, r_5][r_5, r_{16}][r_{16}, r_9][r_9, r_2]$

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List of Real Numbers Generated by the Algothrim

$L = \{r_1, r_2\}$

$L = \{r_1, r_3, r_2\}$

$L = \{r_1, r_4, r_3, r_5, r_2\}$

$L = \{r_1, r_6, r_4, r_7, r_3, r_8, r_5, r_9, r_2\}$

$L = \{r_1, r_{10}, r_6, r_{11}, r_4, r_{12}, r_7, r_{13}, r_3, r_{14}, r_8, r_{15}, r_5, r_{16}, r_9, r_{17}, r_2\}$

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Beginning with one interval, growth of the number of intervals created is exponential and after the fourth iteration we have a total of 16 intervals. If n is the number of iterations and I is the number of intervals, we have $I = 2^n$ and if L_n is the number of list elements then
$L_n = 2^n + 1$.

VI. Proving the List is Complete

The question remains as to whether or not the list L will contain all real numbers in $[r_1, r_2]$. With the help of Cantor's Diagonal Argument we will prove that: **All the real numbers in $[r_1, r_2]$ are contained in the list L**.

Proof:

Let each number in L be represented by its digits so that:

$r_1 = d_1d_2d_3d_4\ldots$

…

$r_3 = d_1d_2d_3d_4\ldots$

…

$r_2 = d_1d_2d_3d_4\ldots$

Because $[r_1, r_2]$ is a closed interval and all numbers in between are $> r_1$ and $< r_2$ we can assign r_1 as the absolute first element of L and r_2 as the absolute last element of L. Transferring the elements of L into a vertical list, that we'll call List B, allows us to employ the diagonal argument to generate a number X that is not contained in List B and then show that X will be contained in L.

Examining X we note that:

1. If $X < r_1$ or $X > r_2$ then X is not in $[r_1, r_2]$ and is of no consequence since we are considering only numbers within $[r_1, r_2]$.

2. If X is in $[r_1, r_2]$ then at any point in time it must be either a member of a sub-interval contained in $[r_1, r_2]$ or the relative bound of a conjoined interval pair in $[r_1, r_2]$.

3. If X is a relative bound of a conjoined interval pair in $[r_1, r_2]$ it is already an element of L.

4. If X is a member of a sub-interval contained in $[r_1, r_2]$ and not a relative bound, then at some point it will designated a relative bound of a conjoined interval pair contained in $[r_1, r_2]$ by the algorithm.

5. Once X becomes a relative bound of a conjoined interval pair it will be included in L and become a member of L.

6. There are no other cases regarding the nature of X to consider, therefore at any point in time, all numbers X_i are or will be elements of L.

7. We can then assert that at infinity L will be complete and this ends the proof.

Having proved that L is complete we can now define a domain D of $f(x) = C$ as

$D = (r_1, \ldots r_3, \ldots r_2)$.

VII. An Unintended Consequence

In this paper we initially set out to accomplish three things:

> 1. Construct a list of real numbers from $[r_1, r_2]$ and
>
> 2. Prove that the list is complete; that is that the list contains all the real numbers in $[r_1, r_2]$.
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> 3. Use L to define the domain of the function $f(x) = C$ for the purpose of proving the Continuum Hypothesis as outlined in The Function f(x) = C and the Continuum Hypothesis.

We have constructed the list L from $[r_1, r_2]$ and have shown that the list is complete, containing all the real numbers in $[r_1, r_2]$. Our construction requires only two givens: the interval

$$[r_1, r_2] \text{ where } r_1 < r_2 \text{ and } r_1, r_2 \text{ are real numbers}$$

and the list

$$L = \{r_1, r_2\}.$$

The interval sieve algorithm constructs L which, with the aid of Cantor's Diagonal Argument, has been shown to be complete at infinity. We can now use the domain D as defined above as the domain of $f(x) = C$.

However (the unintended consequence), to demonstrate that there exists a bijective function from $\mathbb{N}$ to $[r_1, r_2]$,

$$f : \mathbb{N} \leftrightarrow [r_1, r_2]$$

we only have to represent the numbers in the list as their digitized equivalents, drop the r from their condensed forms and change = to ↔ to go from

$r_1 = d_1d_2d_3d_4\ldots$		$1 \leftrightarrow d_1d_2d_3d_4\ldots$
$\ldots$		$\ldots$
$r_3 = d_1d_2d_3d_4\ldots$	to	$3 \leftrightarrow d_1d_2d_3d_4\ldots$
$\ldots$		$\ldots$
$r_2 = d_1d_2d_3d_4\ldots$		$2 \leftrightarrow d_1d_2d_3d_4\ldots$

Since the set comprised of the elements of L is complete and for every element of the set there is a corresponding natural number we can conclude that $f : \mathbb{N} \leftrightarrow [r_1, r_2]$ exists and therefore the Continuum Hypothesis is true for closed intervals of real numbers.

Because we are only considering closed intervals of reals this proof of the CH is restricted to that special case.